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AN ITERATIVE SCHEME FOR FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we introduce a new three steps iteration process, prove that our newly proposed iterative scheme can be used to approximate the fixed point of a contractive-like mapping and establish some convergence results for our newly proposed iterative scheme generated by a mapping satisfying condition (E) in the framework of uniformly convex Banach space. In addition, with the aid of numerical examples, we established that our newly proposed iterative scheme is faster than the iterative process introduced by Ullah et al., [26], Karakaya et al., [16], Abass et. al. [1] and some existing iterative scheme in literature. More so, the stability of our newly proposed iterative process is presented and we also gave some numerical examples to display the efficiency of our proposed algorithm

1. INTRODUCTION

In general, to solve fixed point problems analytically is almost impossible and thus the need to consider and approximate solution is pertinent. Over the years researchers have developed several iterative schemes for solving fixed point problems for different operators but the research is still on going in order to develop a faster and more efficient iterative algorithms.

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The Picard iterative process

$$x_{n+1} = Tx_n,\ \forall n \in \mathbb{N}, \tag{1.1}$$

is one of the earliest iterative process used to approximate a solution of a fixed point problem, where T is the class of contraction mappings. If T is nonexpansive, the Picard iterative process fails to approximate a solution of any fixed point problem even when the existence of the fixed point is guaranteed. In the light of this shortcoming, authors have develop different iterative processes to approximate the fixed points of nonexpansive mappings and other class of mappings of interest. The following iterative schemes are well-known in the literature; Mann iteration [18], Ishikawa iteration [13], Krasnosel'skii iteration [17], Noor iteration [19], and the iteration scheme of Abbas et al. [3].

In [14], Kadioglu et. al. introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than the Normal S-iteration process. The Picard Normal S-iteration process is defined as follows: Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n,\ n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. In addition, Thakur et al., [25] introduced the following iterative scheme in the framework of Banach space. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ x_{n+1} = Ty_n,\ n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They established that their iterative scheme is faster than Picard, Mann, Ishikawa, Agrawal [4], Noor and Abbas et al. [3] iteration process. They gave a numerical example to justify their claim. Hereafter, for brevity, we will call this the *Thakur Algorithm*.

In 2017, Karakaya et al. in [16] introduced a new iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $r_0 \in C$, the sequence $\{r_n\}$ in C is defined by

$$\begin{cases} p_n = Tr_n, \\ q_n = (1-\alpha_n)p_n + \alpha_n Tp_n \\ r_{n+1} = Tq_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al. process and some existing one in literature. Hereafter, for brevity, we will call this the *Karakaya Algorithm*.

In 2018, Ullah et al., in [26] introduce new iteration process called M iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $u_0 \in C$, the sequence $\{u_n\}$ in C is defined by

$$\begin{cases} w_n = (1-\alpha_n)u_n + \alpha_n Tu_n, \\ v_n = Tw_n \\ u_{n+1} = Tv_n, \ n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process, the above listed iterative process and some existing ones in literature.

Motivated by the iterative processes (1.5) and (1.4), Abass et. al. [1] introduced the following iterative process. Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $u_0 \in C$, the sequence $\{u_n\}$ in C is defined by

$$\begin{cases} w_n = Tu_n, \\ v_n = Tw_n \\ u_{n+1} = (1-\alpha_n)v_n + \alpha_n Tv_n, \ n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They established that the rate of convergence of iterative process (1.4), (1.5) and (1.6) are the same, which in turn is faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process, the above listed iterative process and some existing ones in literature.

Question: It is therefore natural question is if one can construct an iterative algorithm which has a better rate of convergence than iteration scheme (1.2), (1.3) (1.4), (1.5), (1.6) and a host of other iterative scheme in literature?

To answer this question, we introduce a new modified iteration process in the framework of Banach space, as follows: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T((1-\alpha_n)y_n + \alpha_n T y_n), \ n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

In [6] Berinde introduced a new class of mapping statisfying

$$\|Tx - Ty\| \leq \delta\|x-y\| + L\|x - Tx\|, \tag{1.8}$$

for all $x, y \in C, \delta \in (0,1)$ and $L \geq 0.$

He was able to establish that the class of mapping satisfying (1.8) is wider than the class of mapping introduced and studied by Zamfirescu in [28].

In [12], Imoru and Olantiwo gave the following contractive-like definition.

Definition 1.1. *Let T be a self-mapping on a Banach space X. The mapping T is called contractive-like mapping if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\xi : [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ such that for all $x, y \in X$,*

$$\|Tx - Ty\| \leq \delta\|x-y\| + \xi(\|x - Tx\|). \tag{1.9}$$

They established that a mapping satsisfying (1.9) is more general than those considered by Berinde [5], Osilike and Udomene [22] and some other contractive like mappings in literature.

Remark 1.1. *If $\xi(t) = Lt$, then (1.9) reduces to (1.8).*

Definition 1.2. *Let C be a nonempty subset of a real Banach space X and T a mapping from C to C. A mapping T is said to*

(1) *be nonexpansive if for each each $x, y \in C$*

$$\|Tx - Ty\| \leq \|x - y\|.$$

(2) *be Suzuki generalized nonexpansive [24] if, for all* $x, y \in C$

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

(3) *satisfy condition* (E_μ) *on* C, *if there exist* $\mu \geq 1$ *and for all* $x, y \in C$

$$\|x - Ty\| \leq \mu\|Tx - x\| + \|x - y\|.$$

We say T *satisfy condition* (E) *on* C *whenever* T *satisfies the condition* (E_μ) *for some* $\mu \geq 1$.

Remark 1.2. *It is established in [24], that every Suzuki generalized nonexpansive mapping satisfy the following inequality*

$$\|x - Ty\| \leq 3\|x - Tx\| + \|x - y\|.$$

From the above fact, it is clear that, the class of Suzuki generalized nonexpansive mapping is a special type of a mapping satisfying condition (E), *when* $\mu = 3$.

Follows introduced of the notion of (α, β)-nonexpansive type 1.

Definition 1.3. *Let* C *be a nonempty subset of a Banach space* X. *A mapping* $T : C \to C$ *will be called generalized* (α, β)*-nonexpansive type 1 mapping if there exist* $\alpha, \beta, \lambda \in [0, 1)$, *with* $\alpha \leq \beta$ *and* $\alpha + \beta < 1$ *such that for all* $x, y \in C$,

$$\begin{aligned} &\lambda\|Tx - x\| \leq \|x - y\| \\ \Rightarrow \quad &\|Tx - Ty\| \leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|. \end{aligned} \tag{1.10}$$

Remark 1.3. *It is easy to see that if*

(1) $\alpha = \beta = 0$ *and* $\lambda = \frac{1}{2}$, *we obtain mapping satisfying condition* (C);
(2) $\alpha = \beta = 0$ *and* $\lambda \in [0, 1)$, *we obtain mapping satisfying condition* (C_λ).

The purpose of this paper is to establish that the iteration process (1.7) converges faster than iteration processes (1.2), (1.3), (1.4), (1.5), (1.6) and some other existing iterative process in literature, for contractive-like mapping. We also present some convergence results for a mapping satisfying condition (E) using the iteration (1.7) and also present a stability result for our newly proposed iterative scheme.

2. Preliminaries

Let X be a Banach space and $S_X = \{x \in X : \|x\| \leq 1\}$ be a unit ball in X. For all $\alpha \in (0,1)$ and $x, y \in S_X$ such that $x \neq y$, if $\|(1-\alpha)x + \alpha y\| < 1$, then we say X is strictly convex. If X is a strictly convex Banach space and $\|x\| = \|y\| = \|(1-\lambda)y + \lambda x\| \, \forall x, y \in X$ and $\lambda \in (0,1)$, then $x = y$.

Definition 2.1. *A Banach space X is said to be smooth if*

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_X$.

In the above definition, the norm of X is called Gateaux differentiable. For all $y \in S_X$, if the limit (2.1) is attained uniformly for $x \in S_X$, then the norm is said to be uniformly Gateaux differentiable or Fréchet differentiable.

Definition 2.2. *A Banach space X satisfies Opial's condition [21], if for any sequence $\{x_n\} \in X, x_n \rightharpoonup x$ implies that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in X$ such that $x \neq y$.

Definition 2.3. *Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$ and $f(t) > 0 \quad \forall t \in (0,\infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T))$ denotes the distance from x to $F(T)$.*

Berinde [7] proposed a method to compare the fastness of two sequences.

Lemma 2.1. [7] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then a_n converges faster than $\{b_n\}$.*

Lemma 2.2. [7] *Suppose that for two fixed point iteration processes $\{u_n\}$ and $\{v_n\}$ both converging to the same fixed point x^*, the error estimates*

$$\|u_n - x^*\| \leq a_n \qquad n \geq 1,$$
$$\|v_n - x^*\| \leq b_n \qquad n \geq 1,$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to x^*.

Definition 2.4. [11] *Let $\{t_n\}$ be any arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be T-stable or stable with respect to T, if for $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$, $\forall n \in \mathbb{N}$, we have $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.*

Lemma 2.3. [27] *Let $\{\Psi_n\}$ and $\{\Phi_n\}$ be nonnegative real sequence satisfying the following inequality:*

$$\Psi_{n+1} \leq (1 - \phi)\Psi_n + \Phi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\lim_{n\to\infty} \frac{\Phi_n}{\phi_n} = 0$, then $\lim_{n\to\infty} \Psi_n = 0$.

Lemma 2.4. [23] *Let $\{\Psi_n\}$ and $\{\Phi_n\}$ be nonnegative real sequence satisfying the following inequality:*

$$\Psi_{n+1} \leq (1 - \phi)\Psi_n + \phi_n\Phi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\Phi_n \geq 0$ for all $n \in \mathbb{N}$ then

$$0 \leq \limsup_{n\to\infty} \Psi_n \leq \limsup_{n\to\infty} \Phi_n.$$

Lemma 2.5. *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1 \quad \forall n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = c$ hold for some $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.5. *Let C be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \geq 1$,*

$$\|x_{n+1} - x\| \leq \|x_n - x\|.$$

Proposition 2.1. *Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect to C, then we have the following:*

(i) *$\{x_n\}$ is bounded;*
(ii) *The sequence $\{\|x_n - x^*\|\}$ is decreasing and converges for all $x^* \in F(T)$.*

Lemma 2.6. [10] *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $F(T) \neq \emptyset$. Then T is quasi-nonexapansive.*

Theorem 2.1. [10] *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Theorem 2.2. [10] *Let C be a nonempty closed subset of a Banach space X with Opial property and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $\lambda = \frac{\gamma}{2}, \gamma \in [0, 1)$. If $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tx = x$. That is $I - T$ is demiclosed at zero, where I is the identity mapping on X.*

3. Rate of Convergence and Stability

In this section, we established that our newly proposed iterative process (1.7), converges faster than iterative process (1.2), (1.4), (1.5) and some existing iterative schemes in literature .The stability result is also presented.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a mapping satisfying (1.9). Let $\{x_n\}$ be the iterative sequence defined in (1.7) with sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. Using (1.7), and (1.9), we have

$$\begin{aligned} \|z_n - x^*\| &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| \\ &= (1 - \beta_n)\|x_n - x^*\| + \beta_n\|Tx^* - Tx_n\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\delta\|x^* - x_n\| + \xi(\|x^* - Tx^*\|) \\ &= (1 - (1 - \delta)\beta_n)\|x_n - x^*\|. \end{aligned} \tag{3.1}$$

Using (1.7), (3.1) and (1.9), we have

$$\begin{aligned} \|y_n - x^*\| &= \|Tz_n -^* \| \\ &\leq \delta\|z_n - x^*\| \\ &\leq \delta(1 - (1 - \delta)\beta_n)\|x_n - x^*\|. \end{aligned} \tag{3.2}$$

Using (1.7), (3.2) and (1.9), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| &= \|Tx^* - T((1-\alpha_n)y_n + \alpha_n Ty_n)\| \\
&\le \delta\|x^* - ((1-\alpha_n)y_n + \alpha_n Ty_n)\| \\
&\le \delta[(1-\alpha_n)\|y_n - x^*\| + \alpha_n\|Ty_n - x^*\|] \\
&\le \delta[(1-\alpha_n)\|y_n - x^*\| + \alpha_n\delta\|y_n - x^*\|] \qquad (3.3)\\
&= \delta(1-(1-\delta)\alpha_n)\|y_n - x^*\| \\
&\le \delta^2(1-(1-\delta)\alpha_n)(1-(1-\delta)\beta_n)\|x_n - x^*\| \\
&= \delta^2[1-(1-\delta)\beta_n - (1-\delta)\alpha_n + (1-\delta)^2\alpha_n\beta_n]\|x_n - x^*\| \\
&\le \delta^2[1-(1-\delta)\alpha_n - (1-\delta)\alpha_n\beta_n + (1-\delta)\alpha_n\beta_n]\|x_n - x^*\| \\
&= \delta^2(1-(1-\delta)\alpha_n)\|x_n - x^*\|.
\end{aligned}
$$

From (3.3), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\le \delta^2(1-(1-\delta)\alpha_n)\|x_n - x^*\| \\
\|x_n - x^*\| &\le \delta^2(1-(1-\delta)\alpha_n)\|x_{n-1} - x^*\| \\
&\vdots \\
\|x_1 - x^*\| &\le \delta^2(1-(1-\delta)\alpha_0)\|x_n - x^*\|. \qquad (3.4)
\end{aligned}
$$

From (3.4), we have that

$$
\|x_{n+1}-x^*\| \le \|x_0 - x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m). \qquad (3.5)
$$

Since $\{\alpha_n\}, \{\beta_n\}$ and δ are in $(0,1)$, we have $1-(1-\delta)\alpha_n \in (0,1)$. We recall the inequality $1-x \le e^{-x}$ for all $x \in [0,1]$, thus from (3.5), we have

$$
\|x_{n+1}-x^*\| \le \frac{\delta^{n+1}\|x_0 - x^*\|}{e^{(1-\delta)\sum_{m=0}^{n}\alpha_m}}.
$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

We now establish that x^* is unique. Let $x^*, x_2^* \in F(T)$, such that $x^* \ne x_2^*$, using the definition of T, we have

$$
\begin{aligned}
\|x^* - x_2^*\| = \|Tx^* - Tx_2^*\| \le \delta\|x^* - x_2^*\| \le \|x^* - x_2^*\| \\
\Rightarrow \|x^* - x_2^*\| \le \|x^* - x_2^*\|.
\end{aligned}
$$

Clearly, we have that $\|x^* - x_2^*\| = \|x^* - x_2^*\|$, if not we get a contradiction $\|x^* - x_2^*\| < \|x^* - x_2^*\|$. Hence, we have that $x^* = x_2^*$. Thus the proof is complete. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a mapping satisfying (1.9). The iterative processes (1.2), (1.3), (1.4), (1.5) and* (1.6) *with sequences $\{\alpha_n\}, \{\beta_n\}$ in $(0,1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, converges strongly to a unique fixed point of T.*

Proof. The proof is the same as Theorem 3.1 and thus the proof is omitted. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T be a mapping satisfying (1.9). Let $\{x_n\}$ be defined by (1.7) with sequences $\{\alpha_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the iteration (1.7) is T-stable.*

Proof. Let $\{t_n\} \subset X$ be any arbitrary sequence in C and suppose that the sequence generated by (1.7) is $x_{n+1} = f(T, x_n)$ converging to a unique fixed point x^* and that $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. To establish that T is stable, we need to prove that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = x^*$.

Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Using the triangular inequality and (3.3), we have that

$$\begin{aligned}
\|t_{n+1} - x^*\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - x^*\| \\
&= \epsilon_n + \|T((1-\alpha_n)T((1-\beta_n)t_n + \beta_n T t_n) \\
&\quad + \alpha_n T(T((1-\beta_n)t_n + \beta_n T t_n))) - x^*\| \\
&\leq \epsilon_n + \delta^2(1 - (1-\delta)\alpha_n)\|t_n - x^*\| \\
&\leq \epsilon_n + (1 - (1-\delta)\alpha_n)\|t_n - x^*\|.
\end{aligned}$$

Let $\Psi_n = \|t_n - x^*\|, \phi_n = (1-\delta)\beta_n\alpha_n \in (0,1)$ and $\Phi_n = \epsilon_n$. By our hypothesis that, $\lim_{n\to\infty} \epsilon_n = 0$, it follows that $\lim_{n\to\infty} \frac{\epsilon_n}{(1-\delta)\alpha_n} = \lim_{n\to\infty} \frac{\Phi_n}{\phi_n} = 0$. Using Lemma (2.3), we have that $\lim_{n\to\infty} t_n = x^*$.

Conversely, suppose that $\lim_{n\to\infty} t_n = x^*$. We have that

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\
&\leq \|t_{n+1} - x^*\| + \|x^* - f(T, t_n)\| \\
&\leq \|t_{n+1} - x^*\| + (1 - (1-\delta)\alpha_n)\|t_n - x^*\|.
\end{aligned}$$

Using our hypothesis that $\lim_{n\to\infty} t_n = x^*$, we then have that $\lim_{n\to\infty} \epsilon_n = 0$. Hence, iteration (1.7) is stable with respect to T. □

In what follows, we give some numerical examples to establish that our newly proposed iterative scheme converges faster than the iterative processes (1.4) and (1.5).

Example 1. *Define a mapping $T : [0,1] \to [0,1]$ as $Tx = \frac{x}{4}$. Clearly, the unique fixed point of T is zero. We need to show that T satisfy (1.9). Now for $\delta = \frac{1}{4}$ and for any increasing function ξ with $\xi(0) = 0$, for all $x, y \in [0,1]$, we have*

$$\begin{aligned}\|Tx - Ty\| - \delta\|x - y\| - \xi(\|x - Tx\|) &= \frac{1}{4}|x - y| - \frac{1}{4}|x - y| - \xi(|x - \frac{x}{4}|)\\ &= -\xi(\frac{3x}{2}) \le 0.\end{aligned}$$

We take $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}}$ and $x_0 = 0.4$. The comparison of the iterative schemes are shown below.

TABLE 1. Comparison of iteration processes for Example 1

Step	*New Algorithm*	*M Iteration*	*Karakaya Algorithm*	*Thakur Algorithm*
0	*0.4*	*0.4*	*0.4*	*0.4*
1	*5.514746e-03*	*1.174175e-02*	*1.174175e-02*	*1.562500e-02*
2	*1.108032e-04*	*4.160889e-04*	*4.160889e-04*	*7.324219e-04*
3	*2.705155e-06*	*1.625347e-05*	*1.625347e-05*	*3.719330e-05*
4	*7.467575e-08*	*6.751182e-07*	*6.751182e-07*	*1.975894e-06*
5	*2.246702e-09*	*2.927539e-08*	*2.927539e-08*	*1.080567e-07*
6	*7.209251e-11*	*1.311038e-09*	*1.311038e-09*	*6.029950e-09*
7	*2.433043e-12*	*6.021226e-11*	*6.021226e-11*	*3.415401e-10*
8	*8.553667e-14*	*2.822450e-12*	*2.822450e-12*	*1.956740e-11*
9	*3.110906e-15*	*1.345654e-13*	*1.345654e-13*	*1.131241e-12*
10	*1.164392e-16*	*6.508480e-15*	*6.508480e-15*	*6.588190e-14*

Comparison shows that our iteration process (1.7) converges faster than the M iteration (1.5), iteration (1.4) and iteration (1.3), while iteration (1.5)and iteration (1.4) converges at the same rate.

Example 2. *Let $X = \mathbb{R}$ and $C = [0,50]$. Let $T : C \to C$ be a mapping defined by $T(x) = \frac{2x}{3}$. Using similar argument as above with $\delta = \frac{2}{3}$, it is easy to see that for all $x, y \in C$, T satisfy (1.9) with a unique fixed point zero. Choose $\alpha_n = \beta_n = \frac{3}{4}$, with an initial value of $x_0 = 30$.*

TABLE 2. Comparison of iteration processes for Example 2

Step	*New Algorithm*	*M Iteration*	*Karakaya Algorithm*	*Thakur Algorithm*
0	30	30	30	30
1	7.500000e+00	1.000000e+01	1.000000e+01	1.083333e+01
2	1.875000e+00	3.333333e+00	3.333333e+00	3.912037e+00
3	4.687500e-01	1.111111e+00	1.111111e+00	1.412680e+00
4	1.171875e-01	3.703704e-01	3.703704e-01	5.101345e-01
5	2.929688e-02	1.234568e-01	1.234568e-01	1.842152e-01
6	7.324219e-03	4.115226e-02	4.115226e-02	6.652216e-02
7	1.831055e-03	1.371742e-02	1.371742e-02	2.402189e-02
8	4.577637e-04	4.572474e-03	4.572474e-03	8.674572e-03
9	1.144409e-04	1.524158e-03	1.524158e-03	3.132484e-03
10	2.861023e-05	5.080526e-04	5.080526e-04	1.131175e-03

Comparison shows that our iteration process (1.7) converges faster than the M iteration (1.5), iteration (1.4) and iteration (1.3), while iteration (1.5) and iteration (1.4) converges at the same rate.

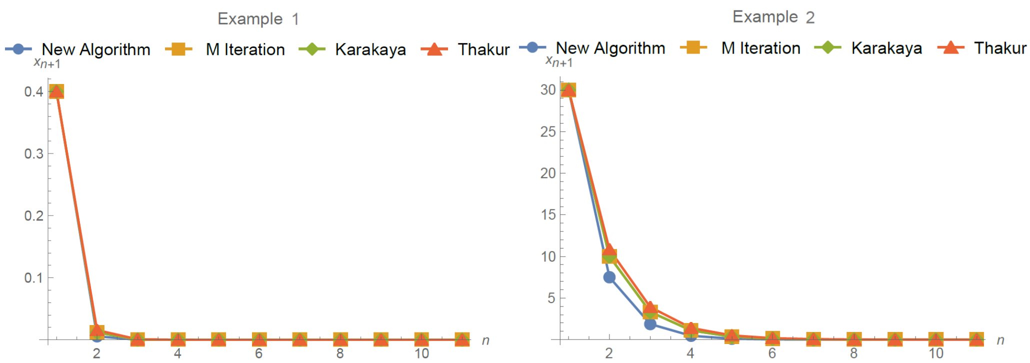

FIGURE 1. Graphical representation of the iteration processes in Examples 1 & 2

4. CONVERGENCE THEOREMS

In this section, we establish some convergence theorems using our newly proposed iterative scheme (1.7).

Lemma 4.1. *Let C be a nonempty closed and convex subset of a Banach space X. Let $T : C \to C$ be a (α, β)-nonexpansive type 1 mapping and $F(T) \neq \emptyset$. Suppose* *that $\{x_n\}$ is defined by (1.7), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$, then the following hold:*

- (i) *$\{x_n\}$ is bounded.*
- (ii) *$\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.*
- (iii) *$\lim_{n\to\infty} d(x_n, F(T))$ exists, where $d(x_n, F(T))$ denotes distance from x_n to $F(T)$.*

Proof. Let $x^* \in F(T)$, then from (1.7) and Lemma 2.6, we have

$$\begin{aligned}\|z_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n T x_n - x^*\| \\ &\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| \\ &\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{4.1}$$

From (1.7) and (4.1), we obtain

$$\begin{aligned}\|y_n - x^*\| &= \|Tz_n - x^*\| \\ &\le \|z_n - x^*\| \\ &\le \|x_n - x^*\|. \end{aligned} \tag{4.2}$$

From (1.7) and (4.2), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|T((1-\alpha_n)y_n + \alpha_n T y_n) - x^*\| \\ &\le \|(1-\alpha_n)y_n + \alpha_n T y_n - x^*\| \\ &\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n\|Ty_n - x^*\| \\ &\le (1-\alpha_n)\|y_n - x^*\| + \alpha_n\|y_n - x^*\| \\ &= \|y_n - x^*\| \\ &\le \|x_n - x^*\|, \end{aligned} \tag{4.3}$$

which implies that $\|x_{n+1} - x^*\| \le \|x_n - x^*\|$ for all $x^* \in F(T)$. Hence, $\{x_n\}$ is Fejer monotone with respect to $F(T)$ and by Proposition 2.1, $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F(T)$ and consequently, $\lim_{n\to\infty} d(x_n, F(T))$ exists. □

Theorem 4.1. *Let X be a uniformly convex Banach space and C be a nonempty closed and convex subset of X. Let $T : C \to C$ be an (α, β)-nonexpansive type 1* *mapping and* $F(T) \neq \emptyset$*. Suppose that* $\{x_n\}$ *is defined by (1.7), where* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$*, then* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. From Lemma 4.1, we have that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$. Suppose that $\lim_{n\to\infty} \|x_n - x^*\| = c$. If we take c = 0, then we are done. Thus, we consider the case where $c > 0$.

By definition of T, we have

$$\|Tx_n - x^*\| \leq \|x_n - x^*\|.$$

Thus,

$$\limsup_{n\to\infty} \|Tx_n - x^*\| \leq c.$$

From (4.1), we have

$$\|z_n - x^*\| \leq \|x_n - x^*\|,$$

which implies that

$$\limsup_{n\to\infty} \|z_n - x^*\| \leq c. \tag{4.4}$$

From (4.3) and (4.2), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|y_n - x^*\| \\ &\leq \|z_n - x^*\|. \end{aligned}$$

Thus, taking $\liminf_{n\to\infty}$, we have that

$$c \leq \liminf_{n\to\infty} \|z_n - x^*\|. \tag{4.5}$$

From (4.4) and (4.5), we obtain that $\lim_{n\to\infty} \|z_n - x^*\| = c$. That is,

$$\lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\| = c.$$

Thus, by Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

□

Theorem 4.2. *Let* X *be a uniformly convex Banach space which satisfies the Opial's condition and* C *a nonempty closed convex subset of* X*. Let* $T : C \to C$ *be an* (α, β)*-nonexpansive type 1 mapping with* $F(T) \neq \emptyset$ *and* $\{x_n\}$ *be a sequence defined by iteration (1.7). Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof. In Lemma 4.1, we established that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and that $\{x_n\}$ is bounded. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let u and v be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Theorem 4.1, we have that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Theorem 2.2, we therefore have that $Tu = u$. Using similar approach, we can show that $v = Tv$. In what follows, we establish uniqueness. From Lemma 4.1, we have that $\lim_{n\to\infty}\|x_n - v\|$ exists. Now, suppose that $u \neq v$, then by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{k\to\infty}\|x_{n_k} - u\| \\
&< \lim_{k\to\infty}\|x_{n_k} - v\| \\
&= \lim_{n\to\infty}\|x_n - v\| \\
&= \lim_{j\to\infty}\|x_{n_j} - v\| \\
&< \lim_{j\to\infty}\|x_{n_j} - u\| \\
&= \lim_{n\to\infty}\|x_n - u\|.
\end{aligned}$$

This is a contradiction, so $u = v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a generalized (α,β)-nonexpansive mapping type 1 mapping on C, $\{x_n\}$ defined by (1.7) and $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.*

Proof. Let $\{x_n\}$ converges to x^* a fixed point of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Lemma 4.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $x^* \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{u_k\}$ is a sequence in $F(T)$ such that for all $n \in \mathbb{N}$,

$$\|x_{n_k} - u_k\| < \frac{1}{2^k}$$

it follows from (4.3) that $\|x_{n+1} - u_k\| \leq \|x_n - u_k\| < \frac{1}{2^k}$, hence

$$\begin{aligned}\|u_{k+1} - u_k\| &\leq \|u_{k+1} - x_{n+1}\| + \|x_{n+1} - u_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Thus, we have that $\{u_k\}$ is a Cauchy sequence in $F(T)$. Also, by Theorem 2.1, we have that $F(T)$ is closed. Thus $\{u_k\}$ is a convergent sequence in $F(T)$. Now, suppose that $\{u_k\}$ converges to $p \in F(T)$. Therefore, since

$$\|x_{n_k} - p\| \leq \|x_{n_k} - u_k\| + \|u_k - p\| \to 0 \text{ as } k \to \infty,$$

we obtain that $\lim\limits_{k\to\infty} \|x_{n_k} - p\| = 0$ and so $\{x_{n_k}\}$ converges strongly to $p \in F(T)$. Since $\lim\limits_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be an (α, β)-nonexpansive type 1 mapping and $\{x_n\}$ defined by (1.7) and $F(T) \neq \emptyset$. Let T satisfy condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 4.1, we have $\lim_{n\to\infty} \|x_n - F(T)\|$ exist and by Theorem 4.1, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Using the fact that

$$0 \leq \lim_{n\to\infty} f(d(x, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \; \forall x \in C$$

we have that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, by Theorem 4.3 $\{x_n\}$ converges strongly to $x^* \in F(T)$. □

In view or Remark 1.2, we have the following corollaries.

Corollary 4.1. *Let X be a uniformly convex Banach space which satisfies the Opial's condition and C a nonempty closed convex subset of X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$ and $\{x_n\}$ be a sequence defined by iteration (1.7). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Corollary 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a Suzuki generalized nonexpansive mapping on C, $\{x_n\}$ defined by (1.7) and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf \|x - x^*\| : x^* \in F(T)$.*

Corollary 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a Suzuki generalized nonexpansive mapping, $\{x_n\}$ defined by (1.7) and $F(T) \neq \emptyset$. Let T satisfy condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

5. Numerical Examples

Example 3 (Cubic Equation)**.** *It is well known that finding the roots of a cubic equation $x^3 + x^2 - 1 = 0$, means finding the fixed point of the function $Tx = (1 - x^3)^{1/2}$ as $x^3 + x^2 - 1 = 0$ can be written as $(1 - x^3)^{1/2} = x$, with a fixed point $x^* = 0.7548777$. We take $\alpha_n = \beta_n = \frac{1}{\sqrt[4]{n+1}}$ and $x_0 = 0.8$. The comparison of the iterative scheme is shown below.*

TABLE 3. Comparison of iteration processes for Example 3

Step	*New Algorithm*	*M Iteration*	*Karakaya Algorithm*	*Thakur Algorithm*
0	0.8	0.8	0.8	0.8
1	0.7937786	0.7029093	0.7069398	0.7205547
2	0.7761471	0.7902225	0.7921357	0.7630997
3	0.7624492	0.7288931	0.7312731	0.7540768
4	0.7566954	0.7677311	0.7680462	0.7548292
5	0.7551865	0.7485711	0.7489116	0.7548696
6	0.7549160	0.7573176	0.7572899	0.7548757
7	0.7548812	0.7540288	0.7540556	0.7548770
8	0.7548779	0.7551279	0.7551220	0.7548774
9	0.7548777	0.7548137	0.7548154	0.7548776
10	0.7548777	0.7548917	0.7548913	0.7548776
11	0.7548777	0.7548750	0.7548751	0.7548776
12	0.7548777	0.7548781	0.7548781	0.7548777
13	0.7548777	0.7548776	0.7548776	0.7548777
14	0.7548777	0.7548777	0.7548777	0.7548777
15	0.7548777	0.7548777	0.7548777	0.7548777

Comparison shows that our iteration process (1.7) converges to the fixed point in seven iterations, while the iteration M iteration (1.5) and iteration (1.4) converges in fourteen iterations to the fixed point and (1.3) converges in twelve iterations to the fixed point.

Example 4 (Increasing Function). *Let* $T : [0, 8] \to [0, 8]$ *be defined as* $Tx = \frac{x}{2} + 3$. *Clearly,* T *is an increasing function with a fixed point* 6. *We take* $\alpha_n = \beta_n = \frac{1}{\sqrt[4]{5n+1}}$ *and* $x_0 = 7$. *The comparison of the iterative schemes is shown below.*

TABLE 4. Comparison of iteration processes for Example 4

Step	*New Algorithm*	*M Iteration*	*Karakaya Algorithm*	*Thakur Algorithm*
0	7	7	7	7
1	6.115780	6.170132	6.170132	6.198969
2	6.015233	6.030856	6.030856	6.042243
3	6.002142	6.005785	6.005785	6.009241
4	6.000315	6.001109	6.001109	6.002058
5	6.000048	6.000216	6.000216	6.000464
6	6.000007	6.000043	6.000043	6.000106
7	6.000001	6.000008	6.000008	6.000024
8	6.000000	6.000002	6.000002	6.000006
9	6.000000	6.000000	6.000000	6.000001
10	6.000000	6.000000	6.00000	6.00000
11	6.000000	6.000000	6.000000	6.00000

Comparison shows that our iteration process (1.7) converges to the fixed point in seven iterations while the iteration M iteration (1.5), iteration (1.4) converges in nine iterations to the fixed point and iteration (1.3) converges in ten iterations to the fixed point.

Example 5 (Decreasing Function). *Let* $T : [0, 8] \to [0, 8]$ *be defined as* $Tx = (1 - x)^2$. *Clearly,* T *is an increasing function with a fixed point* 0.3819660 *We take* $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}}$ *and* $x_0 = 0.8$. *The comparison of the iterative schemes is shown below.*

Comparison shows that our iteration process (1.7) converges to the fixed point in four iterations while the iteration M iteration (1.5) and iteration (1.4) converges to the fixed point in nine iterations, while the iteration (1.3) never converge to the fixed point.

TABLE 5. Comparison of iteration processes for Example 5

Step	New Algorithm	M Iteration	Karakaya Algorithm	Thakur Algorithm
0	0.8	0.8	0.8	0.8
1	0.4810099	0.2081545	0.1133094	0.4403650
2	0.3919118	0.4864851	0.4112892	0.4064884
3	0.3821678	0.3714770	0.3760525	0.3987498
4	0.3819660	0.3820412	0.3819362	0.3962508
5	0.3819660	0.3819760	0.3819620	0.3957312
6	0.3819660	0.3819684	0.3819651	0.3963442
7	0.3819660	0.3819668	0.3819657	0.3978619
8	0.3819660	0.3819663	0.3819659	0.4002992
9	0.3819660	0.3819660	0.3819660	0.4038169
10	0.3819660	0.3819660	0.3819660	0.4087068
11	0.3819660	0.3819660	0.3819660	0.4154137

Example 6 (Oscillatory Function). *Let* $T : (0, 8] \to (0, 8]$ *be defined as* $Tx = \frac{1}{x}$. *The fixed point of* T *is one. We take* $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}}$ *and* $x_0 = 0.8$. *The comparison of the iterative schemes is shown below.*

TABLE 6. Comparison of iteration processes for Example 6

Step	New Algorithm	M Iteration	Karakaya Algorithm	Thakur Algorithm
0	0.8	0.8	0.8	0.8
1	0.9500117	1.1181981	1.0731894	1.025000
2	0.9985350	0.9889281	0.9867460	1.008537
3	1.0000000	1.0000620	0.9999110	1.004286
4	1.0000000	1.0000065	0.9999906	1.002575
5	1.0000000	1.0000012	0.9999983	1.001718
6	1.0000000	1.0000003	0.9999996	1.001228
7	1.0000000	1.0000001	0.9999999	1.000921
8	1.0000000	1.0000000	1.0000000	1.000716

Comparison shows that our iteration process (1.7) converges to the fixed point in three iterations while the iteration M iteration (1.5) and iteration (1.4) converges to the fixed point in eight iterations, while the iteration (1.3) is yet to converge after eight iterations.

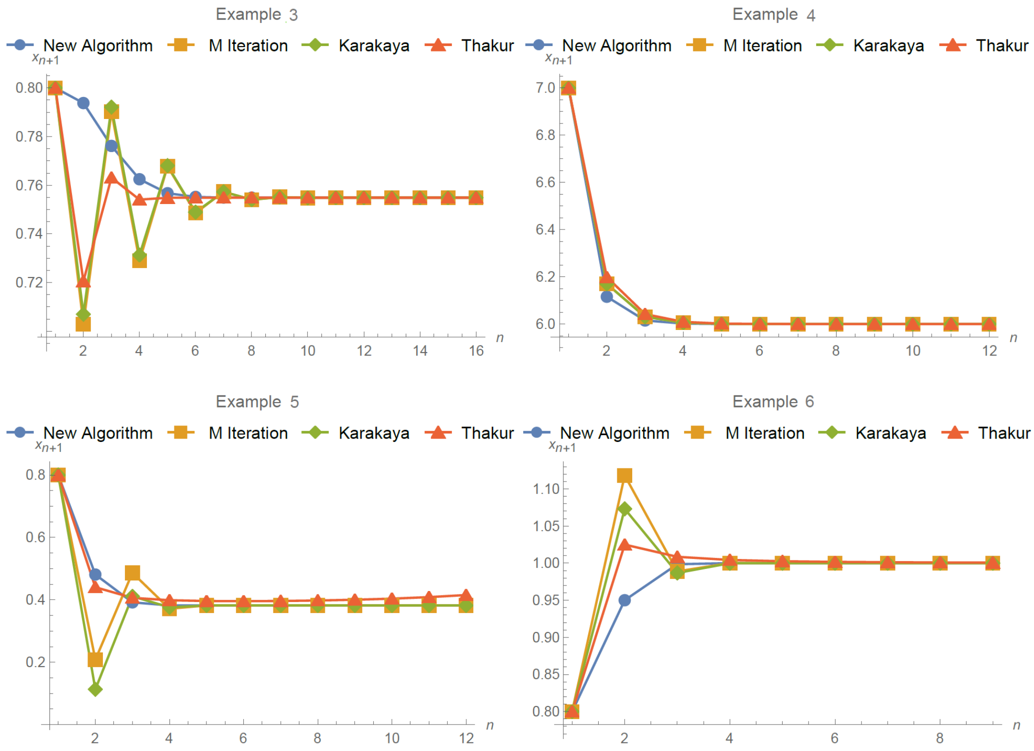

FIGURE 2. Graphical representation of the iteration processes in Examples 3 - 6

6. CONCLUSION

We have established that our newly proposed iterative scheme is more efficient than recently introduced iterative algorithms in literature, from Section 5, it is clear that our newly proposed iterative algorithm have good potentials for further applications.

Data dependency is an interesting area of research in fixed point theory which has been studied by the likes of Espinola et al. [8], Soltuz et al. [23] and the references therein. An interesting open problem : Is it possible to compute the data dependency of iteration (1.7) using a contractive-like mapping?

REFERENCES

[1] H. A. ABASS, A. A. MEBAWONDU, O. T. MEWOMO: *Some results for a new three iteration scheme in Banach spaces,* Bulletin University, Transilvania Brasov, Series III: Mathematics, Information, Physics, **11** (2018), 1-18.

[2] M. ABBAS, Z. KADELBURG, D. R. SAHU: *Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces,* Math. Comp. Modeling, **55** (2012), 1418-1427.

[3] M. ABBAS, T. NAZIR: *A new faster iteration process applied to constrained minimization and feasibility problems,* Matematiqki Vesnik, **66** (2) (2014), 223-234.

[4] R. P. AGARWAL, D. O'REGAN, D. R. SAHU: *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings,* J. Convex Anal., **8**(1) (2007), 61-79.

[5] V. BERINDE: *On the stability of some fixed point procedures,* Bul. Stiint.Univ. BaiaMare, Ser. B, Matematica Informatica, XVIII, **1** (2002), 7-14.

[6] V. BERINDE: *On the approximation of fixed points of weak contractive mappings,* Carpath. J. Math., **19** (2003), 7-22.

[7] V. BERINDE: *Picard iteration converges faster than Mann iteration for a class of quasicontractive operators,* Fixed Point Theory A., **2** (2004), 97-105.

[8] R. ESPÍNOLA, A. PETRUSEL: *Existence and data dependence of fixed points for multivalued operators on gauge spaces,* J. Math. Anal. Appl., **309** (2005), 420-432.

[9] J. G. FALSET, E. L. FUSTER AND T. SUZUKI: *Fixed point theory for a class of generalized nonexpansive mapping,* J. Math. Anal. Appl., **309** (2011), 185-195.

[10] A. FRANCIS, O. K. NARAIN: *On generalized (α, β)-nonexpansive Mappings in Banach Spaces with Application,* Nonlinear Funct. Anal. Appl. (Accepted).

[11] A. M. HARDER: *Fixed point theory and stability results for fixed point iteration procedures. PhD thesis,* University of Missouri-Rolla, Missouri.

[12] C. O. IMORU, M.O. OLANTIWO: *On the stability of Picard and Mann iteration processes,* Carpath. J. Math., **19**(2) (2003), 155-160.

[13] S. ISHIKAWA: *Fixed points by new iteration method,* Proc. Am. Math. Soc., **149** (1974), 147-150.

[14] N. KADIOGLU, I. YILDIRIM: *Approximating fixed points of non- expansive mappings by faster iteration process,* arXiv:1402. 6530v1 [math.FA] (2014).

[15] V. KARAKAYA, K. DOGAN, F. GURSOY, M. ERTURK: *Fixed point of a new three step iteration algorithm under contractive like operators over normed space,* Abstract and Applied Analysis Volume 2013, Article ID 560258.

[16] V. KARAKAYA, Y. ATALAN, K. DOGA: *On fixed point result for a three steps iteration process in Banach space,* Fixed Point Theory, **18**(2) (2017), 625-640.

[17] M. A. KRASNOSEL'SKII: *Two remarks on the method of successive approximations,* Usp. Mat. Nauk., **10** (1955), 123-127.

[18] W. R. MANN: *Mean value methods in iteration,* Proc. Am. Math. Soc., **4** (1953), 506-510.

[19] M. A. NOOR: *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl., **251** (2000), 217-229.

[20] M. O. OLATINWO: *Some results on the continuous dependence of the fixed points in normed linear space,* Fixed Point Theory Appl., **10** (2009), 51-157.

[21] Z. Opial, *Weak convergence of the sequence of the successive approximation for nonexapansive mapping,* Bull Amer. Math. Soc., **73** (1967), 591-597.

[22] M. O. OSILIKE, A. UDOMENE; *Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings,* Indian J. Pure Ap. Mat., **30**(12) (1999), 1229–1234.

[23] S. M. SOLTUZ, T. GROSAN: *Data dependence for Ishikawa iteration when dealing with contractive like operators,* Fixed Point Theory A., doi:10.1155/2008/242916

[24] T. SUZUKI; *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,* J. Math. Anal. Appl., **340** (2008), 1088-1095.

[25] B. S. THAKUR, D. THAKUR, M. POSTOLACHE: *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings,* App. Math. Comp., **275** (2016), 147-155.

[26] K. ULLAH, M. ARSHAD: *Numerical reckoning fixed points for Suzuki generalized nonexpansive mappings via new iteration process,* Filomat, **32**(1) (2018), 187-196.

[27] X. WENG: *Fixed point iteration for local strictly pseudocontractive mapping,* Proc. Am. Math. Soc., **113** (1991), 727-731.

[28] T. ZAMFIRESCU: *Fix point theorems in metric spaces,* Arch. Math. (Basel), **23** (1972), 292-298.

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