Advances in Mathematics: Scientific Journal **10** (2021), no.4, 2269–2283
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.40

FORCED DIFFUSION EQUATION WITH PIECEWISE CONTINUOUS TIME DELAY

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ABSTRACT. In this article, we study the boundary value problem (BVP) for forced diffusion equation with piecewise constant arguments. We give explicit formula for solving BVP. The problem of finding periodic solutions of some differential equations with piecewise constant arguments (DEPCA) is reduced to solving a system of algebraic equations. Using the method of finding periodic solutions of DEPCA, the solutions of BVP are given in several examples which are periodic in time.

1. INTRODUCTION

Differential equations with piecewise constant arguments are usually referred to as a hybrid system, and could model certain harmonic oscillators with almost periodic forcing [3], [5]. This type of equation, in which techniques of differential and difference equations are combined, models, among others, some biological phenomena (see [1], [2] and references therein), the stabilization of hybrid control systems with feedback discrete controller [4]. For a survey of

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2020 *Mathematics Subject Classification.* 35-xx, 47-xx, 47A10.

Key words and phrases. Forced heat equation, Piecewise constant argument, Differential equation, Periodic solution.

Submitted: 06.04.2021; *Accepted:* 26.04.2021; *Published:* 30.04.2021.

work on ordinary and partial differential equations with piecewise constant arguments (DEPCA) we refer the reader to [7], [8] and [16]. In [6] investigated DEPCA coupled with nonlinear boundary value conditions, the existence of extremal solutions and extremal quasi-solutions is obtained. A recently published papers [9], [10] has studied the second order DEPCA. The periodic solvable problem are reduced to the study a system of linear equations. Furthermore, by applying the well-known properties of linear system in algebra, all existence conditions are described for n-periodic solutions that yield explicit formula for the solutions of DEPCA.

With respect to the PDEs with piecewise constant time, it has been shown in [12] that they naturally arise in the process of approximating PDEs using piecewise constant arguments. And it is important to investigate boundary value problems (BVP) and initial-value problems for EPCA in partial derivatives and explore the influence of certain discontinuous delays on the behavior of solutions to some typical problems of mathematical physics. For example, the measuring the lateral heat change at discrete moments of time leads to the equation with piecewise continuous delay [15]:

$$\begin{aligned}
&u_t(x,t) - a^2 u_{xx}(x,t) + bu(x,[t]) = 0, \quad 0 < x < 1,\, t > 0,\\
&u(0,t) = u(1,t) = 0, \quad t > 0,\\
&u(x,0) = v(x), \quad 0 < x < 1.
\end{aligned}$$

Partial differential equations with piecewise constant argument (PDEPCA) were introduced in [13] to study the existence, oscillation, and asymptotic bounds of the solutions of initial value problems with piecewise constant delays. The diffusion equation with boundary value problems, and initial value problems assuming piecewise constant arguments were investigated in [15] and [14], respectively.

In 2015, Veloz and Pinto [11] studied the following equation with piecewise constant argument of generalized type of the form

$$u_t(x,t) = a^2(t) u_{xx}(x,t) - b(t)u(x,\gamma(t)),$$

where $\gamma(t)$ is a step function. The authors establish conditions where the convergence of the solution can be verified computing a finite number of terms of the series in each constancy interval, without requiring any regularity on the initial condition.

In the present paper, we consider the boundary value problem (BVP) in [15] for forced diffusion equation with piecewise constant arguments

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t]) + F(x,t), 0 < x < 1, t > 0, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = v(x), \tag{1.3}$$

where $F(x,t)$ is a continuous function on $[0,1] \times [0,\infty)$.

By adapting the method [15], we first represent formal series solution of the BVP, which reduces to solving the first order DEPCA with respect time. Then we give convergence condition for this series solution. Further, applying the method used in [9] and [10] we obtain existence condition and explicit formula for the periodic solutions of DEPCA. That allowed to find the exact solutions of BVP in a several examples which are periodic in time. Recall that for the case when $F(x,t) = 0$, the series solution coincide with the results of [15]. Moreover, existence conditions are obtained for an explicit form k-periodic (with respect to t) of any terms of the series solution (see Remark 4.1), which improves the results of article [15].

2. Definition of solution. Examples

A solution of (1.1-1.3) is defined in [12], [15] as follows

Definition 2.1. *A function $u(x,t)$ is called a solution of (1.1-1.3) if the following conditions are satisfied:*

- (i) *$u(x,t)$ is continuous on $\Omega = [0,1] \times \mathbf{R}_+$, $\mathbf{R}_+ = [\mathbf{0},\infty)$;*
- (ii) *u_t and u_{xx} exist and are continuous in Ω, with possible exception at points $(x,[t]) \in \Omega$, where one-sided derivatives exist with respect to second argument;*
- $(iii\ )$ *$u(x,t)$ satisfies Eq. (1.1) in Ω, with the possible exception at the points $(x,[t]) \in \Omega$ and conditions (1.2), (1.3).*

Example 1. *Let* $a\pi = 1$, $b(e-1) = 1$. *Then the function* $u(x,t) = T(t)\sin\pi x$ *is solution of the problem*

$$\begin{aligned}
&u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t]) + \sin\pi t \sin\pi x, \quad 0 < x < 1, t > 0,\\
&u(0,t) = u(1,t) = 0,\\
&u(x,0) = T(0)\sin\pi x,
\end{aligned}$$

where $T(\cdot)$ *is the* 2*-periodic solution of the differential equation with piecewise constant argument*

$$T^{'}(t) + T(t) = -\frac{1}{e-1}T([t]) + \sin\pi t, \quad t > 0,$$

defined as

$$T(t) = \begin{cases} e^{-t}\big(T(0) + \Phi(t) - \frac{1}{e-1}T(0)(e^t - 1)\big), & t \in [0,1),\\ e^{-t}\big(\Phi(t) - \frac{1}{e-1}T(1)(e^t - e)\big), & t \in [1,2]. \end{cases}$$

Here

$$\Phi(t) = \int_0^t e^s \sin\pi s ds = \frac{e^t}{1+\pi^2}(\sin\pi t - \pi\cos\pi t) + \frac{\pi}{1+\pi^2}$$

and

$$T(0) = \frac{\Phi(2) - \Phi(1)}{e^2}, \quad T(1) = \frac{\Phi(1)}{e}.$$

Example 2. *Let* $a\pi = 1$, $b = -4$. *Then the function* $u(x,t) = -\frac{1}{3}\sin\pi x + (\sin\pi t + T_0)\sin 2\pi x$ *is solution of the problem*

$$\begin{aligned}
&u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t]) + \sin\pi x + (4\sin\pi t + \pi\cos\pi t)\sin 2\pi x,\\
&\qquad 0 < x < 1, t > 0,\\
&u(0,t) = u(1,t) = 0,\\
&u(x,0) = -\tfrac{1}{3}\sin\pi x + T_0 \sin 2\pi x.
\end{aligned}$$

Example 3. *Let* $b = -(an\pi)^2$. *Then the function* $u(x,t) = T_0 \sin n\pi x$ *is solution of the problem*

$$\begin{aligned}
&u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t]), \quad 0 < x < 1, t > 0,\\
&u(0,t) = u(1,t) = 0,\\
&u(x,0) = T_0 \sin n\pi x.
\end{aligned}$$

Example 4. *Let* $an\pi = 1$*,* $b = \frac{e+1}{e-1}$*. Then the function* $u(x,t) = T(t)\sin n\pi x$ *is solution of the problem*

$$\begin{aligned}
&u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t]) + (4\sin \pi t + \pi\cos \pi t)\sin n\pi x,\\
&\qquad 0 < x < 1, t > 0,\\
&u(0,t) = u(1,t) = 0,\\
&u(x,0) = T(0)\sin n\pi x,
\end{aligned}$$

where $T(\cdot)$ *is the* 2*-periodic solution of the differential equation with piecewise constant argument*

$$T^{'}(t) + T(t) = -\frac{e+1}{e-1}T([t]) + 4\sin \pi t + \pi\cos \pi t, \quad t > 0,$$

defined as

$$T(t) = \begin{cases} e^{-t}\big(T(0) + e^t \sin \pi t - \frac{e+1}{e-1}T(0)(e^t - 1)\big), & t \in [0,1),\\ e^{-t}\big(e^t \sin \pi t - eT(0) + \frac{e+1}{e-1}T(0)(e^t - e)\big), & t \in [1,2]. \end{cases}$$

3. The solution of the problem

For the non-homogeneous problem (1.1) we look for a solution in the form

$$u(x,t) = \sum_{j=1}^{\infty} T_j(t)\sin j\pi x. \tag{3.1}$$

We can notice from the initial data (1.3) that

$$u(x,0) = \sum_{j=1}^{\infty} T_j(0)\sin j\pi x = v(x),$$

where $v_j = T_j(0)$ are the Fourier coefficients of v, i.e.,

$$v(x) = \sum_{j=1}^{\infty} v_j \sin j\pi x, \qquad v_j = 2\int_0^1 v(x)\sin j\pi x dx.$$

Next we find $\{F_j(t)\}$ so that

$$F(x,t) = \sum_{j=1}^{\infty} F_j(t)\sin j\pi x, \tag{3.2}$$

by setting

$$F_j(t) = 2\int_0^1 F(x,t)\sin j\pi x dx.$$

Substituting (3.1) and (3.2) into (1.1) we obtain

$$\sum_{j=1}^{\infty}\Big(T_j'(t)+a^2\pi^2j^2T_j(t)+bT_j([t])-F_j(t)\Big)\sin j\pi x=0.$$

Then using the orthogonality of the functions $\sin n\pi x$ we obtain an infinite sequence of the DEPCA

$$T_j'(t)+a^2\pi^2j^2T_j(t)+bT_j([t])=F_j(t), t>0,\quad j=1,2,\dots. \tag{3.3}$$

with

$$T_j(0)=v_j. \tag{3.4}$$

Let $T_{nj}(t)$ denote a solution of (3.3) on the interval $[n,n+1)$, i.e.,

$$T_j(t)=T_{nj}(t),\, t\in[n,n+1),\quad n=0,1,2,\dots.$$

Then

$$T_{nj}'(t)+a^2\pi^2j^2T_{nj}(t)=F_{nj}(t)-bT_{nj}(n),\quad t\in[n,n+1), \tag{3.5}$$

where $F_{nj}(t)=F_j(t)$.

The solution of (3.5) is

$$T_{nj}(t)=e^{-a^2\pi^2j^2t}\int_n^t e^{a^2\pi^2j^2s}F_{nj}(s)ds-\frac{bT_{nj}(n)}{a^2\pi^2j^2}\big(1-e^{-a^2\pi^2j^2(t-n)}\big)+T_{nj}(n)e^{-a^2\pi^2j^2(t-n)}.$$

That is,

$$T_{nj}(t)=e^{-a^2\pi^2j^2t}\Phi_{nj}(t)+E_j(t-n)T_{nj}(n), \tag{3.6}$$

where

$$\Phi_{nj}(t)=\int_n^t e^{a^2\pi^2j^2s}F_{nj}(s)ds,$$

$$E_j(t)=e^{-a^2\pi^2j^2t}-\frac{b}{a^2\pi^2j^2}\big(1-e^{-a^2\pi^2j^2t}\big).$$

Setting $t=n+1$ in (3.6), we have

$$T_{nj}(n+1)=e^{-a^2\pi^2j^2(n+1)}\Phi_{nj}(n+1)+E_j(1)T_{nj}(n).$$

It follows from continuity $T_j(t)$ in $t>0$ that

$$T_{nj}(n+1)=T_{n+1,j}(n+1).$$

Hence,

$$T_{n+1,j}(n+1) = e^{-a^2\pi^2 j^2(n+1)}\Phi_{nj}(n+1) + E_j(1)T_{nj}(n).$$

This gives

$$\begin{aligned}T_{nj}(n) &= e^{-a^2\pi^2 j^2 n}\Phi_{n-1,j}(n) + E_j(1)T_{n-1,j}(n-1)\\ &= e^{-a^2\pi^2 j^2 n}\Phi_{n-1,j}(n) + E_j(1)e^{-a^2\pi^2 j^2(n-1)}\Phi_{n-2,j}(n-1) + E_j^2(1)T_{n-2,j}(n-2)\\ &= e^{-a^2\pi^2 j^2 n}\Phi_{n-1,j}(n) + E_j(1)e^{-a^2\pi^2 j^2(n-1)}\Phi_{n-2,j}(n-1)\\ &\quad + E_j^2(1)e^{-a^2\pi^2 j^2(n-2)}\Phi_{n-3,j}(n-2)y\\ &\quad + \cdots + E_j^{n-1}(1)e^{-a^2\pi^2 j^2}\Phi_{0j}(1) + E_j^n(1)T_{0j}(0)\end{aligned}$$

or

$$T_{nj}(n) = e^{-a^2\pi^2 j^2 n}\Phi_{n-1,j}(n) + \Psi_{nj} + E_j^n(1)T_{0j}(0),$$

where

$$\begin{aligned}\Psi_{nj} &= E_j(1)e^{-a^2\pi^2 j^2(n-1)}\Phi_{n-2,j}(n-1) + E_j^2(1)e^{-a^2\pi^2 j^2(n-2)}\Phi_{n-3,j}(n-2)\\ &\quad + \cdots + E_j^{n-1}(1)e^{-a^2\pi^2 j^2}\Phi_{0j}(1).\end{aligned}$$

Therefore, the solution $T_{nj}(t)$ defined by (3.6) represents as

$$T_{nj}(t) = \Phi_n(t) + E_j(t-n)E_j^n(1)T_{0j}(0), \tag{3.7}$$

where

$$\Phi_n(t) = \Phi_n(t,F) = e^{-a^2\pi^2 j^2 t}\Phi_{nj}(t) + E_j(t-n)e^{-a^2\pi^2 j^2 n}\Phi_{n-1,j}(n) + E_j(t-n)\Psi_{nj},$$

This allows to represent the proposed formal series solution $u(x,t) = u_n(x,t)$ on $[n, n+1)$, in terms of $F(\cdot)$ and $T_{0j}(0)$, of the form

$$u_n(x,t) = \sum_{j=1}^{\infty}[\Phi_n(t) + E_j(t-n)E_j^n(1)T_{0j}(0)]\sin j\pi x,\ t\in[n,n+1), \tag{3.8}$$

where $T_j(t) = T_{nj}(t), t\in[n,n+1)$.

Assumption 3.1 *We assume that F is continuous on $[0,1]\times[0,\infty)$ and $\frac{\partial F}{\partial t}$ exists, are continuous and bounded on $[0,1]\times(0,\infty)$. In addition, we assume that*

$$\sum_{j=1}^{\infty}|F_j(t)| < \infty \quad \textit{for any} \quad t > 0. \tag{3.9}$$

Note that the condition (3.9) provides the absolute convergence of the series

$$u_t(x,t)+u_{xx}(x,t)+bu(x,t)=\sum_{j=1}^{\infty}\Big(T_j'(t)+a^2\pi^2j^2T_j(t)+bT_j([t])\Big)\sin j\pi x,$$

since

$$\Big|\sum_{j=1}^{\infty}\Big(T_j'(t)+a^2\pi^2j^2T_j(t)+bT_j([t])\Big)\sin j\pi x\Big|\le\sum_{j=1}^{\infty}|F_j(t)|.$$

The following theorem provides us to determine the convergence of the series $\sum_{j=1}^{\infty}T_j'(t)\sin j\pi x$, $\sum_{j=1}^{\infty}j^2T_j(t)\sin j\pi x$ and $\sum_{j=1}^{\infty}T_j([t])\sin j\pi x$.

Theorem 3.1. *Let for F the Assumption 3.1 be fulfilled and $v(x)$ be integrable continuous function on $[0,1]$. Then the function $u(x,t)$ defined by (3.8) is solution of (1.1).*

Proof. We show that u_{xx} exists at every $(t,x)\in(n,n+1)\times(0,1)$ and continuous on $[n,n+1]\times[0,1]$ for any $n=0,1,\ldots$. Let $t\in[n,n+1]$. Using integrating by part we obtain

$$\begin{aligned}\Phi_{nj}(t)&=\int_n^t e^{a^2\pi^2j^2s}F_j(s)ds\\&=F_j(t)\frac{e^{a^2\pi^2j^2t}}{a^2\pi^2j^2}-F_j(n)\frac{e^{a^2\pi^2j^2n}}{a^2\pi^2j^2}-\frac{1}{a^2\pi^2j^2}\int_n^t e^{a^2\pi^2j^2s}F_j'(s)ds\\&=\frac{F_j(t)}{a^2\pi^2j^2}\Big(e^{a^2\pi^2j^2t}-e^{a^2\pi^2j^2n}\Big)+\Big(F_j(t)\\&\quad-F_j(n)\Big)\frac{e^{a^2\pi^2j^2n}}{a^2\pi^2j^2}-\frac{1}{a^2\pi^2j^2}\int_n^t e^{a^2\pi^2j^2s}F_j'(s)ds.\end{aligned}$$

This gives

$$\begin{aligned}|\Phi_{nj}(t)|\le{}&\frac{|F_j(t)|}{a^2\pi^2j^2}\Big(e^{a^2\pi^2j^2t}-e^{a^2\pi^2j^2n}\Big)+\Big|F_j(t)-F_j(n)\Big|\frac{e^{a^2\pi^2j^2n}}{a^2\pi^2j^2}\\&+\frac{F'_{max}}{a^4\pi^4j^4}\Big(e^{a^2\pi^2j^2t}-e^{a^2\pi^2j^2n}\Big),\quad t\in[n,n+1],\end{aligned}$$

where

$$F'_{\max}=\max|F_t'(t,x)|.$$

Hence

$$|e^{-a^2\pi^2j^2t}\Phi_{nj}(t)| \le \tfrac{|F_j(t)|}{a^2\pi^2j^2}\Big(1-e^{-a^2\pi^2j^2(t-n)}\Big) + \big|F_j(t)-F_j(n)\big|\tfrac{e^{-a^2\pi^2j^2(t-n)}}{a^2\pi^2j^2} + \tfrac{F'_{max}}{a^4\pi^4j^4}\Big(1-e^{-a^2\pi^2j^2(t-n)}\Big), \tag{3.10}$$

$t \in [n, n+1]$. Note that

$$e^{-a^2\pi^2j^2n}\Phi_{n-1,j}(n) = \frac{1}{a^2\pi^2j^2}\Big(F_j(n) - e^{a^2\pi^2j^2}F_j(n-1)\Big) - \frac{e^{-a^2\pi^2j^2n}}{a^2\pi^2j^2}\int_{n-1}^{n} e^{a^2\pi^2j^2s}F'_j(s)ds.$$

This gives

$$|e^{-a^2\pi^2j^2n}\Phi_{n-1,j}(n)| \le \frac{1}{a^2\pi^2j^2}\Big|F_j(n) - e^{-a^2\pi^2j^2}F_j(n-1)\Big| + \frac{F'_{max}}{a^4\pi^4j^4}(1-e^{-a^2\pi^2j^2}). \tag{3.11}$$

Since

$$E_j(t-n) = e^{-a^2\pi^2j^2(t-n)} - \frac{b}{a^2\pi^2j^2}\big(1-e^{-a^2\pi^2j^2(t-n)}\big)$$

and

$$E_j(1) = e^{-a^2\pi^2j^2} - \frac{b}{a^2\pi^2j^2}\big(1-e^{-a^2\pi^2j^2}\big),$$

we obtain

$$|E_j(t-n)e^{-a^2\pi^2j^2n}\Phi_{n-1,j}(n)| \le C_0\frac{|F_j(n)|}{a^2\pi^2j^2} + C_1\frac{1}{a^4\pi^4j^4}. \tag{3.12}$$

$$|E_j(t-n)E_j^n(1)T_{0j}(0)| \le 2\frac{|b|^n}{(a\pi)^{2n}}\frac{|T_{0j}(0)|}{j^{2n}} \quad \text{for large } j. \tag{3.13}$$

Since $|e^{-a^2\pi^2j^2(n-k)}\Phi_{n-k-1,j}(n-k)| \le \frac{C_2}{a^2\pi^2j^2}$,

$$|\Psi_{nj}| = \sum_{k=1}^{n-1}|E_j^k(1)e^{-a^2\pi^2j^2(n-k)}\Phi_{n-k-1,j}(n-k)| \le \frac{C_3}{a^2\pi^2j^2}\sum_{k=1}^{n-1}\frac{1}{j^{2k}} = \frac{C_3}{a^2\pi^2j^2}\frac{\frac{1}{j^2}-\frac{1}{j^{2n}}}{1-j^{-2}}.$$

Hence,

$$|\Psi_{nj}| \le \frac{C_4}{j^4}. \tag{3.14}$$

The estimations (3.10-3.14) give

$$|T_{nj}(t)| \leq \frac{C_5}{j^2}, \quad \text{for } t \in [n, n+1] \quad \text{and large } j.$$

This gives the convergence of the series

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -\sum_{j=1}^{\infty} \pi^2 j^2 T_{nj}(t) \sin j\pi x,\ t \in [n, n+1].$$

□

4. Periodic solutions of differential equation with piecewise constant argument

In this section we give a method of finding N-periodic solutions of the differential equation with piecewise constant argument (DEPCA)

(4.1) $$T'(t) + a^2\pi^2 j^2 T(t) + bT([t]) = f(t), t > 0, \quad j = 1, 2, \ldots,$$

where $f(t)$ is N-periodic and continuous function on $[0, \infty)$.

Let us define a definition of solution for (4.1)

Definition 4.1. *A function $T(t)$ is called a solution of (4.1) if the following conditions are satisfied:*

- (i) *$T(t)$ is continuous on $\mathbf{R}_+$;*
- (ii) *$T'(t)$ exists and is continuous in $\mathbf{R}_+$, with possible exception at points $[t] \in \mathbf{R}_+$, where one-sided derivatives exist;*
- (iii) *$T(t)$ satisfies Eq. (4.1) in $\mathbf{R}_+$, with the possible exception at the points $[t] \in \mathbf{R}_+$.*

2-periodic solutions DEPCA. We first give a method of finding periodic solutions of (4.1) and their existence conditions for the case when $f(\cdot)$ is 2-periodic function.

We seek a function T as a 2-periodic function that solves (4.1).

Integrating (4.1) we have

$$T(t) = e^{-(aj\pi)^2 t}[T(0) + \Phi(t) - \int_0^t be^{(aj\pi)^2 s} T([s]) ds].$$

where

$$\Phi(t) = \int_0^t e^{(aj\pi)^2 s} f(s) ds.$$

The function $T(t)$ can be represented as

$$T(t) = \begin{cases} e^{-(aj\pi)^2 t}[\Phi(t) + (1 + \frac{b}{(aj\pi)^2}(1 - e^{(aj\pi)^2 t}))T(0)], & t \in [0,1), \\ e^{-(aj\pi)^2 t}[\Phi(t) + (1 + \frac{b}{(aj\pi)^2}(1 - e^{(aj\pi)^2}))T(0) & \\ \quad +(\frac{b}{(aj\pi)^2}(e^{(aj\pi)^2} - e^{(aj\pi)^2 t})T(1)], & t \in [1,2). \end{cases} \tag{4.2}$$

Let we denote

$$\begin{aligned} P_i(t) &= \tfrac{b}{(aj\pi)^2}(e^{(aj\pi)^2(i-t)} - 1), \quad i = 0,1,2\ldots, \\ N_i(t) &= \tfrac{b}{(aj\pi)^2}(e^{(aj\pi)^2(i-t)} - e^{(aj\pi)^2(1+i-t)}), \quad i = 0,1,2\ldots. \end{aligned}$$

Then (4.2) represents as

$$T(t) = \begin{cases} (e^{-(aj\pi)^2 t} + P_0(t))T(0) + e^{-(aj\pi)^2 t}\Phi(t), & t \in [0,1), \\ (e^{-(aj\pi)^2 t} + N_0(t))T(0) + P_1(t)T(1) + e^{-(aj\pi)^2 t}\Phi(t), & t \in [1,2). \end{cases} \tag{4.3}$$

This shows that the right-hand side of (4.3) contains only unknown numbers $T(0)$ and $T(1)$. Since $T(\cdot)$ is continuous and periodic, $T(0) = T(2)$. We apply (4.3) to get the system of equations

$$\begin{cases} (\frac{b}{(aj\pi)^2}(1 - e^{-(aj\pi)^2}) - e^{-(aj\pi)^2})T(0) + T(1) = e^{-(aj\pi)^2}\Phi(1), \\ (1 + \frac{b}{(aj\pi)^2}(e^{-(aj\pi)^2} - e^{-2(aj\pi)^2}) - e^{-2(aj\pi)^2})T(0) \\ \quad + \frac{b}{(aj\pi)^2}(1 - e^{-(aj\pi)^2})T(1) = e^{-2(aj\pi)^2}\Phi(2), \end{cases}$$

or

$$\begin{cases} (E_1 - e^{-(aj\pi)^2})T(0) + T(1) = e^{-(aj\pi)^2}\Phi(1), \\ (1 + E_2 - e^{-2(aj\pi)^2})T(0) + E_1 T(1) = e^{-2(aj\pi)^2}\Phi(2), \end{cases} \tag{4.4}$$

where

$$E_k = \frac{b}{(aj\pi)^2}(e^{-(k-1)(aj\pi)^2} - e^{-k(aj\pi)^2}),\ k = 1,2,\ldots. \tag{4.5}$$

Let $\Delta_2(a,b)$ be the determinant of the matrix $M_2(a,b)$, where

$$M_2(a,b) = \begin{pmatrix} E_1 - e^{-(aj\pi)^2} & 1 \\ 1 + E_2 - e^{-2(aj\pi)^2} & E_1 \end{pmatrix}.$$

Thus, for 2-periodic solution of (4.1) we obtain

Theorem 4.1. *Let $f(\cdot)$ be 2-periodic function.*

(i) *If $\Delta_2(a,b) \neq 0$, then equation (4.1) has a unique 2-periodic solution having the form (4.3), where (T(0), T(1)) is the unique solution of (4.4).*

(ii) *If $\Delta_2(a,b) = 0$ and $\Phi(1) = \Phi(2) = 0$, then equation (4.1) has infinite number of 2-periodic solutions having the form*

$$T_\alpha(t) = \begin{cases} \alpha(e^{-(aj\pi)^2 t} + P_0(t))T(0) + e^{-(aj\pi)^2 t}\Phi(t), & t \in [0,1), \\ \alpha(e^{-(aj\pi)^2 t} + N_0(t))T(0) + \alpha P_1(t)T(1) + e^{-(aj\pi)^2 t}\Phi(t), & t \in [1,2], \end{cases}$$

where $(T(0), T(1))$ is an eigenvector of $M_2(a,b)$ corresponding to 0, α is any number.

(iii) *If $\Delta_2(a,b) = 0$ and the $rank[M_2(a,b)] < rank[M_2(a,b), \Phi^T], \Phi = (\Phi(1), \Phi(2))$, then equation (4.1) does not have any 2-periodic solution.*

Application Theorem 4.1 . Let F satisfy the Condition 3.1 and the functions $\{F_j(t)\}$ in the series presentation (3.2) of F be 2-periodic functions. In addition, we assume a and b such that $\Delta_2(a,b) \neq 0$ for any $j = 1, 2, \ldots$. Then by Theorem 4.1 the equation

$$T'(t) + a^2\pi^2 j^2 T(t) + bT([t]) = F_j(t), t > 0, \quad j = 1, 2, \cdots$$

has 2-periodic solutions $T(t) = T_j(t)$ for any $j = 1, 2, \ldots$.

Let we assume v such that $v_j = T_j(0)$. Then the function

$$u(x,t) = \sum_{j=1}^{\infty} T_j(t) \sin j\pi x$$

is solution of the problem (1.1-1.3), where $u(x,t)$ is 2-periodic function with respect time t.

***n*-periodic solution**. We next solve equation (4.1), where f is periodic with positive integer period $n \geq 3$. It is clear that to seek a function $T(\cdot)$ as a periodic function, we assume that $T(t) = T(t+n)$. One can see that the solution $T(\cdot)$ has a form

(4.6)

$$T(t) = \begin{cases} (e^{-(aj\pi)^2 t} + P_0(t))T(0) + e^{-(aj\pi)^2 t}\Phi(t), & t \in [0,1), \\ (e^{-(aj\pi)^2 t} + N_0(t))T(0) + P_1(t)T(1) + e^{-(aj\pi)^2 t}\Phi(t), & t \in [1,2), \\ \cdots \quad \cdots \quad \cdots \\ e^{-(aj\pi)^2 t}T(0) + \sum\limits_{k=0}^{n-2} N_k(t)T(k) + P_{n-1}(t)T(n-1) + e^{-(aj\pi)^2 t}\Phi(t), \\ \quad t \in [n-1, n] \end{cases}$$

The right-hand side of (4.6) contains only unknown numbers $T(0), \cdots, T(n-1)$. Using the periodicity conditions $T(\cdot)$ from (4.6) we have n system of equations

$$
\begin{aligned}
&(E_1 - e^{-(aj\pi)^2})T(0) + T(1) = e^{-(aj\pi)^2}\Phi(1)\\
&(E_2 - e^{-2(aj\pi)^2})T(0) + E_1 T(1) + T(2) = e^{-2(aj\pi)^2}\Phi(2)\\
&\vdots \quad \vdots \quad \vdots\\
&(E_{n-1} - e^{-(n-1)(aj\pi)^2}T(0) + \sum_{k=2}^{n-1} E_{n-k}T(k-1) + T(n-1)\\
&\quad = e^{-(n-1)(aj\pi)^2}\Phi(n-1)\\
&(1 + E_n - e^{-n(aj\pi)^2})T(0) + \sum_{k=1}^{n-1} E_{n-k}T(k) = e^{-n(aj\pi)^2}\Phi(n),
\end{aligned}
\tag{4.7}
$$

where E_k is defined in (4.5).

We denote by $D_n(a,b)$ the determinant of the equation (4.7). Then, $D_n(a,b)$ is the determinant of the matrix

$$
M_n(a,b) = \begin{pmatrix}
E_1 - e^{-(aj\pi)^2} & 1 & 0 & \dots & 0\\
E_2 - e^{-2(aj\pi)^2} & E_1 & 1 & \dots & 0\\
\vdots & \vdots & \vdots & \dots & \vdots\\
E_{n-1} - e^{-(n-1)(aj\pi)^2} & E_{n-2} & E_{n-3} & \dots & 1\\
1 + E_n - e^{-n(aj\pi)^2} & E_{n-1} & E_{n-2} & \dots & E_1
\end{pmatrix}.
$$

Summarizing these results, we get the following result.

Theorem 4.2. . *Let $f(\cdot)$ be n-periodic function.*

(i) *If $\Delta_n(a,b) \neq 0$, then equation (4.1) has a unique n-periodic solution having the form (4.6), where $(T(0), \cdots, T(n-1))$ is the unique solution of (4.7).*

(ii) *If $\Delta_n(a,b) = 0$ and $\Phi(1) = \cdots = \Phi(n) = 0$, then equation (4.1) has infinite number of n-periodic solutions having the form*

$$
T(t;\alpha) = \begin{cases}
\alpha(e^{-(aj\pi)^2 t} + P_0(t))T(0) + e^{-(aj\pi)^2 t}\Phi(t), \quad t \in [0,1),\\
\alpha(e^{-(aj\pi)^2 t} + N_0(t))T(0) + \alpha P_1(t)T(1) + e^{-(aj\pi)^2 t}\Phi(t), \quad t \in [1,2),\\
\vdots \quad \vdots \quad \vdots\\
\alpha\big(e^{-(aj\pi)^2 t}T(0) + \sum_{k=0}^{n-2} N_k(t)T(k) + P_{n-1}(t)T(n-1)\big)\\
\quad + e^{-(aj\pi)^2 t}\Phi(t), \quad t \in [n-1, n]
\end{cases}
$$

where $(T(0), \cdots, T(n-1))$ *is an eigenvector of* $M_n(a,b)$ *corresponding to* 0*,* α *is any number.*

(iii) *If* $\Delta_n(a,b) = 0$ *and the* $rank[M_n(a,b)] < rank[M_n(a,b), \Phi^T], \Phi = (\Phi(1), \cdots, \Phi(n))$*, then equation (4.1) does not have any* n*-periodic solution.*

Remark 4.1. *Let* $F(x,t) = 0$*, then* $\Phi(1) = \cdots = \Phi(n) = 0$*. Moreover, in this case, if for any real or complex* a *and* b*,* $\Delta_n(a,b) = 0$ *then the function* $T_j(t)$ *in (3.1) is* n*-periodic with* $T_j(t) = T(t;\alpha)$*, where* α *from the boundary condition (3.4) is* $\alpha = v_j$*.*

Example 5. *We consider equation (4.1) with* $b = -a^2\pi^2 j^2$ *and* $f(t) = a^2\pi^2 n^2 \sin 2\pi t + 2\pi \cos 2\pi t$*. So,* $f(t)$ *is* n*-periodic function. One can see that*

$$\Phi(t) = \int_0^t e^{(an\pi)^2 s} f(s) ds = e^{(an\pi)^2 t} \sin 2\pi t \quad \textit{and} \quad \Delta_n(a,b) = 0,$$

where

$$M_n(a,b) = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ -e^b & -1+e^b & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -e^{(n-2)b} & -e^{(n-3)b} + e^{(n-2)b} & -e^{(n-4)b} + e^{(n-3)b} & \dots & 1 \\ 1 - e^{(n-1)b} & -e^{(n-2)b} + e^{(n-1)b} & -e^{(n-3)b} + e^{(n-2)b} & \dots & -1+e^b \end{pmatrix}.$$

The eigenvector $T_n = (T(0), \cdots, T(n-1))$ *of* $M_n(a,b)$ *corresponding to* 0 *is* $T_n = (1, \cdots, 1)$*. Moreover, the* n*-periodic solution* $T(t,\alpha)$ *of the equation is* $T(t,\alpha) = \alpha + \sin 2\pi t$*.*

Acknowledgements. This work was supported by the Malaysian Ministry of Education (MOE) through the Research Management Center (RMC), Universiti Teknologi Malaysia (FRGS, Votes: 5F235)

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