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AN EFFICIENT NUMERICAL APPROACH FOR SOLVING ABEL'S INTEGRAL EQUATIONS BY USING MODIFIED TAYLOR WAVELETS

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ABSTRACT. In this paper, a modified Taylor wavelet method (MTWM) is developed for numerical solutions of various types of Abel's integral equations. This method is based on the modified Taylor wavelet (MTW) approximation. The purpose behind using the MTW approximation is to transform the introduction problems into an equivalent set of algebraic equations. To check the accuracy and applicability of the proposed method, some examples have been solved and compared with other existing methods.

1. INTRODUCTION

In the realm of scientific research, and applied mathematics, integral equation (IE) is one of the best tools. Abel's integral equation (AIE) is one of the major equations that are directly linked to an existing problem of physics without passing through the differential equation. Integral equations (IEs) are arising naturally in distinct areas of basic sciences and engineering. Such as physical electronics, solid state physics, plasma physics, mathematical physics, astrophysics,

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soiled mechanics, microscopy, chemical reactions, X-ray radiography, heat conduction, fluid mechanics, fluid flow, Scattering theory, semiconductors, plasma diagnostics, mathematical biology etc. [1]- [12]. Abel computed a special IE of the Volterra type to solve the following problem.

Let a substance point of mass m moving under the influence of gravity on a flat curve lying in a vertical plane. Let the time t that is expected for substance point to move across the curve from the height z to the lowest point of the curve by a given function f of z, which yields the IE.

$$f(z) = \int_0^z \frac{u(t)}{\sqrt{2g(z-t)}} dt,$$

where g is the acceleration due to the gravity.

Abel (1823) derived the following equation after generalization of the tautochrone problem

$$f(z) = \int_0^z \frac{u(t)}{\sqrt{(z-t)}} dt, \tag{1.1}$$

where, $f(z)$ are known, and function $u(z)$ is unknown to be determined. This is a special case of first type linear volterra integral equation.

Many mathematical models of real life and physical phenomena usually results in IEs. In this study, our focus about the numerical solution of famous AIE given in equation (1.1).

Currently, many numerical techniques or methods are available for famous AIE (1.1). Among these methods, wavelet based method is more efficient and attractive. There are, different kinds of wavelet methods and techniques are applied to get the approximate numerical solution of famous AIE (1.1) namely, Haar wavelet [13], Legendre wavelet [14], B-polynomial multiwavelet [15], Bernoulli wavelet [16], Chebyshev wavelet first kind [17], second kind Chebyshev wavelet [18], block-pulse operational matrix method [19], collocation method [20], product integration method [21], Homotopy analysis transform method [22], New approach [23]. Operational matrix method [24], Euler-Maclaurin summation formula [25] etc. The goal of proposed MTWM is to find more convenient approximate solution for the famous AIE (1.1).

The rest structure of present article is summarized as follows: in section 2, preliminaries of modified Taylor wavelet with function approximation is discussed. Section 3 introduces convergence analysis. In Section 4 description of introduce method for solving famous AIE is discussed with algorithm and Pseudocode. In section 5, numerical Problems are presented. Finally, in last section we draw conclusions.

2. METHODS AND PRELIMINARIES OF MODIFIED TAYLOR WAVELETS

2.1. **Basic definition of wavelet.** A family of wavelets is made up of a mother wavelets and dilated and translated forms of mother wavelet. When dilation parameter r and translation parameter s vary continuously, we can get the following family of continuous wavelets [26]- [30]:

$$\psi_{r,s}(z) = |r|^{-1/2}\psi\left(\frac{z-s}{r}\right); \qquad r, s \in R;\ r \neq 0.$$

If the parameters r and s taken to the discrete values as $r = r_0^{-k},\ \ s = ns_0 r_0^{-k}$, $r_0 > 1\,, s_0 > 0$ and $n\,, k \in Z^+$ then family of discrete wavelets is obtained as,

$$\psi_{k,n}\,(z) = \left|r_0^{-k}\right|^{-1/2}\psi\left(\frac{z - ns_0 r_0^{-k}}{r_0^{-k}}\right) = |r_0|^{k/2}\psi(r_0^k z - ns_0),$$

where $\psi_{k,n}\,(z)$ form a wavelet basis for $L^2(R)$.

2.2. **Taylor wavelet.** Taylor wavelets $\psi_{n,m}(z) = \psi(k, \widehat{n}, m, z)$ have four arguments: $\widehat{n}= n-1,\ n = 1, 2, \ldots, 2^{k-1}\,,\ k \in Z^+$, m is the order and z is the normalized time for Taylor polynomial, Phan Than Toan, Thieu N. Vo., and Mohsen Razzaghi [31], have defined them on the interval $[0, 1]$ as follows,

$$\psi_{n,m}(z) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{\tau}_m(2^{k-1}z - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \le z < \frac{\hat{n}+1}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases},$$

with $\tilde{\tau}_m(z\,) = \sqrt{2m+1}\,\tau_m(z)$, where $m = 0, 1, 2, M-1$ and $n = 1, 2, \ldots, 2^{k-1}$.

The coefficient $\sqrt{2m+1}$ is for normality. The dilation parameter is $r = 2^{-(k-1)}$ and translation parameter is $s = \hat{n}2^{-(k-1)}$. Here $\tau_m(z)$ are the Taylor polynomials of order m, which can be defined by $\tau_m(z) = z^m$.

2.3. **Modified Taylor wavelet (MTW).** Modified Taylor wavelet $\Psi_{n,m}(z) = \Psi(k, \widehat{n}, m, z)$ have four arguments: $\widehat{n}= n-1,\ n = 1, 2, \ldots, 2^{k-1}\,,\ k \in Z^+$, m is the order and z is the normalized time for modified Taylor polynomial (MTP) they are defined on the interval $[0, 1]$ as:

$$\Psi_{n,m}(z) = \begin{cases} \sqrt{2^{k-1}}\, \tilde{\xi}_m(2^{k-1}z - n + 1), & \frac{n-1}{2^{k-1}} \le z < \frac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases}, \tag{2.1}$$

where $\tilde{\xi}_m(2^{k-1}z - n + 1)$ modified Taylor polynomials (MTPs) of order m which are orthogonal in interval $[0, 1]$. These MTPs are obtained after applying famous Gram-smith orthogonalization processes on $\tilde{\tau}_m(z)$. Also these MTPs of degree m can be calculated directly by the following recursive formula:

$$\tilde{\xi}_m(2^{k-1}z - n + 1) = (2m + 1)^{\frac{1}{2}} \left(\frac{(m!)^2}{(2m)!}\right) \Re_m(2^{k-1}z - n + 1), \tag{2.2}$$

where, symbol ! represent factorial sign and$\Re_m(2^{k-1}z - n + 1)$ are classical Legendre function of degree m [32].

Thus some MTPs for $k = 1$ are $\tilde{\xi}_0(z) = 1$, $\tilde{\xi}_1(z) = \sqrt{3}\,(z - \frac{1}{2})$, $\tilde{\xi}_2(z) = \sqrt{5}\,(z^2 - z + \frac{1}{6})$, $\tilde{\xi}_3(z) = \sqrt{7}\,(z^3 - \frac{3}{2}z + \frac{3}{5}z - \frac{1}{20})$.

The set of the MTWs are orthogonal in $[0, 1]$ i.e.,

$$\left\langle \Psi_{n,m}(z),\, \Psi_{n',m'}(z) \right\rangle = \begin{cases} \left(\frac{(m!)^2}{(2m)!}\right)^2, & n = n',\, m = m' \\ g0, & \text{otherwise} \end{cases}. \tag{2.3}$$

Now substitute $k = 1$, $n = 1$ and $M = 12$ in equation (2.1) and using recurrence relation (2.2), then the twelve MTWs are depicted in the following figure (1).

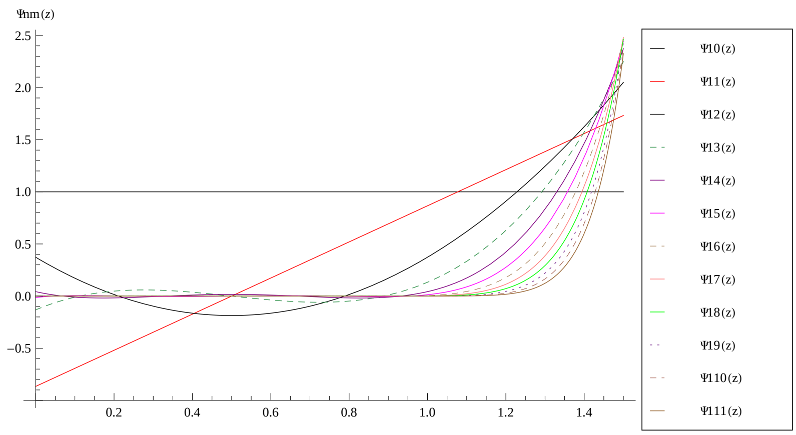

FIGURE 1. Twelve MTWS for $k = 1$, $n = 1$ and $M = 12$.

2.4. **Function approximation.** Any function $u(z) \in L^2(R)$defined over $[0,\, 1]$ may be expand as linear combination of MTWs series as

$$u(z) \approx \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \sigma_{n,m} \Psi_{n,m}(z), \tag{2.4}$$

where $\sigma_{n,m} = \langle u(z),\ \Psi_{n,m}(z)\rangle$in which $\langle .,.\rangle$denotes the inner product. If the infinite series given in equation (2.4) is truncated then equation (2.4) can be written as

$$u(z) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} \sigma_{n,m}\, \Psi_{n,m}(z) = \sigma^T \Psi(z)\ , \tag{2.5}$$

where σ and $\Psi(z)$ are $2^{k-1}M\ \times 1$ matrices given by

$$\sigma = \left[\sigma_{1,0},\ \ldots,\ \sigma_{1,M-1},\ \sigma_{2,0},\ \ldots,\ \sigma_{2,M-1},\ \ldots,\ \sigma_{2^{k-1},0},\ldots,\ \sigma_{2^{k-1},M-1}\right]^T, \tag{2.6}$$

$$\begin{aligned} \Psi(z) = [&\Psi_{1,0}(z),\ \ldots,\ \Psi_{1,M-1}(z),\ \Psi_{2,0}(z),\ \ldots,\ \Psi_{2,M-1}(z),\ \ldots, \\ &\Psi_{2^{k-1},0}(z),\ \ldots,\ \Psi_{2^{k-1},M-1}(z)]^T . \end{aligned} \tag{2.7}$$

3. Convergence analysis

Theorem 3.1. *If $u(z) \in L^2(R)$ be a continuous and bounded function defined on the interval $[0,1)$, such that $u(z) \le k$ then the Modified Taylor Wavelets (MTW) series of $u(z)$ converges uniformly.*

Proof. Let $u(z)$ be the bounded function on $[0,1)$ then Modified Taylor Wavelets coefficients of $u(z)$ is defined by

$$\sigma_{n,m} = \int_0^1 u(z)\ \Psi_{n,m}(z)\ dz = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} u(z)\,\sqrt{2^{k-1}}\tilde{\xi}_m(2^{k-1}z - n + 1)\,dz.$$

Now, substitute $2^{k-1}z - n + 1 = \ell$, then,

$$\begin{aligned} \sigma_{n,m} &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} u\,(\frac{\ell - 1 + n}{2^{k-1}})\sqrt{2^{k-1}}\tilde{\xi}_m(\ell)2^{-k+1}d\ell \\ &= \sqrt{2^{-k+1}}\ \int_0^1 u\,(\frac{\ell - 1 + n}{2^{k-1}})\tilde{\xi}_m(\ell)d\ell. \end{aligned} \tag{3.1}$$

Next, by generalized mean value theorem for integrals,

$$= \sqrt{2^{-k+1}}\,u\,(\frac{w - 1 + n}{2^{k-1}})\ \int_0^1 \tilde{\xi}_m(\ell)\,d\ell,$$

for some $w \in (0,\, 1)$

$$= \sqrt{2^{-k+1}}\, u\,(\frac{w-1+n}{2^{k-1}})\Im, \tag{3.2}$$

where $\Im = \int_0^1 \tilde{\xi}_m(\ell)\, d\ell$ then,

$$|\sigma_{n,\,m}| = \left|\sqrt{2^{-k+1}}\right| \left|u\,(\frac{w-1+n}{2^{k-1}})\Im\right|. \tag{3.3}$$

Since $u\,(z)$ is bounded, therefore series $\sum_{n,m=0}^{\infty} \sigma_{n,\,m}$ is absolutely convergent. Hence the modified Taylor series of $u\,(z)$ converges uniformly. □

Theorem 3.2. *If a function $u(z)$ described on $[0,\,1)$ in such a way $|u''(z)| \leq \gamma$ (constant) then it is expressed as an infinite series of MTWs, and this series uniformly converges to $u(z)$, i.e.,*

$$u(z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \sigma_{n,m}\Psi_{n,m}(z), \tag{3.4}$$

where, $\sigma_{n,m} = \langle u(z)\ ,\Psi_{n,m}(z)\rangle$ and $\langle .\ ,\ .\rangle$ indicate inner product in $L^2\,[0,\,1)$.

Proof. We have

$$\begin{aligned} \sigma_{n,m} &= \langle u(z)\ ,\Psi_{n,m}(z)\rangle \\ &= \int_0^1 u(z)\, \Psi_{n,m}(z)\, dz. \end{aligned} \tag{3.5}$$

Using MTWs definition described in the eq. (2.1), we get

$$\sigma_{n,m} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} u(z)\, \sqrt{2^{k-1}}\, \tilde{\xi}_m(2^{k-1}z - n + 1)\, dz. \tag{3.6}$$

Now, using recursive relation described in the eq. (2.2), we obtain

$$\sigma_{n,m} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} u(z)\, \sqrt{2^{k-1}}\sqrt{2m+1}\, \left(\frac{(m!)^2}{(2m)!}\right) \Re_m(2^{k-1}z - n + 1)\, dz. \tag{3.7}$$

Here $\Re_m(z)$ are the famous classical Legendre functions given in [32].

Putting $2^{k-1}z - n + 1\ = \phi$ in the above equation (3.7), we have

$$\begin{aligned} \sigma_{n,m} &= \int_0^1 u(\frac{\phi+n-1}{2^{k-1}})\, \sqrt{2^{k-1}}\sqrt{2m+1}\, \left(\frac{(m!)^2}{(2m)!}\right) \frac{1}{2^{k-1}}\Re_m(\phi)d\phi \\ &= \sqrt{\frac{2m+1}{2^{k-1}}} \left(\frac{(m!)^2}{(2m)!}\right) \int_0^1 u(\frac{\phi+n-1}{2^{k-1}})\, \Re_m(\phi)d\phi. \end{aligned}$$

Now inserting the property of classical Legendre function which is demonstrated in equation (12) in ref. [32], we get,

$$\sigma_{n,m} = \sqrt{\frac{2m+1}{2^{k-1}}} \left(\frac{(m!)^2}{(2m)!}\right) \int_0^1 u(\frac{\phi+n-1}{2^{k-1}}) \frac{1}{(4m+2)} d\left[\Re_{m+1}(\phi) - \Re_{m-1}(\phi)\right] d\phi.$$

Using the formula integration by parts, we get,

$$\sigma_{n,m} = \sqrt{\frac{2m+1}{2^{k-1}}} \left(\frac{(m!)^2}{(2m)!}\right) \frac{(-1)}{(4m+2)\left(2^{k-1}\right)} \int_0^1 u^{'} \left(\frac{\phi+n-1}{2^{k-1}}\right) d\left[\Re_{m+1}(\phi) - \Re_{m-1}(\phi)\right] d\phi.$$

By using eq. (12) from ref. [32], we get

$$\sigma_{n,m} = \left(\frac{(m!)^2}{(2m)!}\right) \frac{(-1)}{\sqrt{2m+1}\sqrt{2^{3k-1}}} \int_0^1 u^{'} \left(\frac{\phi+n-1}{2^{k-1}}\right) \left\{ d\frac{\left[\Re_{m+2}(\phi) - \Re_{m}(\phi)\right]}{(4m+6)} - d\frac{\left[\Re_{m}(\phi) - \Re_{m-2}(\phi)\right]}{(4m-2)} \right\} d\phi.$$

Again apply integration by parts formula, we have

$$\sigma_{n,m} = \left(\frac{(m!)^2}{(2m)!}\right) \frac{(-1)^2}{\sqrt{2m+1}\sqrt{2^{5k-3}}} \int_0^1 u^{''} \left(\frac{\phi+n-1}{2^{k-1}}\right) \left\{ \frac{\left[\Re_{m+2}(\phi) - \Re_{m}(\phi)\right]}{(4m+6)} - \frac{\left[\Re_{m}(\phi) - \Re_{m-2}(\phi)\right]}{(4m-2)} \right\} d\phi.$$

Now,

$$\begin{aligned} |\sigma_{n,m}| &\leq \left(\frac{(m!)^2}{(2m)!}\right) \frac{1}{\sqrt{2m+1}\sqrt{2^{5k-3}}} \int_0^1 \left| u^{''} \left(\frac{\phi+n-1}{2^{k-1}}\right)\right| \left|\left\{ \frac{\left[\Re_{m+2}(\phi) - \Re_{m}(\phi)\right]}{(4m+6)} - \frac{\left[\Re_{m}(\phi) - \Re_{m-2}(\phi)\right]}{(4m-2)} \right\}\right| d\phi \\ &\leq \frac{\gamma}{\sqrt{2m+1}\sqrt{2^{5k-3}}} \left(\frac{(m!)^2}{(2m)!}\right) \int_0^1 \left|\left\{ \frac{\left[\Re_{m+2}(\phi) - \Re_{m}(\phi)\right]}{(4m+6)} - \frac{\left[\Re_{m}(\phi) - \Re_{m-2}(\phi)\right]}{(4m-2)} \right\}\right| d\phi, \end{aligned}$$

where,

$$\left| u^{''}(\frac{\phi+n-1}{2^{k-1}}) \right| \leq \gamma.$$

Using Lemma 1, given in reference [32], we get

$$\begin{aligned}|\sigma_{n,m}| &< \frac{\gamma}{\sqrt{2m+1}\sqrt{2^{5k-3}}}\left(\frac{(m!)^2}{(2m)!}\right)\frac{\sqrt{3}}{2}\frac{\sqrt{2m+1}}{(2m-3)^2}\\ &< \left(\frac{(m!)^2}{(2m)!}\right)\frac{\sqrt{6}}{16}\frac{\gamma}{(n)^{\frac{5}{2}}(m-\frac{3}{2})^2}.\end{aligned}$$

By the method of principal of mathematical induction, we get,

$$< \frac{1}{2^{m-1}}\frac{\sqrt{6}}{16}\frac{\gamma}{(n)^{\frac{5}{2}}(m-\frac{3}{2})^2}. \tag{3.8}$$

Hence the series $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sigma_{n,m}$ is absolutely convergent.

Also, the series $\sum_{n=1}^{\infty}\sigma_{n,0}\Psi_{n,0}(z)$ is convergent for $m=0$. Consequently, it follows that $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sigma_{n,m}\Psi_{n,m}(z)$ converges to $u(z)$ uniformly. □

Theorem 3.3. *Let the hypothesis of theorem* (3.2) *hold, and then we have the following accuracy estimation*

$$\|\varepsilon_n\|^2_{L^2(0,1)} < \frac{\gamma}{4\sqrt{2}}\left[-\kappa^{(4)}\left(2^{k-1}+1\right)\right]^{\frac{1}{2}}, \tag{3.9}$$

where,

$$\|\varepsilon_n\|_{L^2(0,1)} = \left(\int_0^1\left|u(z)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sigma_{n,m}\Psi_{n,m}(z)\right|^2 dz\right)^{\frac{1}{2}},$$

and $\kappa(z)$ is digamma function [33] and $\kappa^q(z)$ is the q^{th}derivative of digamma function which is known as polygamma function.

Proof. We have

$$\begin{aligned}\|\varepsilon_n\|^2_{L^2(0,1)} &= \int_0^1\left|u(z)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sigma_{n,m}\Psi_{n,m}(z)\right|^2 dz\\ &= \int_0^1\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sigma_{n,m}\Psi_{n,m}(z)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sigma_{n,m}\Psi_{n,m}(z)\right|^2 dz\\ &= \int_0^1\sum_{2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|\sigma_{n,m}\Psi_{n,m}(z)|^2 dz.\end{aligned}$$

By orthogonality condition of MTWs given in Eq. (2.3), we obtained

$$\|\varepsilon_n\|^2_{L^2(0,1)} = \left(\frac{(m!)^2}{(2m)!}\right)^2 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |\sigma_{n,m}|^2.$$

Using Eq. (3.8) of theorem (3.2), we get

$$\|\varepsilon_n\|^2_{L^2(0,1)} < \left(\frac{(m!)^2}{(2m)!}\right)^2 \frac{6\gamma^2}{256} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{(n)^5\left(m-\frac{3}{2}\right)^4}.\frac{1}{2^{2m-2}}.$$

By principal of mathematical induction

$$\begin{aligned}
\|\varepsilon_n\|^2_{L^2(0,1)} &< \frac{6\gamma^2}{256} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{(n)^5\left(m-\frac{3}{2}\right)^4}.\frac{1}{2^{2m-2}}.\frac{1}{2^{2m-2}} \\
&< \frac{6\gamma^2}{256}.16. \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{(n)^5\left(m-\frac{3}{2}\right)^4}.\frac{1}{2^{4m}} \\
&< \frac{3\gamma^2}{8}. \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{(n)^5\left(m-\frac{3}{2}\right)^4}.\frac{1}{2^{4m}} \\
&< \frac{3\gamma^2}{8} \sum_{n=2^{k-1}+1}^{\infty} \frac{1}{(n)^5} \sum_{m=M}^{\infty} \frac{1}{\left(m-\frac{3}{2}\right)^4}.\frac{1}{2^{4m}}.
\end{aligned}$$

Since $\sum_{m=M}^{\infty} \frac{1}{\left(m-\frac{3}{2}\right)^4}.\frac{1}{2^{4m}} < \sum_{m=M}^{\infty} \frac{1}{2^{4m}} \leq \sum_{m=0}^{\infty} \frac{1}{2^{4m}} \leq \sum_{m=0}^{\infty} \frac{1}{2^{m}}$, therefore, $\|\varepsilon_n\|^2_{L^2(0,1)} < \frac{3\gamma^2}{8} \sum_{n=2^{k-1}+1}^{\infty} \frac{1}{(n)^5}. \sum_{m=0}^{\infty} \frac{1}{2^m}$.

Since $\sum_{m=0}^{\infty} \frac{1}{2^m} = 2$, so

$$\begin{aligned}
\|\varepsilon_n\|^2_{L^2(0,1)} &< \frac{3\gamma^2}{4} \sum_{n=2^{k-1}+1}^{\infty} \frac{1}{(n)^5} < \frac{3\gamma^2}{4}\left[-\frac{1}{24}\kappa^{(4)}\left(2^{k-1}+1\right)\right] \\
&< \frac{3\gamma^2}{96}\left[-\kappa^{(4)}\left(2^{k-1}+1\right)\right].
\end{aligned}$$

Here, $\kappa^{(4)}\left(2^{k-1}+1\right) < 0$, for all k.

Taking square root on both sides of above inequality to obtain

$$\|\varepsilon_n\|^2_{L^2(0,1)} < \frac{\gamma}{4\sqrt{2}}\left[-\kappa^{(4)}\left(2^{k-1}+1\right)\right]^{\frac{1}{2}}.$$

□

4. PROCEDURE OF SOLUTION OF ABEL INTEFRAL EQUATION

Consider the Abel's integral equation, given bellow

$$\lambda\, u\;(z)\;=f(z)+\int_0^z \frac{u(t)}{\sqrt{(z-t)}}dt, \qquad 0\leq z\,t\leq 1, \tag{4.1}$$

where $\lambda\;=\;0\,or\,1$. Now we approximate $u\;(z)$ as truncated series defined in equation (2.5) by MTWs as

$$u\,(z)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sigma_{n,\,m}\,\Psi_{n\,,m}\,(z)=\sigma^T\Psi\,(z)\,, \tag{4.2}$$

where σ and $\Psi\,(z)$ are defined in equation (2.6) and (2.7). Then substituting equation (4.2) in equation (4.1), we get

$$\lambda\,\sigma^T\Psi\,(z)\;=f(z)+\int_0^z \frac{\sigma^T\Psi\,(t)}{\sqrt{(z-t)}}dt. \tag{4.3}$$

Now evaluating equation (4.3) at the collocation points $z_i=\frac{2i-1}{2^kM}$, $i=1,\,2,\,\ldots,$ $2^{k-1}M$. Thus we obtained

$$\lambda\,\sigma^T\Psi\,(z_i)\;=f(z_i)+\int_0^{z_i} \frac{\sigma^T\Psi\,(t)}{\sqrt{(z_i-t)}}dt. \tag{4.4}$$

Therefore after applying collocation points we get required system of algebric equation with $2^{k-1}M$ unknown. By solving this system we get MTW coefficients and then putting these coefficients in equation (4.2), we get approximated solution of equation (4.1).

Algorithm 1 Algorithm (MTWM): for solving AIEs.

Input: $k,\,n,\,m,\,M,\;z\geq 0,\,z_i,\;i=1,\,2,\,\ldots,\,2^{k-1}M.$

Step 1. Define the MTWs $\Psi(z)$ through eq. (2.7).

Step 2. Introduced unknown MTWs coefficient vector 'σ' by using eq. (2.6).

Step 3. In this step Approximate $u(z)$ in terms of MTWs series from eq. (4.2).

Step 4. Substituting the approximated form of function $u(z)$ in eq. (4.1) according to procedure of solution by using step 3.

Step 5. Introduce $z_i=\frac{2i-1}{2^kM}$, $i=1,\,2,\,\ldots,\,2^{k-1}M$

Step 6. Take set of $2^{k-1}M$ algebraic equations from eq. (4.4) using step 5.

Step 7. Solve this complete system which is obtained in step 6 and find σ.

Step 8. Increases: Mor k to get high accuracy in approximate solution $u(z)$.

Output: The approximated MTWs solution: $u(z)\cong\sigma^T\Psi(z)$.

Algorithm 2 Pseudo code for MTWM for solving AIEs.

Input: Taking independent arguments $k,\ M$, and dependent arguments: $\widehat{n}= n-1$, $n=1,2,\ldots,2^{k-1}, m=0,\ 1,\ 2,\ \ldots,\ M-1$. Also, consider MTPs $\tilde{\xi}_m(2^{k-1}z-n+1)$ in equation (2.2).

Step 1: (Construction of MTWs): by using basic inputs from the definition of wavelets discuss in section 2.

Print $R=\Psi_{n,m}(z)$

Step 2: Take R and approximates$u(z)$, in the linear combination of unknown MTWs coefficients 'σ' in equations (4.2).

Step 3: Print $S=\frac{2i-1}{2^k M}$, $i=1,\ 2,\ \ldots,\ 2^{k-1}M$.

Step 4: Takes S, and using step 2, to be transform of eq. (4.1) into the system of algebraic equations which is represented in terms of 'σ'.

Step 6: Print $T=\sigma$, for fixed k and M.

Output: $u(z)=\sigma^T\Psi(z)$.

5. Numerical problems

In this section, we apply proposed method for solving typical AIEs.

Example 1. *Consider the AIE of the type ([24, 34])*

$$\frac{2}{105}\sqrt{z}\left(105-56z^2+48z^3\right)=\int_0^z\frac{u\left(t\right)}{\sqrt{z-t}}dt,\quad 0\le z\le 1. \tag{5.1}$$

Exact solution of the given problem is $u\left(z\right)=z^3-z^2+1$.

On applying proposed method to solve above problem for $k=1\, and\, M=4$ then we get truncating approximate solution as,

$$u\left(z\right)\approx\sum_{m=0}^{3}\sigma_{1,\,m}\,\Psi_{1\,,m}\left(z\right)=\sigma^T\Psi\left(z\right). \tag{5.2}$$

Now applying procedure of solution discussed in above section (4) we get the MTWs coefficients as

$$\sigma_{1,\,0}=0.9166666666666666,\quad \sigma_{1,\,1}=-0.05773502691896264,$$
$$\sigma_{1,\,2}=0.2236067977499791,\quad \sigma_{1,\,3}=0.37796447300922276.$$

Substituting these MTWs coefficient in equation (5.2) we get approximate solution,

$$\begin{aligned}u(z)\ &\approx 0.9166666666666666\ \Psi_{1\,,0}\left(z\right)-0.05773502691896264\ \Psi_{1\,,1}\left(z\right)\\&+0.2236067977499791\ \Psi_{1\,,2}\left(z\right)+\ 0.37796447300922276\,\Psi_{1\,,3}\left(z\right).\end{aligned}$$

TABLE 1. Numerical results of Example 1.

z	Exact	Method [34] (k=1, M=8)	MTWM (k=1, M=4)	Abs. Error by Method [34]	Abs. Error by MTWM
0.1	0.9910	0.9910	0.9910	1.16E-12	1.11E-16
0.2	0.9680	0.9680	0.9680	4.71E-13	2.22E-16
0.3	0.9370	0.9370	0.9370	2.27E-12	3.33E-16
0.4	0.9040	0.9040	0.9040	3.04E-12	2.22E-16
0.5	0.8750	0.8750	0.8750	2.52E-12	0
0.6	0.8560	0.8560	0.8560	9.67E-13	0
0.7	0.8530	0.8530	0.8530	9.64E-13	2.22E-16
0.8	0.8720	0.8720	0.8720	2.36E-12	2.22E-16
0.9	0.9190	0.9190	0.9190	2.36E-12	1.11E-16

Comparison of proposed method approximated solution with exact solution and existing method with the absolute error $= |u_a(z) - u_e(z)|$, (where $u_a(z)$ and $u_e(z)$ are approximate numerical solution and exact solution respectively) analysis shown in table (1). Graph of comparison between exact and approximated solution drawn in figure (2).

Example 2. *Consider the Abel integral equation of the following type ([24, 35, 36]),*

$$u\left(z\right) = 2\sqrt{z} - \int_0^z \frac{u(t)}{\sqrt{z-t}}dt = z, \quad 0 \leq z < 1, \tag{5.3}$$

Exact solution of the given problem is $u\left(z\right) = 1 - \exp\left(\pi\, z\right) erfc\left(\sqrt{\pi\, z}\right)$. We solve this problem by the proposed MTWM and get approximate solution. Error analysis comparison of introduced method and other existing methods are shown in table (2). Graph of comparison between exact and approximated solution drawn in figure (3).

Example 3. *Consider the another Abel integral equation of the type ([24,35,36]),*

$$4u\left(z\right) = \frac{4}{\sqrt{z+1}} - \arcsin\left(\frac{1-z}{1+z}\right) + \frac{\pi}{2} - \int_0^z \frac{u(t)}{\sqrt{z-t}}\, dt, \qquad 0 \leq z < 1. \tag{5.4}$$

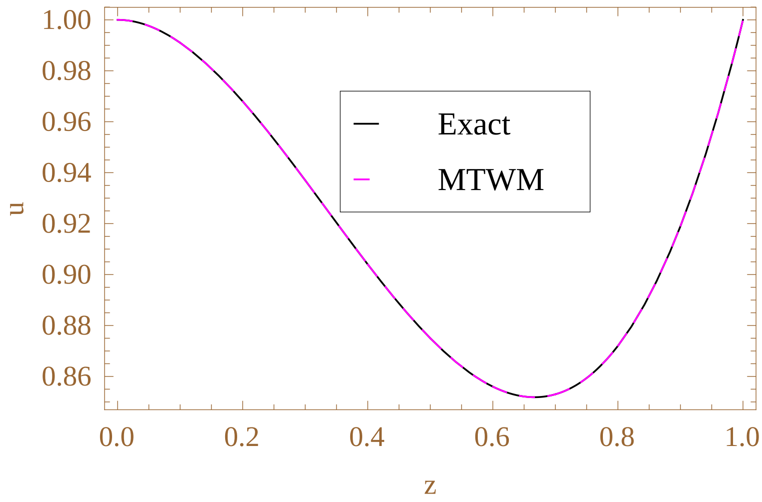

FIGURE 2. Graph of comparison of MTWM and Exact solution for Example (1)

TABLE 2. Error analysis of Example 2.

z	MTWM (k=1, M=10)	MTWM (k=1, M=12)	Method [24] (k=0, M=16)	Method [35] (m=16)	Method [36] (k=32)
0.1	2.82E-03	1.55E-03	1.62E-03	1.15E-02	1E-02
0.2	7.79E-04	6.48E-04	2.82E-03	1.13E-02	1E-03
0.3	6.23E-04	4.25E-04	1.89E-03	9.55E-03	1E-03
0.4	4.21E-04	3.15E-04	1.43E-03	1.68E-03	3E-03
0.5	3.46E-04	2.42E-04	1.32E-03	7.61E-03	3E-03
0.6	2.63E-04	1.98E-04	1.21E-03	1.53E03	1E-03
0.7	2.42E-04	1.63E-04	9.86E-04	3.09E-03	4E-04
0.8	1.50E-04	1.34E-04	2.45E-04	2.98E-03	5E-04
0.9	3.87E-04	1.91E-04	1.40E-03	7.08E-04	1E-03

Exact solution of the given problem is $u(z) = \frac{1}{\sqrt{z+1}}$. We solve this problem by the proposed MTWM and get approximate solution. Error analysis comparison of introduced method and other existing methods are shown in table (3). Graph of comparison between exact and approximated solution drawn in figure (4).

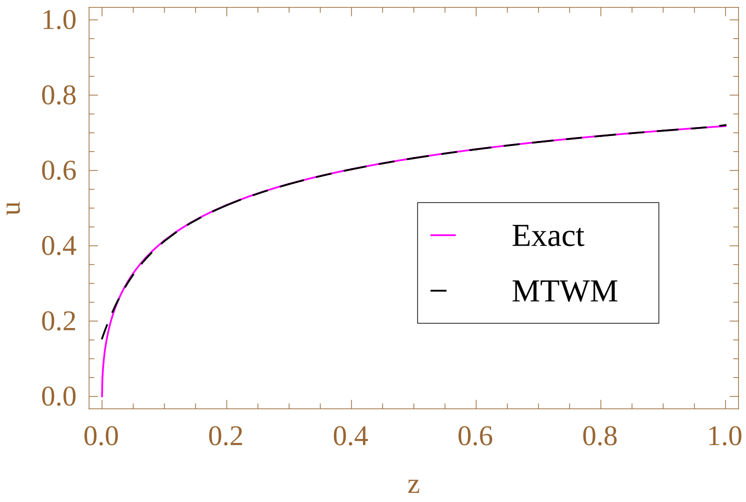

FIGURE 3. Graph of comparison of MTWM and Exact solution for Example (2)

TABLE 3. Error analysis of Example 3.

z	MTWM (k=1, M=10)	MTWM (k=1, M=12)	Method [35] (m=16)	Method [36] (k=32)
0.1	1.52E-08	4.12E-10	2.99E-03	3E-03
0.2	6.92E-10	5.92E-11	6.86E-03	1E-03
0.3	1.79E-09	5.01E-11	6.55E-03	1E-03
0.4	5.08E-10	4.86E-11	1.75E-03	2E-03
0.5	1.04E-09	3.39E-11	8.27E-03	4E-03
0.6	3.79E-10	3.65E-11	1.64E-03	2E-03
0.7	1.14E-09	2.68E-11	4.11E-03	6E-04
0.8	9.93E-10	1.52E-11	3.98E-03	6E-04
0.9	8.86E-09	2.30E-10	1.33E-03	1E-03

Example 4. *Consider the Abel integral equation of the type ([24, 35, 36]),*

(5.5)
$$z = \int_0^z \frac{u(t)}{\sqrt{z-t}} dt, \quad 0 \le z \le 1.$$

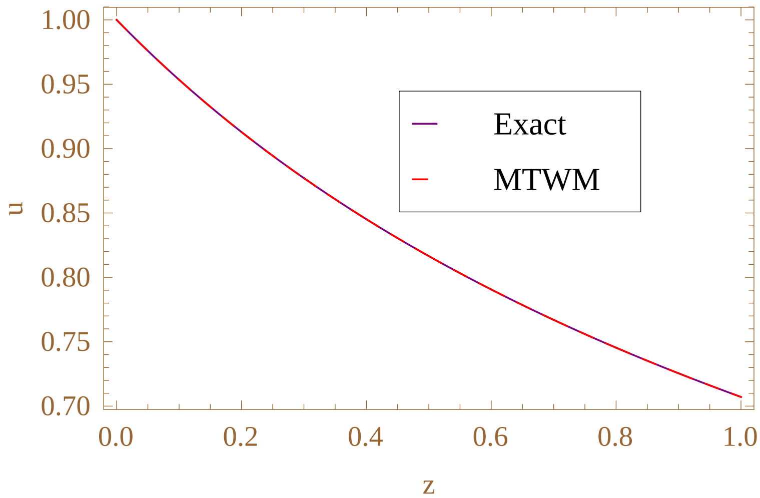

FIGURE 4. Graph of comparison of MTWM and Exact solution for Example (3)

Exact solution of the given problem is $u(z) = \frac{2}{\pi}\sqrt{z}$. we solve this problem by the proposed MTWM and get approximate solution. Error analysis comparison of introduced method and other existing methods are shown in table (4). Graph of comparison between exact and approximated solution drawn in figure (5).

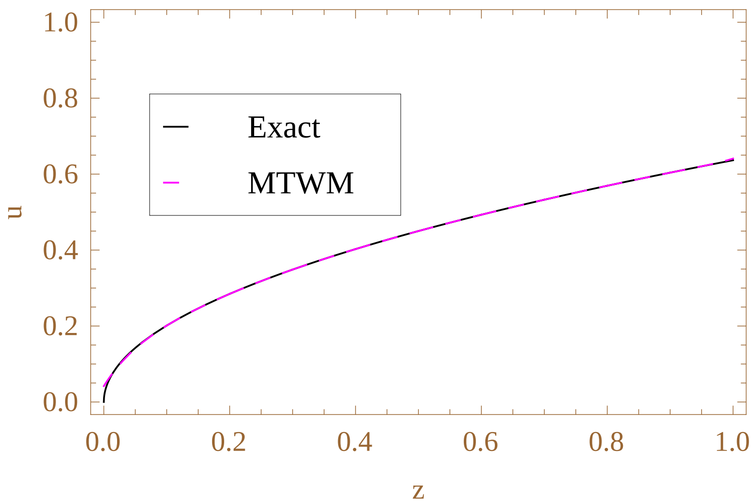

FIGURE 5. Graph of comparison of MTWM and Exact solution for Example (4)

TABLE 4. Error analysis of Example 4.

z	MTWM (k=1, M=8)	MTWM (k=1, M=12)	Method [24] (k=1, M=8)	Method [35] (m=16)
0.1	1.61E-03	1.00E-04	1.18E-03	8.57E-04
0.2	1.77E-04	7.38E-05	1.38E-03	1.32E-02
0.3	1.40E-04	2.67E-05	1.29E-03	1.11E-02
0.4	5.52E-05	2.32E-05	1.52E-03	3.13E-03
0.5	3.31E-06	1.38E-05	5.90E-03	1.38E-02
0.6	8.01E-05	1.11E-05	4.19E-03	2.57E-03
0.7	4.56E-05	1.04E-05	3.19E-03	7.08E-03
0.8	7.09E-05	3.70E-06	2.54E-03	6.71E-03
0.9	3.38E-04	1.11E-04	2.08E-03	2.09E-03

Example 5. *Consider the Abel integral equation of the type [36]:*
(5.6)

$$u(z) = -4\sqrt{z} + 4\sqrt{z}\log(2) - 2\sqrt{z}\log\left(\frac{1}{z}\right) + \log(z) - \int_0^z \frac{u(t)}{\sqrt{z-t}}dt, \qquad 0 \le z \le 1.$$

Exact solution of the given problem is $u(z) = \log(z)$. We solve this problem by the proposed MTWM and get approximate solution. Error analysis comparison of proposed method and other existing methods are shown in table (5). Graph of comparison between exact and approximated solution drawn in figure (6).

6. CONCLUSION

In this paper, MTWM presented for solving AIEs which is the advancement in the field of new research. The proposed MTWM is tested for the results of some numerical problems of the type AIEs and gives more accurate result. If we increase no. of wavelet bases then we get more accuracy in approximation solution. Proposed method is computationally efficient and gives improved performance which is approved by the approximate solution for the discussed

TABLE 5. Error analysis of Example 5.

z	Exact	MTWM (k=1, M=12)	Abs. Error by Method [36] (k=32)	Abs. Error by MTWM (k=1, M=12)
0.1	-2.30258509	-2.33166408	7E-02	2.90E-02
0.2	-1.60943791	-1.62233421	8E-03	1.29E-02
0.3	-1.20397280	-1.21245110	1E-02	8.47E-03
0.4	-0.91629073	-0.92254575	2E-02	6.25E-03
0.5	-0.69314718	-0.69796546	2E-02	4.81E-03
0.6	-0.51082563	-0.51475561	1E-02	3.93E-03
0.7	-0.35667494	-0.35992653	2E-03	3.25E-03
0.8	-0.22314355	-0.22585616	5E-03	2.71E-03
0.9	-0.10536051	-0.10864939	1E-02	3.28E-03

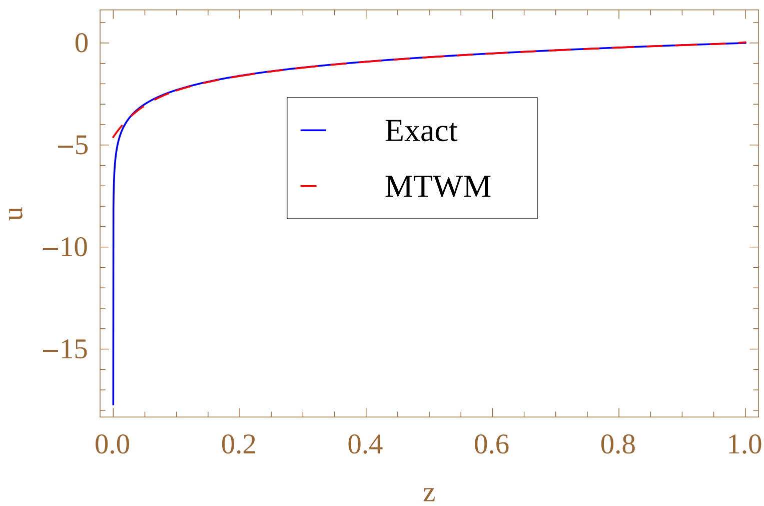

FIGURE 6. Graph of comparison of MTWM and Exact solution for Example (5)

problem as well as the proposed method algorithm easily implemented on computer. So, proposed method is more accurate and efficient for the approximate solution of Abel's type integral equations.

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