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STABILITY OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS IN THE FRAME OF ATANGANA-BALEANU OPERATOR

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ABSTRACT. Theory of fractional calculus with singular and non-singular kernels is pioneering and has garnered significant interest recently. Fair amount of literature on the qualitative properties of fractional differential and integral equations involving different types of operators is available. This manuscript aims to analyze the stability of a class of nonlinear fractional differential equation in terms of Atangana-Baleanu-Caputo operator. Sufficient conditions for the existence and uniqueness of solutions are obtained by employing classical fixed point theorems and Banach contraction principle. Also adequate conditions for Hyers-Ulam stability are established. To substantiate our analytic results, an example is provided with numerical simulation.

1. INTRODUCTION

The concept of fractional derivative emerged in 1695 in a dialogue between G.A. de L'Hopital and G.W. Leibniz. In the field of applied sciences as well as in pure mathematics, theory and applications of Fractional Calculus (FC) experienced an exponential growth in the past three decades. To describe the

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phenomena involving long time memory effects and nonlocality, FC is an excellent tool. An extensive amount of research articles, monographs, and books on FC have been published [1–5]. Nowadays, one can identify the progress of FC in both theoretical front and application areas such as physics, engineering, biology, medicine, economy and finance [2].

There are different types of Fractional Derivatives (FD) such as Riemann-Liouville FD [6], the generalized FD [7], the conformable FD [8], the non conformable FD [9], the Caputo-Fabrizio FD [10], the Caputo FD [11], the Hadamard and Hilfer FD [12], the Atangana-Baleanu FD [13] and others. In 2015 [14], Caputo and Fabrizio proposed a new FD which does not contain any singular kernel. Also in 2016 [15], Atangana-Baleanu established another form of FD, where they considered Mittag-Leffler function as a kernel. They introduced the Atangana-Baleanu Derivative(ABD) based on the approaches of Caputo and Riemann-Liouville, where the complexity of the differentiation is associated with the generalized Mittag-Leffler function. The new fractional derivative contains additional encouraging properties in comparison to the former derivatives. For instance, they have shown that it can represent substance heterogeneities and configurations with different scales, which clearly cannot be overseen with the prominent local theories and also the known fractional derivative [16].

Nowadays, ample research on analysis of Fractional Order Differential Equation (FODE) using ABD is seen in literature. Jarad et.al [17] analyzed the ODE with ABD. Fractional integro-differential equations with numerical results using ABD is studied by Ravichandran et.al [18]. Koca [19] with ABD analyzed the coupled fractional differential equations numerically. Yadav et.al [20] discussed about the ABD and their applications numerically by employing approximation methods.

The qualitative properties of the solutions of differential equations with fractional order differential operators appear to be very important. Various types of stability analysis of FODE using different types of derivatives have been analyzed by many researchers. The main objective of this manuscript is to establish results for the solutions of a new class of FODE to be Hyers-Ulam stable with Atangana-Baleanu derivative. The analysis is based on fixed points theorems. The structure of this manuscript is as follows: Section 2, recalls basic definitions and introduces the hypothesis, proposition and theorems which are necessary to establish the main results. In section 3, the existence and uniqueness of the solution of FODE is derived. Hyers-Ulam stability of the FODE is discussed in section 4. In section 5, suitable example is provided with numerical illustration and conclusion is presented in section 6.

2. PREREQUISITES

This section presents the basic definitions and theorems which are needed to obtain the main results.

Definition 2.1. *[18] Let $L = [\omega, T], L' = (\omega, T) \subset \mathbb{R}$ and $G(x, y)$ be the Banach space of all continuous functions from L to $\mathbb{R}$ with the norm*

$$\|p\|_\infty = sup\left\{|p(\omega)| : \omega \in L\right\}. \tag{2.1}$$

Definition 2.2. *[18] The AB derivative in Riemann-Liouville sense is*

$$\left({}_0^{\mathbb{ABR}}\mathbb{D}^\beta p\right)(\omega) = \frac{\Upsilon(\beta)}{1-\beta}\frac{d}{dt}\left(\int_0^t p(s)E_\beta\left[-\beta\frac{(\omega-s)^\beta}{1-\beta}\right]ds\right). \tag{2.2}$$

Here $\beta \in [0, 1], p' \in G'(x, y)$ and $y > x$.

Definition 2.3. *[18] The AB derivative in Caputo sense is*

$$\left({}_0^{\mathbb{ABC}}\mathbb{D}^\beta p\right)(\omega) = \frac{\Upsilon(\beta)}{1-\beta}\int_0^t p'(s)E_\beta\left[-\beta\frac{(\omega-s)^\beta}{1-\beta}\right]ds. \tag{2.3}$$

Here $\beta \in [0, 1], p' \in G'(x, y)$ and $y > x$. The corresponding fractional integral is

$$\left({}_0^{\mathbb{AB}}I^\beta p\right)(\omega) = \frac{1-\beta}{\Upsilon(\beta)}p(\omega) + \frac{\beta}{\Upsilon(\beta)}\left({}_0I^\beta p\right)(\omega),$$

where $\left({}_0I^\beta p\right)(\omega) = \dfrac{1}{\Gamma(\beta)}\int_0^t(\omega-s)^{\beta-1}\omega(s)ds$

Theorem 2.1. ***Arzela Fixed Point Theorem.*** *Let Ω be a compact Hausdorff metric space. Then $\Lambda \subset K(\Omega)$ is said to be relatively compact whenever Λ is equicontinuous and bounded uniformly.*

Theorem 2.2. ***Krasnoselskii Fixed Point Theorem.*** *Let N be a bounded closed convex subset on a real Banach space X, and let R_1 and R_2 be two operators on N. If R_1 is contraction and R_2 is completely continuous. Then either*

(a) *There is a $p \in N$ such that $R_1p + R_2p = p$; or*
(b) *The set $\epsilon = \left\{P \in X : \rho R_1\left(\dfrac{y}{\rho}\right) + \rho R_2(y)\right\}$ is unbounded for $\rho \in (0, 1)$.*

In this present article, we analyze the following nonlinear fractional differential equation with Atangana-Baleanu-Caputo operator

$$\begin{aligned} \mathbb{D}^{\beta}[p(\omega) - S(\omega, p(\omega))] &= g(\omega, p(\omega)) \\ p(0) &= p_0, \quad 0 < \beta < 1. \end{aligned} \tag{2.4}$$

Let $p(0) = p_0$ and $p \in \Omega[0,1]$ is a solution of (2.4), then there exists, $g \in [\Omega[0,1] \times L \times L, L]$, where $p \in [0,1]$ and

$$p(\omega) = p_0 - S(0, p(0)) + S(\omega, p(\omega)) +^{\mathbb{ABC}} I^{\beta} g(\omega, p(\omega))$$

is satisfied.

Let us introduce the following hypothesis which are needed to establish the existence results.

(F_1) If $p \in \Omega[0,1]$, then there exist an continuous function $g \in [\Omega[0,1] \times L \times L, L]$ with constants ξ_1, ξ_2 and ξ such that

$$\|g(\omega, y_1) - g(\omega, y_2)\| \le \xi_1(\|y_1 - y_2\|), \quad \forall y_1, y_2 \in Y,$$

$\xi_2 = \max_{\omega \in L} \|g(\omega, 0)\|$ and $\xi = \max\{\xi_1, \xi_2\}$. Here $Y = \Omega[L, X]$ is continuous with respect to L on the Banach spaces X.

(F_2) There is a continuous function $S \in (\Omega[0,1] \times L \times L, L)$ with constants ϱ_1, ϱ_2 and ϱ such that

$$\|S(\omega, y_1) - S(\omega, y_2)\| \le \varrho_1(\|y_1 - y_2\|), \quad \forall y_1, y_2 \in Y,$$

$\varrho_2 = \max_{\omega \in L} \|S(\omega, 0)\|$ and $\varrho = \max\{\varrho_1, \varrho_2\}$.

(F_3) For each η, let $B_\eta \in \{p \in Y : \|p\| \le \eta\} \subseteq Y$, hence B_η is closed and bounded. Also it is a convex subset in $(\Omega[0,1], E)$, where $\eta \ge (1 - 2P)^{-1}$ $[\|p_0\| + P]$ and consider $P = \max\{\varrho, \xi\}$ and $P < \dfrac{1}{2}$.

(F_4) Suppose that $e, f \in \varrho * (L, \mathbb{R}_+)$ such that
  (i) $|g(\omega, y)| \le e\,[\zeta(\|y\|)]$ for each $(\omega, y) \in L \times E$;
  (ii) $|k(\omega, y)| \le f\,[\zeta(\|y\|)]$ for each $(\omega, y) \in L \times E$.

Here $\zeta : [0, \infty] \to (0, \infty)$ is non-decreasing and continuous function and E is measurable function.

(F_5) Suppose that $\xi_* > 0$ such that

$$\frac{(1 - \varrho - \varrho^*)\xi_*}{[(\phi + \varrho^*) + (\varrho e + \varrho f^*)(\zeta \xi_*)]} > 1.$$

Proposition 2.1. *For* $0 \leq \beta \leq 1$,

$$\begin{aligned}\left({}^{\mathbb{AB}}I^{\beta}\left({}^{\mathbb{ABC}}\mathbb{D}^{\beta}p\right)\right)(\omega) &= p(\omega) - p(0)E_{\beta}(\rho\omega^{\beta}) - \frac{\beta}{1-\beta}p(0)E_{\beta,\beta+1}(\rho\omega^{\beta}) \\ &= p(\omega) - p(0).\end{aligned}$$

Definition 2.4. *If* $p(0) = p_0$ *and* $p \in \Omega[0,1]$ *is said to be a solution of* (2.4), *then there exists* $g \in (\Omega[0,1] \times L \times L, L)$, *where* $g \in [0,1]$ *and*

$$p(\omega) = p_0 - S(0,p(0)) + S(\omega,p(\omega)) +^{\mathbb{AB}} I^{\beta}g(\omega,p(\omega)) \tag{2.5}$$

is satisfied.

3. Main Results

This section presents the existence and uniqueness results of (2.4).

Theorem 3.1. *If* $p(\omega) \in \Omega[0,1]$ *such that* $\left({}_{0}^{\mathbb{ABC}}\mathbb{D}^{\beta}\right)[p(\omega) - S(\omega,p(\omega))] \in \Omega[0,1]$. *Assume that* $F_1 - F_5$ *are satisfied. If* $S(0,p(0)) = g(0,p(0)) = 0$ *and* $\left(\frac{1-\beta}{\Upsilon(\beta)} + \frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\right) \leq 1$, *then* (2.4) *has a unique positive solution.*

Proof. Let us prove that $p(\omega)$ satisfied (2.4) iff $p(\omega)$ satisfies (2.5)

$$p(\omega) = p_0 - S(0,p(0)) + S(\omega,p(\omega)) +_{0}^{\mathbb{AB}} I^{\beta}g(\omega,p(\omega)).$$

Suppose $p(\omega)$ satisfy (2.4), then by using AB-Integral of (2.5), we have

$$\left({}_{0}^{\mathbb{AB}}I^{\beta}\left({}_{0}^{\mathbb{ABC}}\mathbb{D}^{\beta}\right)[p(\omega) - S(\omega,p(\omega))]\right) =_{0}^{\mathbb{AB}} I^{\beta}g(\omega,p(\omega)).$$

Now, using proposition (2.1), we obtain

$$p(\omega) - S(\omega,p(\omega)) - [p(0) - S(0,p(0))] =_{0}^{\mathbb{AB}} I^{\beta}g(\omega,p(\omega)). \tag{3.1}$$

Since, $p(0) = p_0$ and $g(0,p(0)) = 0$, (2.5) satisfied. Now, if $p(\omega)$ satisfy (2.5), then taking $g(0,p(0)) = 0$, it is clear that $p(0) = p_0$. In Riemann-Liouville sense to apply the ABD in (2.5), we obtain

$$\begin{aligned}\left({}_{0}^{\mathbb{ABR}}D^{\beta}p\right)(\omega) &= p_0\left({}_{0}^{\mathbb{ABR}}D^{\beta}1\right)(\omega) + \left({}_{0}^{\mathbb{ABR}}D^{\beta}\right)[S(\omega,p(\omega)) - S(0,p(0))] \\ &\left({}_{0}^{\mathbb{ABR}}D^{\beta}1\right)(\omega) + \left({}_{0}^{\mathbb{ABR}}D^{\beta}\left({}_{0}^{\mathbb{AB}}I^{\beta}\right)\right)g(\omega,p(\omega)),\end{aligned}$$

which implies

$$\left({}_{0}^{\mathbb{ABR}}D^{\beta}\right)[p(\omega)-S(\omega,p(\omega))]=[p_0-S(0,p(0))]\frac{\Upsilon(\beta)}{1-\beta}E_{\beta}\left(\frac{-\beta}{1-\beta}\omega^{\beta}\right)+g(\omega,p(\omega)).$$

Hence we obtain the result, by using Theorem 1 in [14].

Define the operator

$$R\,p(\omega)=p_0-S(0,p(0))+S(\omega,p(\omega))+{}_{0}^{\mathbb{AB}}I^{\beta}g[\omega,p(\omega)].$$

From F_3, $\|p\|<\eta$ and by proposition 2.1, we have

$$\begin{aligned}\|R\,p(\omega)\| &\le \|p_0\|+\varrho\,\|p\|+\frac{1-\beta}{\Upsilon(\beta)}\xi_1\,\|p\|+\frac{1^{\beta}}{\Upsilon(\beta)}\left[\xi_1(\|p\|)\left({}_{0}I^{\beta}\right)(\omega)\right]\\ &\le \|p_0\|+P\,\|p\|+P\left(\frac{1-\beta}{\Upsilon(\beta)}+\frac{1^{\beta}}{\Upsilon(\beta)\Gamma(\beta)}\right)\|p\|\\ &\le \eta(1-2P)+2P\eta\\ &\le \eta\\ \|Rp(\omega)\| &\le \eta\end{aligned}$$

Now to prove uniqueness:

$$\begin{aligned}&\|Rp_1(\omega)-Rp_2(\omega)\|\\ &\le \left\|p_0+S(\omega,p_1(\omega))-S(0,p_1(0))+{}_{0}^{\mathbb{AB}}I^{\beta}g(\omega,p_1(\omega))\right\|\\ &\quad-\left\|p_0+S(\omega,p_2(\omega))-S(0,p_2(0))+{}_{0}^{\mathbb{AB}}I^{\beta}g(\omega,p_2(\omega))\right\|\\ &\le \varrho\,(\|p_1-p_2\|)+\frac{1-\beta}{\Upsilon(\beta)}\left[\xi\,\|p_1-p_2\|\right]+\frac{1^{\beta}}{\Upsilon(\beta)}\left[\xi\,\|p_1-p_2\|\right]\left({}_{0}I^{\beta}1\right)(\omega)\\ &\le P\,\|p_1-p_2\|+P\left[\frac{1-\beta}{\Upsilon(\beta)}+\frac{1^{\beta}}{\Upsilon(\beta)\Gamma(\beta)}\right]\|p_1-p_2\|\\ &\le 2P\,\|p_1-p_2\|\\ &\le \|p_1-p_2\|\,.\end{aligned}$$

Since $P<\frac{1}{2}$, we have $\|Rp_1-Rp_2\|\le l\,\|p_1-p_2\|$, where $0<l<1$. Hence $Rp(\omega)$ has an unique solution. □

Next, let us discuss the equation (2.4), with another fixed point theorem.

Theorem 3.2. *Assume that F_1-F_5 are satisfied and $q(\omega_2-\omega_1)=\xi\,\|p(\omega_2)-p(\omega_1)\|$. Hence (2.4), has a minimum of one solution.*

Proof. Let R_1 and R_2 defined on B_{η_0}, where η_0 be an positive constant and $p \in B_{\eta_0}$ as follows

$$(R_1p)(\omega) = p_0 - S(0, p(0)) + S(\omega, p(\omega))$$
$$(R_2p)(\omega) = {}_0^{\mathbb{AB}}I^{\beta} g(\omega, p(\omega)). \tag{3.2}$$

Clearly, p is a mild solution of (2.4), if the operators $p = R_1p + R_2p$ has a solution $p \in B_{\eta_0}$. Therefore, from the existence of (2.4) we define the constant η_0, such that R_1, R_2 has a fixed point on B_{η_0}.

The result is established in the following four Claims.

Claim - I: $\|R_1p + R_2p\| \leq \eta_0$ whenever $p \in B_{\eta_0}$.
For every $p \in B_{\eta_0}$, we have

$$\begin{aligned}\|(R_1p)(\omega) + (R_2p)(\omega)\| &\leq \|p_0\| + \varrho\|p\| + \frac{1-\beta}{\Upsilon(\beta)}\xi_1\|p\| + \frac{1^{\beta}}{\Upsilon(\beta)}(\xi_1\|p\|)\left({}_0I^{\beta}\right)(\omega) \\ &\leq \|p_0\| + P\|p\| + P\left(\frac{1-\beta}{\Upsilon(\beta)} + \frac{1^{\beta}}{\Upsilon(\beta)\Gamma(\beta)}\right)\|p\| \\ &\leq \eta(1-2P) + 2P\eta \\ &\leq \eta_0.\end{aligned}$$

Hence $\|R_1p + R_2p\| \leq \eta_0$, for every pair of $p \in B_{\eta_0}$.

Claim - II: R_1 is contraction on B_{η_0}.
If for any $p, p_1 \in B_{\eta_0}$, according to (3.2) and F_3, we have

$$\begin{aligned}\|(R_1p)(\omega) - (R_1p_1)(\omega)\| &\leq \|p(0) - p_1(0)\| + \varrho\|p - p_1\| \\ &\leq \varrho\|p - p_1\|,\end{aligned}$$

which implies that $\|R_1p - R_1p_1\| \leq P$. Since $P < \frac{1}{2}$. R_1 is a contraction on B_{η_0}.

Claim - III: R_2 is the operator which is completely continuous.
Let us show that R_2 is continuous on B_{η_0}. For any $p_n, p \subseteq B_{\eta_0}$, $n = 1, 2, \ldots$, with $\lim_{n\to\infty}\|p_n - p\| = 0$. We get $\lim_{n\to\infty} p_n(\omega) = p(\omega)$, for $\omega \in [0, 1]$. Thus by F_1,

$$\lim_{n\to\infty} g(\omega, p_n(\omega)) = g(\omega, p(\omega)) \text{ for } \omega \in [0, 1].$$

Hence

$$Sup_{\omega\in[0,1]}\|g(\omega, p_n(\omega)) - g(\omega, p(\omega))\| \to 0 \text{ as } n \to \infty.$$

On other hand, for $\omega \in [0,1]$,

$$\begin{aligned}
&\|(R_2p_n)(\omega)-(R_2p)(\omega)\| \\
\leq &\frac{1-\beta}{\Upsilon(\beta)} Sup_{\omega\in[0,1]} \|g(\omega,p_n(\omega))-g(\omega,p(\omega))\| \\
&-\frac{\beta}{\Upsilon(\beta)\Gamma(\beta)} Sup_{\omega\in[0,1]} \|g(\omega,p_n(\omega))-g(\omega,p(\omega))\| \\
\leq &\left(\frac{1-\beta}{\Upsilon(\beta)}-\frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\right) Sup_{\omega\in[0,1]} \|g(\omega,p_n(\omega))-g(\omega,p(\omega))\|
\end{aligned}$$

Hence $\|R_2p_n - R_2p\| \to 0$ as $n \to \infty$, i.e., R_2 is continuous on B_{η_0}.

Let us establish that R_2p, $p \in B_{\eta_0}$ is uniformly bounded, equicontinuous and relatively compact $\forall\omega \in [0,1]$. For any $p \in B_{\eta_0}$, we have $\|R_2p\| \leq \eta_0$, which means that $(R_2p)(\omega)$, $p \in B_{\eta_0}$ is uniformly bounded.

Now, verify that $(R_2p)(\omega)$, $p \in B_{\eta_0}$ is a equicontinuous. For any $p \in B_{\eta_0}$ and $0 \leq \omega_1 \leq \omega_2 \leq \omega$, we get

$$\begin{aligned}
\|(R_2p)(\omega_2)-(R_2p)(\omega_1)\| &\leq \frac{1-\beta}{\Upsilon(\beta)}q(\omega_2-\omega_1)+\frac{\beta}{\Upsilon(\beta)}q(\omega_2-\omega_1)\frac{(\omega_2-\omega_1)^\beta}{\beta\Gamma(\beta)} \\
&\leq \left[\frac{1-\beta}{\Upsilon(\beta)}-\frac{(\omega_2-\omega_1)^\beta}{\Upsilon(\beta)\Gamma(\beta)}\right] q(\omega_2-\omega_1),
\end{aligned}$$

$\|(R_2p)(\omega_2)-(R_2p)(\omega_1)\| \to 0$ as $\omega_2 \to \omega_1$, which implies that R_2 is equicontinuous on B_{η_0}. $\Rightarrow$ R_2 is uniformly bounded and equicontinuous. Therefore by theorem (2.2), R_2 is relatively compact subset of X. Hence R is completely continuous.

Claim - IV:

The final step is to prove the existence of the operator $R_1 + R_2$. It is enough to establish that the set

$$\epsilon = \left\{ y \in p : y = \rho A\left(\frac{y}{\rho}\right) + \rho B(y) \right\}$$

is bounded. Let $\rho = (0,1)$ and $p = \rho(R_1 + R_2)$. Then for each $\omega \in L : [0,1]$

$$p(\omega) = \rho R_1\left(\frac{p}{\rho}\right) + \rho R_2(p)(\omega).$$

From hypothesis $F_1 - F_5$, we have

$$\|p(\omega)\| \leq \rho \|p_0\| - \rho \left\| S\left(0, \frac{p(0)}{\rho}\right)\right\| + \rho \left\| S\left(\omega, \frac{p(\omega)}{\rho}\right)\right\| + \rho \left\| {}_0^{\mathbb{AB}}I^{\beta} g(\omega, p(\omega))\right\|$$
$$\leq \|\phi\| + \varrho\left(\|p\| + e\zeta\|p\|\right) + \left(\frac{\beta}{\Upsilon(\beta)} + \frac{1^{\beta}}{\Upsilon(\beta)\Gamma(\beta)}\right)[\xi\|p\| + \zeta\|p\|]$$
$$+ \left(\frac{\beta}{\Upsilon(\beta)} + \frac{1^{\beta}}{\Upsilon(\beta)\Gamma(\beta)}\right)\xi$$

Put $\mu(\omega) = \max\{|p(s)| : 0 \leq s \leq \omega\}$, $\omega \in L : [0,1]$. Then $\|p\| \leq \sigma(\omega) \forall \omega \in L : [0,1]$, and we have

$$\sigma(\omega) \leq \|\phi\| + \varrho\sigma(s) + \varrho^*\sigma(s) + \varrho^* + \varrho e\zeta(\sigma(s)) + \varrho^* f\zeta(\sigma(s))$$
$$\sigma(\omega) \leq \|\phi\| + \varrho^* + (\varrho + \varrho^*)\sigma(s) + \varrho e\zeta(\sigma(s)) + \varrho^* f\zeta(\sigma(s))$$
$$(1 - \varrho - \varrho^*)\sigma(\omega) \leq \|\phi\| + \varrho^* + (\varrho e + \varrho^* f)\left(\zeta(\sigma(s))\right).$$

If $\|p\|_\infty = Sup\,\|p(\omega)\| :\ 0 \leq \omega \leq 1$, then the above inequality becomes

$$(1 - \varrho - \varrho^*)\|p\|_\infty \leq \|\phi\| + \varrho^* + (\varrho e + \varrho^* f)\left(\zeta(\sigma(s))\right).$$

Hence

$$\frac{(1 - \varrho - \varrho^*)\|p\|_\infty}{(\|\phi\|) + \varrho^* + (\varrho e + \varrho^* f)\left(\zeta(\sigma(s))\right)} \leq 1.$$

From F_5, there is an ξ_* such that $\|p\|_\infty \neq \xi_*$.

Let

$$F = \{p \in \Omega\left([0,a] : E\right) : \|p\|_\infty \leq \xi_*\}.$$

In F, there is no $p \in \partial p$, such that $p = \rho R(p)$, here $\rho \in (0,1)$. Hence R has a fixed point p in $\overline{F}$, which proves that p is a solution of (2.4). □

4. Stability Analysis

Definition 4.1. *The ABC fractional differential equation* (2.4) *is said to be Hyers-Ulam stable if there exist constant* $\epsilon > 0$ *satisfying: for every* $\delta > 0$,

$$p(\omega) = \frac{1-\beta}{\Upsilon(\beta)}\left[S(\omega, p(\omega)) + \int_0^s g(\omega, p(\omega))d\omega\right]$$
$$+ \frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\int_0^s (\omega - s)^{\beta-1}\left[S(\omega, p(\omega)) + \int_0^t g(q, p(q))dq\right]ds,$$

there exists $p_1(\omega)$ *satisfying*

$$p_1(\omega) = \frac{1-\beta}{\Upsilon(\beta)}\left[S(\omega, p_1(\omega)) + \int_0^s g(\omega, p_1(\omega))d\omega\right] + \frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\int_0^s (\omega - s)^{\beta-1}\left[S(\omega, p_1(\omega)) + \int_0^t g(q, p_1(q))dq\right] ds, \tag{4.1}$$

such that $|p(\omega) - p_1(\omega)| \leq \delta\epsilon$, *where* $p(0) = 0$.

Theorem 4.1. *Considering the hypothesis* F_2, *the fractional order differential equation* (2.4), *in the sense of ABC is Hyers-Ulam Stable.*

Proof. From theorem (3.1), equation (2.4) has unique solution $p(\omega)$. Let there exists another solution $p_1(\omega)$ of (2.4) satisfying the condition (4.1). Then we have

$$p_1(\omega) = \frac{1-\beta}{\Upsilon(\beta)}\left[S(\omega, p_1(\omega)) + \int_0^s g(\omega, p_1(\omega))d\omega\right] + \left[\frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\int_0^s (\omega - s)^{\beta-1}\left[S(\omega, p_1(\omega)) + \int_0^t g(q, p_1(q))dq\right] ds\right] \tag{4.2}$$

Now

$$\begin{aligned}
&|p(\omega) - p_1(\omega)| \\
&= \left|\frac{1-\beta}{\Upsilon(\beta)}\left[S(\omega, p(\omega)) + \int_0^s g(\omega, p(\omega))d\omega\right]\right| \\
&\quad + \left|\frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\int_0^s (\omega - s)^{\beta-1}\left[S(\omega, p(\omega)) + \int_0^t g(q, p(q))dq\right] ds\right| \\
&\quad - \left|\frac{1-\beta}{\Upsilon(\beta)}\left[S(\omega, p_1(\omega)) + \int_0^s g(\omega, p_1(\omega))d\omega\right]\right| \\
&\quad - \left|\frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\int_0^s (\omega - s)^{\beta-1}\left[S(\omega, p_1(\omega)) + \int_0^t g(q, p_1(q))dq\right] ds\right| \\
&\leq \frac{1-\beta}{\Upsilon(\beta)}\left[|S(\omega, p(\omega)) - S(\omega, p_1(\omega))| + \int_0^s |g(\omega, p(\omega)) - g(\omega, p_1(\omega))|\, d\omega\right] \\
&\quad + \frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\left[\int_0^s (\omega - s)^{\beta-1}\left(|S(\omega, p(\omega)) - S(\omega, p_1(\omega))|\right) ds\right] \\
&\quad + \frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\left[\int_0^s (\omega - s)^{\beta-1}\left(\int_0^t |g(q, p(q)) - g(q, p_1(q))|\, dq\right) ds\right]
\end{aligned}$$

$$\leq \frac{1-\beta}{\Upsilon(\beta)}\left[\varrho\left\|y_1-y_2\right\|+\int_0^s \xi\left\|y_1-y_2\right\| dt\right]$$
$$+\frac{\beta}{\Upsilon(\beta)\Gamma(\beta)}\left[\int_0^s(\omega-s)^{\beta-1}\varrho\left\|y_1-y_2\right\|+\int_0^t \xi\left\|y_1-y_2\right\| dq\right] ds$$
$$\leq \frac{1-\beta}{\Upsilon(\beta)}[\varrho+\xi(y-x)]\left\|y_1-y_2\right\|$$
$$+\frac{\beta(y^\beta-x^\beta)}{\Upsilon(\beta)\Gamma(\beta)}[\varrho+\xi(y-x)]\left\|y_1-y_2\right\|.$$

This implies that

$$|p(\omega)-p_1(\omega)| \leq \delta\epsilon, \tag{4.3}$$

for $\delta=\left(\frac{1-\beta}{\Upsilon(\beta)}+\frac{\beta(y^\beta-x^\beta)}{\Upsilon(\beta)\Gamma(\beta)}\right)[\varrho+\xi(y-x)]$, h $\epsilon=\left\|y_1-y_2\right\|$.

From (4.3), we conclude that the fractional differential equation in the sense of Atangana-Baleanu-Caputo operator is Hyers- Ulam stable. □

5. Example

Example 1. *Let us consider the following fractional differential equation*

$$D^\beta\left[p(\tau)-\frac{1}{4}e^{2\tau}\right]=\frac{1}{4}sin(\tau,p(\tau)), \textit{where } p(0)=0, \tau\in[0,1], \Upsilon(\beta)=1. \tag{5.1}$$

Comparing with Theorem 4.1, *we have* $K=e^{2\tau}$ *and* $g(\tau,p(\tau))=sin(\tau,p(\tau))$, *then the solution of* (5.1) *is*

$$p_n(\tau)=1+\frac{1}{4}K_n(\tau,p(\tau))+\frac{1-\beta}{4}g_n(\tau,p(\tau))+\frac{\beta}{4\Gamma(\beta)}\int_a^t(\tau-s)^{\beta-1}g_n(\tau,p(\tau))ds.$$

Solving the given equation (5.1), *we have*

$$\delta\leq 1 \quad \textit{where } \delta=\frac{(1-\beta)}{4}(s).$$

For the values $s=0.1$ *to* $s=1$ *and* $\beta=0.5$ *to* 0.9, *the corresponding values of* δ *are tabulated in table 1 and plotted in figure 1. The following curve is increasing and stable in* $s\in(0,1]$. *Also from the table, we observe that the values are periodically decreasing and are less than* 1. *Hence the given fractional differential equation* (5.1), *is Hyers-Ulam Stable.*

s	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
	δ	δ	δ	δ	δ
0.1	*0.0125*	*0.0100*	*0.0075*	*0.0050*	*0.0025*
0.2	*0.0250*	*0.0200*	*0.0150*	*0.0100*	*0.0050*
0.3	*0.0375*	*0.0300*	*0.0225*	*0.0150*	*0.0075*
0.4	*0.0500*	*0.0400*	*0.0300*	*0.0200*	*0.0100*
0.5	*0.0625*	*0.0500*	*0.0375*	*0.0250*	*0.0125*
0.6	*0.0750*	*0.0600*	*0.0450*	*0.0300*	*0.0150*
0.7	*0.0875*	*0.0700*	*0.0525*	*0.0350*	*0.0175*
0.8	*0.1000*	*0.0800*	*0.0600*	*0.0400*	*0.0200*
0.9	*0.1125*	*0.0900*	*0.0675*	*0.0450*	*0.0225*
1.0	*0.1250*	*0.1000*	*0.0750*	*0.0500*	*0.0250*

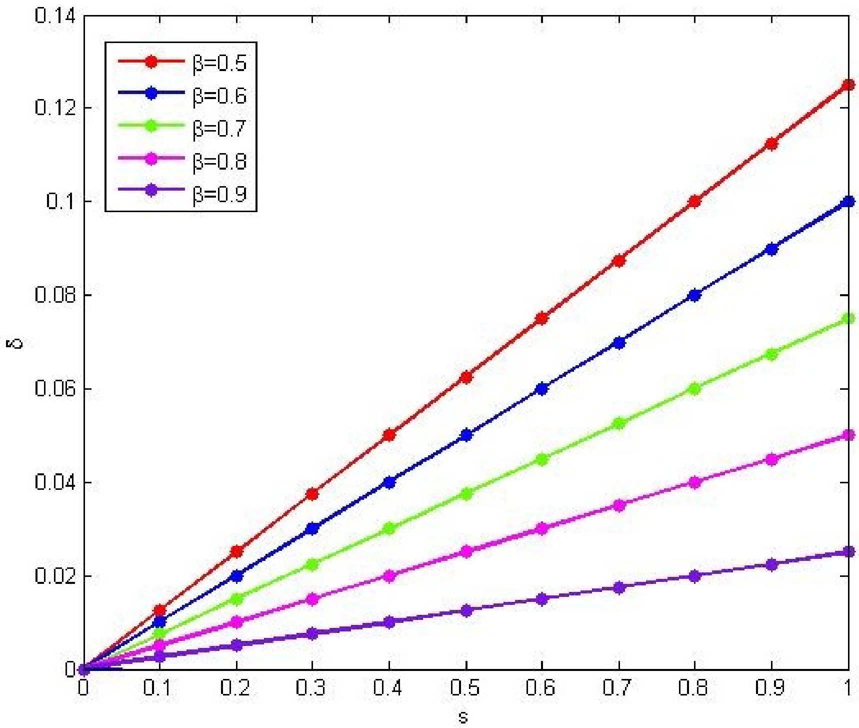

FIGURE 1. Numerical simulation of Example 1, for different values of β

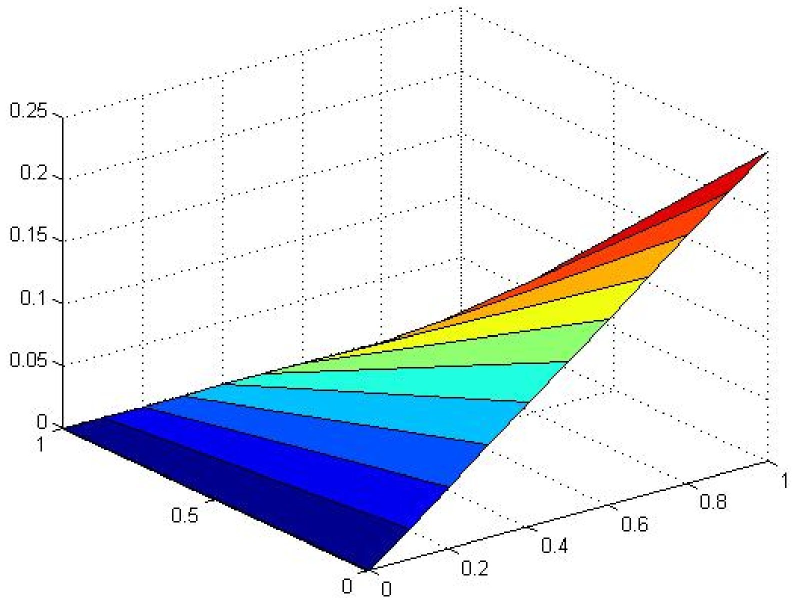

FIGURE 2. Surface plot for Example 1

6. CONCLUSION

It is well known that most of the real world phenomena can be described with FODE. This article, discussed the stability of nonlinear FODE in the frame of ABD. Basic definitions and lemma's and hypothesis are presented in section 2. Necessary and sufficient conditions which ensures the existence and uniqueness of the solutions are derived in section 3. Section 4, established the Hyers-Ulam Stability of the fractional differential equation in the terms of Atangana-Baleanu derivative operator. Example is given in section 5, to validate our analytical results. Important analytic tools used in this work are AB derivative operator, Fixed point theorems and Banach contraction principle.

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