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QUALITATIVE ASPECTS OF THE FRACTIONAL AIR-BORNE DISEASES MODEL WITH MITTAGE-LEFFLER KERNEL

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ABSTRACT. Mathematical models are used to describe transmission and propagation of diseases which have gained momentum over the last hundred years. Formulated mathematical models are currently applied to understand the epidemiology of various diseases including viral diseases viz Influenza, SARS, measles, etc.

In this paper, we shall introduce the fractional air-borne diseases model with Mittage-Leffler kernel and prove some qualitative properties of the fractional the air-borne diseases model with Mittage-Leffler kernel and Ulam-Hyers stability.

1. INTRODUCTION

Recently, Atangana-Baleanu fractional derivative has got much attention of the researchers due to its non- locality and non-singularity. This operator contains an accurate kernel that describes the better dynamics of systems with a memory effect.

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Key words and phrases. The air-borne diseases model, Atangana-Baleanu fractional derivative, Mittage-Leffler kernel, Ulam stability.

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In this paper, we investigate the fractional-order air-borne diseases model (ABDM) under Mittag–Leffler derivative. The existence of a unique solution has been discussed by a direct application of Banach Contraction mapping Theorem. We established the Hyres-Ulam stability of the proposed model under the Mittage–Leffler derivative.

To describe the behaviour of the disease and to enhanced the methods of treatment, various mathematical models have been used. Since 1994. In formulating a simple mathematical model, Mayer and others used ODEs [8] to explain the response of the immune system when pathogens attack the body.

Now, we introduce the definition of the fractional operator with nonlocal kernel (see [2], [4] and the references therein).

Definition 1.1. *Let* $f \in H^1(a,b),\ b > a,\ \alpha \in [0,1]$. *Then, the definition of the new fractional derivative is given as:*

$$ {}^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(x) E_{\alpha}\Big[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha} \Big] dx, \tag{1.1} $$

where $B(\alpha)$ *denotes a normalization function satisfying* $B(0) = B(1) = 1$.

The above definition will be helpful to discuss real world problems and it also will have a great advantage when using the Laplace transform to solve some physical problem with initial condition. However, when α is 0 we do not recover the original function except when at the origin the function vanishes. To avoid this issue, we propose the following definition.

Definition 1.2. *Let* $f \in H^1(a,b),\ b > a,\ \alpha \in [0,1]$. *Then, the definition of the new fractional derivative is given as:*

$$ {}^{ABR}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_b^t f(x) E_{\alpha}\Big[-\alpha \frac{(t-x)^{\alpha}}{1-\alpha} \Big] dx. \tag{1.2} $$

Equations (1.1) and (1.2) have a non-local kernel. Also in equation (1.1) when the function is constant we get zero.

Definition 1.3. *The fractional integral associate to the new fractional derivative with non-local kernel is defined as:*

$$ {}^{AB}I_t^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(y)(t-y)^{\alpha-1} dy. $$

When α is zero we recover the initial function and if also α is 1, we obtain the ordinary integral.

2. THE AIR-BORNE DISEASES MODEL

An airborne transmission is disease transmission through small particulates that can be transmitted through the air over time and distance [11]. The relevant pathogens may be viruses, bacteria, or fungi, and they may be spread through breathing, talking, coughing, sneezing, raising of dust, spraying of liquids, flushing toilets, or any activities which generate aerosol, particles or droplets.

In the air-borne diseases model the individuals were classified as: Susceptible (S) – those who did not have any immunity to the disease; Exposed (E) or latent – those exposed to the virus and incubating it prior to the development of symptoms; "Infectives" (I) – symptomatic and infectious; Asymptomatic (A) – those testing positive in serological tests/blood tests for the disease, but had no symptoms (were assumed to be partially infectious); and recovered population (R). Following assumptions are made where $S,\ E,\ I,\ A,\ R,$ denote the numbers of individuals in the Susceptible, Latent (or exposed), Infective, Asymptomatic and Recovered compartments respectively, with the total population size at all times given by $N = S(t) + E(t) + I(t) + A(t) + R(t),$ as:

(i) Total population at the initial stage was susceptible with no members having immunity through vaccination or any previous exposure. One infective was introduced.

(ii) There is no transmission from individuals at the Latent (Exposed) state.

(iii) A fraction p of the latent (E) individuals proceed to Infective (symptomatic) I compartment at the rate $k.$ The remaining fraction $(1-p)$ goes to the asymptomatic compartment A at the same rate $k.$

(iv) The study population is considered constant and no consideration has been made for the addition or removal of individuals.

(v) Asymptomatic individuals have a reduced capacity to transmit the disease. Let q be the factor that decides reduction in transmissibility of the asymptomatic individuals ($0 < q < 1$) (Poddar et al., 2010 [9], Shil et al., 2011 [10]).

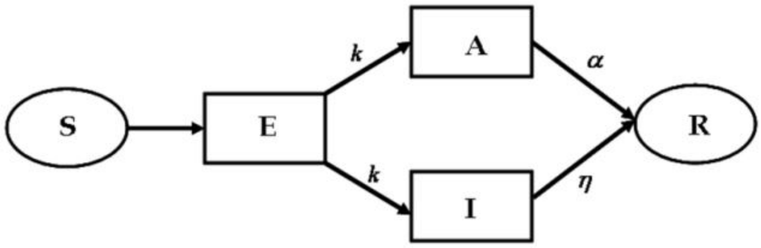

FIGURE 1. The schematic diagram of the SEIAR type transmission model. S, E, I, A and R denote Susceptible, Exposed (latent), Infective, Asymptomatic and Recovered /removed categories of the population, respectively [10].

(vi) Assuming homogeneous mixing within the population, the average member of the population made contact sufficient to transmit infection to βN others per unit time, where β is the transmission rate.

(vii) A fraction α of the infective individuals and a fraction η of the asymptomatic individuals moved to recovered class per unit time.

(viii) No restrictions on human behaviour (such as quarantine, wearing of masks) or interventions (as preventive medicine) are imposed.

Susceptible - Exposed - Infective Asymptomatic-Recovered or SEIAR model can be modified as:

$$\begin{aligned} \frac{dS}{dt} &= -\beta S(t)(I + qA(t)), \\ \frac{dE}{dt} &= \beta S(t)(I(t) + qA(t)) - kE(t), \\ \frac{dI}{dt} &= pkE(t) - \alpha I(t), \\ \frac{dA}{dt} &= (1-p)kE(t) - \eta A(t), \\ \frac{dR}{dt} &= \alpha I(t) + \eta A(t), \\ \frac{dC}{dt} &= \alpha I(t), \end{aligned} \tag{2.1}$$

Here, C denotes the cumulative number of infectives. Also, all variables are positive at all times ($0 < t < \infty$).

The number of susceptible individuals $S(t)$ decreases as the number of incidences (i.e., Infectives $I(t)$) increase. The epidemic peaks then declines as more and more individuals recover and stop transmitting the disease. Considering everyone initially to be susceptible (i.e., at $t = 0,\ S(t) = N$), a newly introduced infected individual can infect on the average $\mathcal{R}_0$ individuals. This is the basic reproduction number, $\mathcal{R}_0$. In other words, $\mathcal{R}_0$ describes the average number of secondary infections generated by one infectious individual when introduced into a fully susceptible population. The severity of the epidemic and rates of increase depend on the value of the basic reproduction number. If $\mathcal{R}_0 > 1$, then the epidemic will continue. If $\mathcal{R}_0 < 1$, then the epidemic will die out. $\mathcal{R}_0$ can be calculated form the growth rate of the epidemic (r) obtained from the cumulative incidences data in the initial growth phase of the outbreak.

Each individual who received the causative agent (pathogen) exist in the Exposed or Latent state (E) during which he/she is incubating the virus or bacteria but they does not transmit the infection to anyone.

If k be the rate of transition from the Exposed state to the Infectious state, then duration of the mean exposed period or latent phase is $1/k$.

3. Formulation of the model with Mittage- Leffler kernel

Now, consider the new fractional system with Mittage- Leffler kernel.

$$
\begin{aligned}
{}^{ABC}D_t^{\gamma} S(t) &= -\beta S(t)(I + qA(t)),\\
{}^{ABC}D_t^{\gamma} E(t) &= \beta S(t)(I(t) + qA(t)) - kE(t),\\
{}^{ABC}D_t^{\gamma} I(t) &= pkE(t) - \alpha I(t),\\
{}^{ABC}D_t^{\gamma} A(t) &= (1-p)kE(t) - \eta A(t),\\
{}^{ABC}D_t^{\gamma} R(t) &= \alpha I(t) + \eta A(t),\\
{}^{ABC}D_t^{\gamma} C(t) &= \alpha I(t),
\end{aligned}
\tag{3.1}
$$

The description of the parameters as given above and

$$S(0) = S_0,\ E(0) = E_0,\ I(0) = I_0,\ A(0) = A_0,\ C(0) = C_0,\ R(0) = R_0.$$

For

$$
\begin{aligned}
u_1(t,S,E,I,A,R,C) &= -\beta S(t)(I+qA(t)),\\
u_2(t,S,E,I,A,R,C) &= \beta S(t)(I(t)+qA(t))-kE(t),\\
u_3(t,S,E,I,A,R,C) &= pkE(t)-\alpha I(t),\\
u_4(t,S,E,I,A,R,C) &= (1-p)kE(t)-\eta A(t),\\
u_5(t,S,E,I,A,R,C) &= \alpha I(t)+\eta A(t),\\
u_6(t,S,E,I,A,R,C) &= \alpha I(t).
\end{aligned} \tag{3.2}
$$

Then system (3.2) may be rewritten in the form:

$$
\begin{aligned}
{}^{ABC}D_t^{\gamma}S(t) &= u_1(t,S,E,I,A,R,C),\\
{}^{ABC}D_t^{\gamma}E(t) &= u_2(t,S,E,I,A,R,C),\\
{}^{ABC}D_t^{\gamma}I(t) &= u_3(t,S,E,I,A,R,C),\\
{}^{ABC}D_t^{\gamma}A(t) &= u_4(t,S,E,I,A,R,C),\\
{}^{ABC}D_t^{\gamma}R(t) &= u_5(t,S,E,I,A,R,C),\\
{}^{ABC}D_t^{\gamma}C(t) &= u_6(t,S,E,I,A,R,C),\\
{}^{ABC}D_t^{\gamma}M(t) &= F(t,M(t)),\ \gamma\in(0,1).
\end{aligned}
$$

Taking

$$M=(S,E,I,A,R,C)^T \text{ and } M_0=(S_0,E_0,I_0,A_0,R_0,C_0)^T.$$

Therefore, system (3.3) can reduced to

$$F(t,M(t))=(u_1,u_2,u_3,u_4,u_5,u_6)^T$$

$$
\begin{aligned}
{}^{ABC}D_t^{\gamma}M(t) &= F(t,M(t)),\\
M(0) &= M_0\geq 0.
\end{aligned} \tag{3.3}
$$

Applying Atangana-Baleanu Caputo integral to (3.3) and using initial conditions, we obtain the equivalent form of (3.3) as

$$M(t)=M_0+\frac{(1-\gamma)}{B(\gamma)}F(t,M(t))+\frac{\gamma}{\Gamma(\gamma)}\frac{1}{B(\gamma)}\int_0^t(t-\theta)^{\gamma-1}F(\theta,M(\theta))d\theta.$$

Now define a Banach space $\Omega=\mathcal{C}(J,\mathbb{R}_+^6),\ J=[0,b]$ with the following norm

$$\|M\|=\sup_{t\in J}M(t): M\in\Omega.$$

Suppose that for each $M \in \Omega$ and $t \in J$, the function $F(t, M(t))$ satisfies the following

(i): $F : J \times \mathbb{R}^6_+$ is continuous functions and there exists a constant $L > 0$ such that

$$|F(t, M_1(t)) - F(t, M_2(t))| \leq L|M_1 - M_2|, \ \forall \ (t, M_1), \ (t, M_2) \in J \times \Omega.$$

Now, define the operator $\mathbb{A}$ such that

$$\mathbb{A}M(t) = M_0 + \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) + \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta.$$

4. Solvability of the model with Mittage- Leffler kernel

At this stage, our target is to prove the existence of unique solution for (3.3). This existence result will be based on the contraction mapping principle.

Theorem 4.1. *Let assumption (i) be satisfied. If* $\frac{L(1-\gamma)}{B(\gamma)} + \frac{\gamma}{B(\gamma)} \frac{Lb^\gamma}{\Gamma(\gamma+1)} < 1,$ *then there exists a unique solution for the equation (3.3).*

Proof. Define the operator $\mathbb{A}$ by:

$$\mathbb{A}M(t) = M_0 + \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) + \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta, \ t \in J.$$

In view of assumptions (i), then $\mathbb{A} : \mathcal{C}(J, \mathbb{R}^6_+) \to \mathcal{C}(J, \mathbb{R}^6_+)$ is continuous operator. Now let M and , $\widetilde{M} \in \mathcal{C}(J, \mathbb{R}^6_+),$ then

$$\begin{aligned}
& | \ \mathbb{A}M(t) - \mathbb{A}\widetilde{M}(t) \ | \\
= \ & \left| \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) - \frac{(1-\gamma)}{B(\gamma)} F(t, \widetilde{M}(t)) \right. \\
+ \ & \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta \\
- \ & \left. \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, \widetilde{M}(\theta)) d\theta \right| \\
\leq \ & \left| \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) - \frac{(1-\gamma)}{B(\gamma)} F(t, \widetilde{M}(t)) \right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}F(\theta,M(\theta))d\theta\right.\\
&- \left.\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}F(\theta,\widetilde{M}(\theta))d\theta\right|\\
&\leq \frac{(1-\gamma)}{B(\gamma)}\left|F(t,M(t))-F(t,\widetilde{M}(t))\right|\\
&+ \frac{\gamma}{B\gamma)}\frac{1}{\Gamma(\gamma)}\left|\int_0^t (t-\theta)^{\gamma-1}F(\theta,M(\theta))d\theta-\int_0^t (t-\theta)^{\gamma-1}F(\theta,\widetilde{M}(\theta))d\theta\right|\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}\left|M(t)-\widetilde{M}(t)\right|\\
&+ \frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}|F(\theta,M(\theta))-F(\theta,\widetilde{M}(\theta))|d\theta\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}|M(t)-\widetilde{M}(t)|\\
&+ \frac{\gamma}{B(\gamma)}\frac{L}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}|M(\theta)-\widetilde{M}(\theta)|d\theta\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}|M(t)-\widetilde{M}(t)|\\
&+ \frac{\gamma}{B(\gamma)}\frac{L}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}\sup_{\theta\in J}|M(\theta)-\widetilde{M}(\theta)|d\theta\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}||M-\widetilde{M}||\\
&+ \frac{\gamma}{B(\gamma)}\frac{L}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}||M-\widetilde{M}||d\theta\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}||M-\widetilde{M}||\\
&+ ||M-\widetilde{M}||\frac{\gamma}{B(\gamma)}\frac{L}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}d\theta\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}||M-\widetilde{M}||\\
&+ ||M-\widetilde{M}||\frac{\gamma}{B(\gamma)}\frac{L}{\Gamma(\gamma)}\frac{(t-\theta)^{\gamma}}{\gamma}\Big|_{t=0}^{t}\\
&\leq \frac{L(1-\gamma)}{B(\gamma)}||M-\widetilde{M}||
\end{aligned}
$$

$$
\begin{aligned}
& + \quad ||M-\widetilde{M}||\frac{\gamma}{B(\gamma)}\frac{L}{\Gamma(\gamma+1)}t^{\gamma} \\
& \leq \quad \frac{L(1-\gamma)}{B(\gamma)}||M-\widetilde{M}|| \\
& + \quad ||M-\widetilde{M}||\frac{\gamma}{B(\gamma)}\frac{Lb^{\gamma}}{\Gamma(\gamma+1)} \\
||\mathbb{A}M-\mathbb{A}\widetilde{M}|| & \leqslant \quad \frac{L(1-\gamma)}{B(\gamma)}||M-\widetilde{M}|| \\
& + \quad ||M(t)-\widetilde{M}(t)||\frac{\gamma}{B(\gamma)}\frac{Lb^{\gamma}}{\Gamma(\gamma+1)} \\
& \leqslant \quad \left[\frac{L(1-\gamma)}{B(\gamma)}+\frac{\gamma}{B(\gamma)}\frac{Lb^{\gamma}}{\Gamma(\gamma+1)}\right]||M-\widetilde{M}||.
\end{aligned}
$$

Since $\frac{L(1-\gamma)}{B(\gamma)}+\frac{\gamma}{B(\gamma)}\frac{Lb^{\gamma}}{\Gamma(\gamma+1)}<1$. Then $\mathbb{A}$ is a contraction. It follows that $\mathbb{A}$ has a unique fixed point which is a solution of the initial value problem (3.3) in $\mathcal{C}(J,\mathbb{R}^6_+)$. □

5. Stability of the model with Mittage- Leffler kernel

5.1. **Ulam-Hyers Stability.**

Definition 5.1. *The equation (3.3) is Ulam-Hyers stable if there exists a real number $c>0$ such that for each $\epsilon>0$ and for each solution $M\in\mathcal{C}(J,\mathbb{R}^6_+)$ of the inequality*

(5.1) $$|^{ABC}\, D^{\gamma}_t M(t)-F(t,M(t))\,|\leqslant\epsilon,\ t\in J,$$

there exists a solution $Y\in\mathcal{C}(J,\mathbb{R}^6_+)$ of the system (3.3) such that $Y(0)=M(0)=M_0$ with

$$|\ M(t)-Y(t)|\leqslant\epsilon c,\ t\in J.$$

Definition 5.2. *The equation (3.3) is generalized Ulam-Hyers stable if there exists $\psi\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\psi(0)=0$, such that for each solution $M\in\mathcal{C}(J,\mathbb{R}^6_+)$ of the inequality (5.1), there exists a solution $Y\in\mathcal{C}(J,\mathbb{R}^6_+)$ of system (3.3) such that $Y(0)=M(0)=M_0$ with*

$$|\ M(t)-Y(t)|\in\psi(\epsilon)\epsilon,\ t\in J.$$

Theorem 5.1. *Let assumptions of Theorem 4.1 be satisfied. Then the fractional order differential equation (3.3) is Ulam-Hyers stable.*

Proof. Let $M \in \mathcal{C}(J, \mathbb{R}_+^6)$ be a solution of the inequality (5.1), Let $Y \in \mathcal{C}(J, \mathbb{R}_+^6)$ be the unique solution of the initial value problem (3.3). This Cauchy problem (3.3) is equivalent to

$$Y(t) = Y_0 + \frac{(1-\gamma)}{B(\gamma)} F(t, Y(t)) + \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, Y(\theta)) d\theta, \ t \in J.$$

Operating AB fractional integral to both sides of (5.1), we get

$$\begin{aligned} | M(t) &- M_0 - \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) - \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta \,| \\ &\leqslant \frac{(1-\gamma)}{B(\gamma)} \epsilon + \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} \epsilon d\theta \\ &\leqslant \frac{\epsilon(1-\gamma)}{B(\gamma)} + \frac{\epsilon\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} d\theta \\ &\leqslant \frac{\epsilon(1-\gamma)}{B(\gamma)} + \frac{\epsilon\gamma \, b^\gamma}{B(\gamma)\Gamma(\gamma+1)}. \end{aligned}$$

Also, we have

$$\begin{aligned} & |M(t) - Y(t)| \\ = & \left| M(t) - Y_0 - \frac{(1-\gamma)}{B(\gamma)} F(t, Y(t)) - \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, Y(\theta)) d\theta \right| \\ = & \left| M(t) - Y_0 - \frac{(1-\gamma)}{B(\gamma)} F(t, Y(t)) - \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, Y(\theta)) d\theta \right. \\ + & \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) + \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta \\ - & \left. \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) - \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta \right| \\ \leq & \left| M(t) - M_0 - \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) - \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta \right| \\ + & \frac{(1-\gamma)}{B(\gamma)} \left| F(t, Y(t)) - F(t, M(t)) \right| \\ + & \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} \left| F(\theta, Y(\theta)) - F(\theta, M(\theta)) \right| d\theta \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\epsilon(1-\gamma)}{B(\gamma)}+\frac{\epsilon\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}+\frac{L(1-\gamma)}{B(\gamma)}\Big|Y(t)-M(t)\Big| \\
&+ \frac{L\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}\Big|Y(\theta)-M(\theta)\Big|d\theta \\
&\leqslant \frac{\epsilon(1-\gamma)}{B(\gamma)}+\frac{\epsilon\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}+\frac{L(1-\gamma)}{B(\gamma)}\Big|Y(t)-M(t)\Big| \\
&+ \frac{L\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}\sup\Big|Y(\theta)-M(\theta)\Big|d\theta \\
&\leqslant \frac{\epsilon(1-\gamma)}{B(\gamma)}+\frac{\epsilon\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}+\frac{L(1-\gamma)}{B(\gamma)}\Big|Y(t)-M(t)\Big| \\
&+ \frac{L\gamma}{B(\gamma)}\frac{||Y-M||}{\Gamma(\gamma)}\int_0^t (t-\theta)^{\gamma-1}d\theta \\
&\leqslant \frac{\epsilon(1-\gamma)}{B(\gamma)}+\frac{\epsilon\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}+\frac{L(1-\gamma)}{B(\gamma)}||Y-M|| \\
&+ \frac{L\gamma\, b^{\gamma}}{B(\gamma)}\frac{||Y-M||}{\Gamma(\gamma+1)} \\
||Y-M|| &\leq \frac{\epsilon(1-\gamma)}{B(\gamma)}+\frac{\epsilon\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)} \\
&+ \frac{L(1-\gamma)}{B(\gamma)}||Y-M|| \\
&+ \frac{L\gamma\, b^{\gamma}}{B(\gamma)}\frac{||Y-M||}{\Gamma(\gamma+1)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left[1 - \frac{L(1-\gamma)}{B(\gamma)} - \frac{L\gamma\, b^{\gamma}}{B(\gamma)}\frac{}{\Gamma(\gamma+1)}\right]||Y-M|| \\
&\leq \frac{\epsilon(1-\gamma)}{B(\gamma)}+\frac{\epsilon\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)} \\
||Y-M|| &\leq \left[1-\frac{L(1-\gamma)}{B(\gamma)}-\frac{L\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}\right]^{-1}\frac{(1-\gamma)}{B(\gamma)}+\frac{\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}\epsilon = c\,\epsilon,
\end{aligned}
$$

thus the Cauchy value problem (3.3) is Ulam-Heyers stable, which completes the proof. ■ □

By putting $\psi(\varepsilon) = c\,\varepsilon,\ \psi(0) = 0$ yields that the equation (3.3) is generalized Ulam-Heyers stable.

5.2. **Ulam-Hyers-Rassias Stability.**

Definition 5.3. *The equation (3.3) is Ulam-Hyers-Rassias stable with respect to* $\varphi \in \mathcal{C}(J, \mathbb{R}_+)$ *if there exists a real number* $c > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $M \in \mathcal{C}(J, \mathbb{R}_+^6)$ *of the inequality*

$$|^{ABC}\, D_t^\gamma M(t) - F(t, M(t))\, | \leqslant \epsilon\, \varphi(t),\ t \in J, \tag{5.2}$$

there exists a solution $Y \in \mathcal{C}(J, \mathbb{R}_+^6)$ *of the system (3.3) such that* $Y(0) = M(0) = M_0$ *with*

$$|\ M(t) - Y(t)| \leqslant \epsilon c \varphi(t),\ t \in J.$$

Theorem 5.2. *Let assumptions of Theorem 4.1 be satisfied, there exists an increasing function* $\varphi \in \mathcal{C}(J, \mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$, *we have*

$${}^{AB}I_t^\gamma \varphi(t) \leqslant \lambda_\varphi \varphi(t),$$

then the equation (3.3) is Ulam-Heyers-Rassias stable.

Proof. Let $M \in \mathcal{C}(J, \mathbb{R}_+^6)$ be a solution of the inequality (5.2) Let $Y \in \mathcal{C}(J, \mathbb{R}_+^6)$ be the unique solution of the initial value problem (3.3). This Cauchy problem (3.3) is equivalent to

$$Y(t) = Y_0 + \frac{(1-\gamma)}{B(\gamma)} F(t, Y(t)) + \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, Y(\theta)) d\theta,\ t \in J.$$

Operating AB fractional integral to both sides of (5.2), we get

$$\begin{aligned} |\ M(t) \ &-\ M_0 - \frac{(1-\gamma)}{B(\gamma)} F(t, M(t)) - \frac{\gamma}{B(\gamma)} \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} F(\theta, M(\theta)) d\theta\ | \\ &\leqslant\ \epsilon\ {}^{AB}I_t^\alpha \varphi(t) \leq \epsilon \lambda_\varphi \varphi(t). \end{aligned}$$

Also, we have

$$
\begin{aligned}
& |M(t)-Y(t)| \\
= & \left|M(t)-Y_0-\frac{(1-\gamma)}{B(\gamma)}F(t,Y(t))-\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}F(\theta,Y(\theta))d\theta\right| \\
= & \left|M(t)-Y_0-\frac{(1-\gamma)}{B(\gamma)}F(t,Y(t))-\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}F(\theta,Y(\theta))d\theta\right. \\
+ & \frac{(1-\gamma)}{B(\gamma)}F(t,M(t))+\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}F(\theta,M(\theta))d\theta \\
- & \left.\frac{(1-\gamma)}{B(\gamma)}F(t,M(t))-\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}F(\theta,M(\theta))d\theta\right| \\
\leq & \left|M(t)-M_0-\frac{(1-\gamma)}{B(\gamma)}F(t,M(t))\right. \\
- & \left.\frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}F(\theta,M(\theta))d\theta\right| \\
+ & \frac{(1-\gamma)}{B(\gamma)}\Big|F(t,Y(t))-F(t,M(t))\Big| \\
+ & \frac{\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}\Big|F(\theta,Y(\theta))-F(\theta,M(\theta))\Big|d\theta \\
\leqslant & \epsilon\lambda_\varphi\varphi(t)+\frac{L(1-\gamma)}{B(\gamma)}\Big|Y(t)-M(t)\Big| \\
+ & \frac{L\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}\Big|Y(\theta)-M(\theta)\Big|d\theta \\
\leqslant & \epsilon\lambda_\varphi\varphi(t)+\frac{L(1-\gamma)}{B(\gamma)}\sup\Big|Y(t)-M(t)\Big| \\
+ & \frac{L\gamma}{B(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}\sup\Big|Y(\theta)-M(\theta)\Big|d\theta \\
\leqslant & \epsilon\lambda_\varphi\varphi(t)+\frac{L(1-\gamma)}{B(\gamma)}||Y-M|| \\
+ & \frac{L\gamma}{B(\gamma)}\frac{||Y-M||}{\Gamma(\gamma)}\int_0^t(t-\theta)^{\gamma-1}d\theta \\
\leqslant & \epsilon\lambda_\varphi\varphi(t)+\frac{L(1-\gamma)}{B(\gamma)}||Y-M|| \\
+ & \frac{L\gamma\, b^\gamma}{B(\gamma)}\frac{||Y-M||}{\Gamma(\gamma+1)}
\end{aligned}
$$

$$
\begin{aligned}
||Y-M|| \quad &\leq \quad \epsilon\lambda_{\varphi}\varphi(t) + \frac{L(1-\gamma)}{B(\gamma)}||Y-M|| \\
&+ \quad \frac{L\gamma\, b^{\gamma}}{B(\gamma)}\frac{||Y-M||}{\Gamma(\gamma+1)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left[1 \quad - \quad \frac{L(1-\gamma)}{B(\gamma)} - \frac{L\gamma\, b^{\gamma}}{B(\gamma)}\frac{}{\Gamma(\gamma+1)}\right]||Y-M|| \\
&\leq \quad \epsilon\lambda_{\varphi}\varphi(t) \\
||Y-M|| \quad &\leq \quad \left[1-\frac{L(1-\gamma)}{B(\gamma)} - \frac{L\gamma\, b^{\gamma}}{B(\gamma)\Gamma(\gamma+1)}\right]^{-1}\epsilon\lambda_{\varphi}\varphi(t) = c\,\epsilon\varphi(t),
\end{aligned}
$$

thus the Cauchy problem (3.3) is Ulam-Heyers-Rassias stable, which completes the proof. □

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