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A GENERALIZATION OF AN EXTENDED *b*-METRIC SPACE AND SOME FIXED POINT THEOREMS

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ABSTRACT. In this paper, we present fixed point theorems for contraction mappings in a generalization of an extended b-metric space where the product of the Lipschitz constant and functions of the underlying space in the limit are bounded by one for sequences in an orbit. Futhermore, we prove fixed point results in which the contraction involves b-comparison functions.

1. INTRODUCTION

The concept of a b-metric was as a result of the works of Bourbaki, [2], and Bakhtin, [1]. Czerwik, [3], gave an axiom which was weaker than the triangle inequality and defined a b-metric with the intention of generalizing the Banach contraction mapping theorem. Databases have been used to store and retrieve textual and numerical information. Applications such as multimedia has led to the development of databases that can handle images. This raises the issues of how to define the measure of 'distances' between shapes. Fagin et. al., [5], discovered some kind of relaxation in the triangular inequality and called the distance measure a non-linear elastic matching for pattern matching,

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shape matching. There is an extensive literature on various ways to define distances between images and these include the method that used strings to code trademark contours and string match was applied for similarity measuring [7], measure ice floes, [6], on Hausdorff distances by Huttenbocher et. al. [8].

Definition 1.1. *Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is an α, β b-metric on X if there exists real numbers $\alpha, \beta \geq 1$ such that the following conditions hold for all $x, y, z \in X$:*

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq \alpha d(x, z) + \beta d(z, y)$

The pair (X, d) is a called an α, β b-metric space, [9]. An α, β b-metric with $\alpha = \beta$ is the usual b- metric, [4].

Definition 1.2. *Let X be a non-empty set. A function $d_\omega : X \times X \to [0, \infty)$ is an extended b-metric on X if there exists a function $\omega : X \times X \to [1, \infty)$, such that the following conditions hold for all $x, y, z \in X$, [10]:*

(i) $d_\omega(x, y) = 0 \iff x = y$

(ii) $d_\omega(x, y) = d_\omega(y, x)$

(iii) $d_\omega(x, y) \leq \omega(x, y) [d_\omega(x, z) + d_\omega(z, y)]$

The pair (X, d_ω) is an extended b-metric space.

Definition 1.3. *Let (X, ρ) be an α, β extended b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:*

(i) *The sequence $\{x_n\}$ converges $\iff$ there exists $x \in X$ such that $\rho(x_n, x) \to 0$ as $n \to \infty$.*

(ii) *The sequence $\{x_n\}$ is a Cauchy in (X, ρ) $\iff$ $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$.*

(iii) *The space (X, ρ) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.*

2. Main result

Definition 2.1. *Let X be a non-empty set. A function $\rho : X \times X \to [0, \infty)$ is an α, β extended b-metric on X if there exists functions $\alpha, \beta : X \times X \to [1, \infty)$ such that the following conditions hold for all $x, y, z \in X$:*

(i) $\rho(x,y) = 0 \iff x = y$

(ii) $\rho(x,y) = \rho(y,x)$

(iii) $\rho(x,y) \leq \alpha(x,y)\rho(x,z) + \beta(x,y)\rho(z,y)$

The pair (X,ρ) is a called an α,β extended b-metric space. The α,β extended b-metric space with $\alpha(x,y) = \alpha, \beta(x,y) = \beta$ is an α,β b-metric, [9].

If $\alpha(x,y) = \alpha(y,x)$ and $\beta(x,y) = \beta(y,x)$ then taking $\omega(x,y) = \frac{\alpha(x,y)+\beta(y,x)}{2}$, we obtain an extended b-metric as in definition 1.2.

Example 1. *Let $X = [0,1]$ then define $\rho : X \to X$ such that*

$$\rho(x,y) = \begin{cases} \frac{1}{xy} & \textit{if } x \neq y, 0 < x,y < 1 \\ 0 & \textit{if } x = y \\ \rho(x,0) = \dfrac{1}{x} & \textit{if } y = 0 \\ \rho(0,y) = \dfrac{1}{y} & \textit{if } x = 0 \end{cases}$$

and define functions $\alpha,\beta : X \to [0,\infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 1 + \dfrac{1}{y} & \textit{if } y \neq 0 \\ \alpha(x,0) = 1 & \textit{if } y = 0 \end{cases}, \quad \beta(x,y) = \begin{cases} 1 + \dfrac{1}{x} & \textit{if } x \neq 0 \\ \beta(0,y) = 1 & \textit{if } x = 0 \end{cases}.$$

To show that ρ is an α,β b-metric it suffices to verify property (iii) of definition 2.1. Let $x,y \in (0,1]$. For $z \in (0,1]$ we obtain

$$\rho(x,y) \leq \alpha(x,y)\rho(x,z) + \beta(x,y)\rho(z,y) \quad \Leftrightarrow \quad \frac{1}{xz} \leq \left(1 + \frac{1}{y}\right)\frac{1}{xz} + \left(1 + \frac{1}{x}\right)\frac{1}{yz}$$
$$\Leftrightarrow \quad z \leq 1 + 1 + x + y,$$

if $z = 0$

$$\rho(x,y) \leq \alpha(x,y)\rho(x,0) + \beta(x,y)\rho(0,y) \quad \Leftrightarrow \quad \frac{1}{xy} \leq \left(1 + \frac{1}{y}\right)\frac{1}{x} + \left(1 + \frac{1}{x}\right)\frac{1}{y}$$
$$\Leftrightarrow \quad 1 \leq 1 + 1 + x + y,$$

for $x \in (0,1]$ and $y = 0$, let $z \in (0,1]$,

$$\rho(x,0) \leq \alpha(x,0)\rho(x,z) + \beta(x,0)\rho(z,0) \quad \Leftrightarrow \quad \frac{1}{x} \leq \frac{1}{xz} + \left(1 + \frac{1}{x}\right)\frac{1}{z}$$
$$\Leftrightarrow \quad z \leq 1 + 1 + x.$$

In conclusion for all $x,y,z \in X$, we obtain $\rho(x,y) \leq \alpha(x,y)\rho(x,z) + \beta(x,y)\rho(z,y)$. Hence, (X,ρ) is an α,β extended b-metric space.

Example 2. *Let $X = [0, \infty)$ then define a function $\rho : X \to X$ by:*

$$\rho(x, y) = |x - y|^2 ,$$

with $\alpha, \beta : X \to [1, \infty)$ defined as: $\alpha(x, y) = 2 + x + y$ and $\beta(x, y) = 2e^{xy}$. It suffices to verify property (iii) of definition 2.1.

Let $x, y, z \in X$ we get

$$\begin{aligned}\rho(x, y) &= |x - y|^2 \\ &\leq [|x - z| + |z - y|]^2 \\ &\leq |x - z|^2 + |z - y|^2 + 2\,|x - z|\,|z - y| \\ &\leq \alpha(x, y)\,|x - z|^2 + \beta(x, y)\,|z - y|^2 .\end{aligned}$$

Thus we conclude that (X, ρ) is an α, β extended b-metric space.

Example 3. *Let $X = (1, 3)$ and let $\rho : X \times X \to [0, \infty)$ be a function defined by*

$$\rho(x, y) = \begin{cases} 2^{|x-y|}, & \text{if } x \neq y \\ 0, & \text{iff } x = y. \end{cases}$$

For $x \neq y,\ z \in X$,

$$\begin{aligned}\rho(x, y) &\leq 2^{|x-z|+|z-y|} \\ &= 2^{\frac{1}{3}|x-z|+\frac{2}{3}|z-y|}2^{\frac{2}{3}|x-z|+\frac{1}{3}|z-y|} \\ &\leq \sup_{x,y,z\in X} 2^{\frac{1}{3}|x-z|+\frac{2}{3}|z-y|}\left(\tfrac{2}{3}2^{|x-z|} + \tfrac{1}{3}2^{|z-y|}\right) \\ &\leq \tfrac{2}{3}2^2 e^{|x-z|} + \tfrac{1}{3}2^2 2^{|z-y|} \\ &= \tfrac{2}{3}2^2\rho(x, z) + \tfrac{1}{3}2^2\rho(z, y).\end{aligned}$$

with $\alpha(x, y) = \frac{2}{3}2^2$ and $\beta(x, y) = \frac{1}{3}2^2$.

3. Fixed point theorem for generalized b-metric spaces

Theorem 3.1. *Let (X, ρ) be a complete α, β extended b-metric space such that ρ is a continuous functional. Let $T : X \to X$ satisfy*

$$\rho(Tx, Ty) \leq \lambda\rho(x, y) \tag{3.1}$$

for all $x, y \in X$, where $0 \leq \lambda < 1$ be such that for each $x_0 \in X$,

$$\lim_{n,m\to\infty} \alpha(x_n, x_m)\beta(x_n, x_m)\lambda < 1,$$

where $x_n = T^n x_0$. Then T has a unique fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ be arbitrary and define the sequence $\{x_n\}$ by

$$x_n = T^n x_0. \tag{3.2}$$

Then successively applying the inequality (3.6) we get

$$\rho(x_n, x_{n+1}) \leq \lambda^n \rho(x_0, x_1). \tag{3.3}$$

For $m > n$, we get

$$\begin{aligned}
&\rho(x_n, x_m)\\
&\leq \alpha(x_n, x_m)\rho(x_n, x_{n+1}) + \beta(x_n, x_m)\rho(x_{n+1}, x_m)\\
&\leq \alpha(x_n, x_m)\rho(x_n, x_{n+1}) + \beta(x_n, x_m)[\alpha(x_{n+1}, x_m)\rho(x_{n+1}, x_{n+2})\\
&\quad + \beta(x_{n+1}, x_m)\rho(x_{n+2}, x_m)]\\
&\leq \alpha(x_n, x_m)\rho(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)\beta(x_n, x_m)\rho(x_{n+1}, x_{n+2}) + \cdots\\
&\quad + \alpha(x_{m-2}, x_m)\beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-3}, x_m)\rho(x_{m-2}, x_{m-1})\\
&\quad + \beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-2}, x_m)\rho(x_{m-1}, x_m)\\
&\leq \alpha(x_n, x_m)\lambda^n\rho(x_0, x_1) + \alpha(x_{n+1}, x_m)\beta(x_n, x_m)\lambda^{n+1}\rho(x_0, x_1) + \cdots\\
&\quad + \alpha(x_{m-2}, x_m)\beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-3}, x_m)\lambda^{m-2}\rho(x_0, x_1)\\
&\quad + \beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-2}, x_m)\lambda^{m-1}\rho(x_0, x_1)\\
&\leq \prod_{i=1}^{n} \alpha(x_i, x_m)\beta(x_i, x_m)\lambda^n\rho(x_0, x_1) + \prod_{i=1}^{n+1} \alpha(x_i, x_m)\beta(x_i, x_m)\lambda^{n+1}\rho(x_0, x_1) + \cdots\\
&\quad + \prod_{i=1}^{m-2} \alpha(x_i, x_m)\beta(x_i, x_m)\lambda^{m-3}\rho(x_0, x_1) + \prod_{i=1}^{m-1} \alpha(x_i, x_m)\beta(x_i, x_m)\lambda^{m-1}\rho(x_0, x_1).
\end{aligned}$$

Since $\lim_{n,m\to\infty} \alpha(x_n, x_m)\beta(x_n, x_m)\lambda < 1$ so the series $\sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{n} \alpha(x_i, x_m)$ $\beta(x_i, x_m)$ converges by the ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} \lambda^n \prod_{i=1}^{n} \alpha(x_i, x_m)\beta(x_i, x_m)$$

and $S_n = \sum_{j=1}^{n} \lambda^j \prod_{i=1}^{j} \alpha(x_i, x_m)\beta(x_i, x_m)$. Thus for $m > n$ we get

$$\rho(x_n, x_m) \leq \rho(x_0, x_1) \left[S_{m-1} - S_n\right].$$

Letting $n \to \infty$ we conclude that $\{x_n\}$ is a Cauchy sequence. By the completeness of (X, ρ) it follows that there exists $x^* \in X$ such that $\lim_{n\to\infty} \rho(x_n, x^*) = 0$. Furthermore, we get

$$\begin{aligned}
\rho(Tx^*, x^*) &\leq \alpha(Tx^*, x^*)\rho(Tx^*, x_n) + \beta(Tx^*, x^*)\rho(x_n, x^*) \\
&\leq \alpha(Tx^*, x^*)\lambda\rho(x^*, x_{n-1}) + \beta(Tx^*, x^*)\rho(x_n, x^*) \\
\rho(Tx^*, x^*) &\leq 0
\end{aligned}$$

as $n \to \infty$ we obtain $\rho(Tx^*, x^*) = 0$ thus $Tx^* = x^*$. Hence x^* is a fixed point of T. Since $\lambda < 1$ the uniqueness can be easily verified. □

Definition 3.1. *Let $T : X \to X$ and for some $x_0 \in X$, $\mathcal{O}(x_0) = \{x_0, Tx_0, T^2x_0, \cdots\}$ be the orbit of x_0. A function $f : X \to \mathbb{R}$ is T- orbitally lower semi-continuous at $x \in X$ if $\{x_n\} \subseteq \mathcal{O}(x_0)$ and $x_n \to x$ implies that $f(x) \leq \lim_{n\to\infty} \inf_{m\geq n} f(x_m)$.*

Theorem 3.2. *Let (X, ρ) be a completed an α, β extended b-metric space such that ρ is a continuous functional. Let $T : X \to X$ such that there exists $x_0 \in X$ and for $x \in \mathcal{O}(x_0)$*

$$\rho(Tx, T^2x) \leq \lambda\rho(x, Tx), \tag{3.4}$$

where $0 \leq \lambda < 1$ be such that $\lim_{n,m\to\infty} \alpha(x_n, x_m)\beta(x_n, x_m)\lambda < 1$ where $x_n = T^n x_0$. Then $T^n x_0 \to x^* \in X$ as $n \to \infty$. Furthermore, x^* is a fixed point of T, if, and only if, $G(x) = \rho(x, Tx)$ is T-orbitally lower semi-continuous at x_0.

Proof. For $x_0 \in X$, define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ then successively applying inequality (3.4) we get

$$\rho(T^n x_0, T^{n+1} x_0) \leq \lambda^n \rho(x_0, x_1). \tag{3.5}$$

Following a similar procedure as the proof of 3.1 we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete there exists $x_0 \in X$ such that $T^n x_0 \to x^*$. Assume that G is orbitally lower semi-continuous at $x^* \in X$ then $\rho(x^*, Tx^*) \leq \lim_{n\to\infty} \inf_{m\geq n} \rho(T^m x_0, T^{m+1} x_0) \leq \lim_{n\to\infty} \lambda^n \rho(x_0, x_1) = 0$.

Conversely, let $Tx^* = x^*$ and $x_n \in \mathcal{O}(x_0)$ with $x_n \to x^*$. Then $G(x^*) = \rho(x^*, Tx^*) = 0 \leq \lim_{n\to\infty} \inf_{m\geq n} G(x_m) = \rho(T^n x_0, T^{n+1} x_0)$ □

Example 4. *Let $X = [0, \infty)$. Define $\rho : X \times X \to [0, \infty)$ by*

$$\rho(x, y) = |x - y|^2$$

and $\alpha(x, y) = 2 + x$, $\beta(x, y) = 2 + y$.The space (X, ρ) is a complete α, β extended b-metric space.

Define $T : X \to X$ by

$$Tx = \tfrac{x}{3}.$$

For $x, y \in X$ we get

$$\rho(Tx, Ty) = \left|\tfrac{x}{3} - \tfrac{y}{3}\right|^2 \leq \tfrac{1}{9}\left|x - y\right|^2 = \tfrac{1}{9}\rho(x, y).$$

For $x \in X$, $T^n x = \frac{x}{3^n}$ and for $x_0 \in X$,

$$\lim_{n,m\to\infty} \alpha(T^n x_0, T^m x_0)\beta(T^n x_0, T^m x_0)\tfrac{1}{9} = \lim_{n,m\to\infty} \left(2 + \tfrac{x_0}{3^n}\right)\left(2 + \tfrac{x_0}{3^m}\right)\tfrac{1}{9} = \tfrac{4}{9} < 1.$$

Thus from theorem 3.1 we get that T has a unique fixed point.

Definition 3.2. *Let (X, ρ) be an α, β b-metric space. A function $\psi : [0, \infty) \to [0, \infty)$ is a b-comparison function if it is increasing and there exists a mapping $T : X \to X$ such that for some $x_0 \in X$, $\mathcal{O}(x_0) \subset X$, $\sum_{n=0}^{\infty} \psi^n(t) \prod_{i=1}^{n} \alpha(x_i, x_m)\beta(x_i, x_m)$ converges for all t and for every $m \in \mathbb{N}$ and $x_n = T^n x_0$.*

Theorem 3.3. *Let (X, ρ) be a complete α, β extended b-metric space such that ρ is a continuous functional. Let $T : X \to X$ such that there exists $x_0 \in X$ and for $x \in \mathcal{O}(x_0)$*

$$\rho(Tx, T^2x) \leq \psi\left(\rho(x, Tx)\right), \tag{3.6}$$

where ψ is a b- comparison function for T such that $x_n = T^n x_0$. Then $T^n x_0 \to x^* \in X$ as $n \to \infty$. Furthermore, x^* is a fixed point of $T \iff G(x) = \rho(x, Tx)$ is T-orbitally lower semi-continuous at x_0.

Proof. Let $x_0 \in X$ be arbitrary and define the sequence $\{x_n\}$ by

$$x_n = T^n x_0. \tag{3.7}$$

Then successively applying the inequality (3.6) we get

$$\rho(x_n, x_{n+1}) \leq \psi^n(\rho(x_0, x_1)). \tag{3.8}$$

For $m > n$, we get

$$
\begin{aligned}
&\rho(x_n, x_m)\\
&\le \alpha(x_n, x_m)\rho(x_n, x_{n+1}) + \beta(x_n, x_m)\rho(x_{n+1}, x_m)\\
&\le \alpha(x_n, x_m)\rho(x_n, x_{n+1}) + \beta(x_n, x_m)[\alpha(x_{n+1}, x_m)\rho(x_{n+1}, x_{n+2})\\
&\quad + \beta(x_{n+1}, x_m)\rho(x_{n+2}, x_m)]\\
&\le \alpha(x_n, x_m)\rho(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)\beta(x_n, x_m)\rho(x_{n+1}, x_{n+2}) + \cdots\\
&\quad + \alpha(x_{m-2}, x_m)\beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-3}, x_m)\rho(x_{m-2}, x_{m-1})\\
&\quad + \beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-2}, x_m)\rho(x_{m-1}, x_m)\\
&\le \alpha(x_n, x_m)\psi^n(\rho(x_0, x_1)) + \alpha(x_{n+1}, x_m)\beta(x_n, x_m)\psi^{n+1}(\rho(x_0, x_1)) + \cdots\\
&\quad + \alpha(x_{m-2}, x_m)\beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-3}, x_m)\psi^{m-2}(\rho(x_0, x_1))\\
&\quad + \beta(x_n, x_m)\beta(x_{n+1}, x_m)\cdots\beta(x_{m-2}, x_m)\psi^{m-1}(\rho(x_0, x_1))\\
&\le \prod_{i=1}^{n}\alpha(x_i, x_m)\beta(x_i, x_m)\psi^n(\rho(x_0, x_1)) + \prod_{i=1}^{n+1}\alpha(x_i, x_m)\beta(x_i, x_m)\psi^{n+1}(\rho(x_0, x_1))\\
&\quad + \cdots + \prod_{i=1}^{m-2}\alpha(x_i, x_m)\beta(x_i, x_m)\psi^{m-3}(\rho(x_0, x_1))\\
&\quad + \prod_{i=1}^{m-1}\alpha(x_i, x_m)\beta(x_i, x_m)\psi^{m-1}(\rho(x_0, x_1))
\end{aligned}
$$

Since $\lim_{n,m\to\infty}\alpha(x_n, x_m)\beta(x_n, x_m)\psi(\rho(x_1, x_0)) < 1$. So, the series

$$\sum_{n=1}^{\infty}\psi^n(\rho(x_1, x_0))\prod_{i=1}^{n}\alpha(x_i, x_m)\beta(x_i, x_m)$$

converges by the ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty}\psi^n(\rho(x_1, x_0))\prod_{i=1}^{n}\alpha(x_i, x_m)\beta(x_i, x_m)$$

and $S_n = \sum_{j=1}^{n}\psi^j(\rho(x_1, x_0))\prod_{i=1}^{j}\alpha(x_i, x_m)\beta(x_i, x_m)$. Thus for $m > n$ we get

$$\rho(x_n, x_m) \le \rho(x_0, x_1)\left[S_{m-1} - S_n\right].$$

Letting $n \to \infty$ we conclude that $\{x_n\}$ is a Cauchy sequence. By the completeness of (X, ρ) it follows that there exists $x^* \in X$ such that $\lim_{n\to\infty}\rho(x_n, x^*) = 0$.

Assume that G is orbitally lower semi-continuous at $x^* \in X$ then

$$\rho(x^*, Tx^*) \leq \lim_{n\to\infty} \inf_{m\geq n} \rho(T^m x_0, T^{m+1}x_0) \leq \lim_{n\to\infty} \psi^n(\rho(x_0, x_1)) = 0.$$

Conversely, let $Tx^* = x^*$ and $x_n \in \mathcal{O}(x_0)$ with $x_n \to x^*$. Then $G(x^*) = \rho(x^*, Tx^*) = 0 \leq \lim_{n\to\infty} \inf_{m\geq n} G(x_m) = \rho(T^n x_0, T^{n+1}x_0)$. $\square$

4. Conclusion

In this paper, we have presented an α, β extended b-metric type and proved some fixed point results for contractions involving b-comparison functions.

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