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SOME FIXED POINT RESULTS FOR A GENERALIZED ψ-WEAK CONTRACTION MAPPINGS IN b-METRIC SPACES

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ABSTRACT. In this paper, we will present the definitions and notation of generalized ψ-weak contraction mappings in b-metric spaces, and establish some results besides the most important properties of fixed point in orbitally complete b-metric spaces. Our results generalize several well-known comparable results in the literature. As an application of our results we generalize the results of Shatanawi [7]. Some examples are given to illustrate the useability of our results.

1. INTRODUCTION

The Banach contraction principle [2] is one of the basic results in fixed point theory which asserts that every contraction function in a complete metric space has a unique fixed point. Subsequently, many authors generalized Banach contraction principle in different ways (see [1–3, 5].

Kamran et al. in [4] introduced the definition of extended b-metric spaces as a generalization of b-metric spaces.

In the sequel, we need the following definitions

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Definition 1.1. *[4] Let X be anon empty set and $s \geq 1$ be a real number. A function $d : x \times x \to [0, \infty)$ is called a $b - metric$ space [1, 3], if it satisfies the following properties for each x, y, $z \in X$:*

(b1): $d(x, y) = 0 \; iff \; x = y$.
(b2): $d(x, y) = d(y, x)$.
(b3): $d(x, z) \leq s\,[d(x, y) + d(y, z)]$.

Definition 1.2. *[6] Let (X, d) be a $b - metric$ space. A sequence $\{x_n\}$ on X is said to be*

(I) *Cauchy iff $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*
(II) *Convergent iff there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim\limits_{n\to\infty} x_n = x$.*
(III) *The $b - metric$ (X, d) is complete if every cauchy sequance is convergent.*

Example 1. *[8, 10] Let $X = l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $d : X \times X \to [0, \infty)$ as:*

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = \{x_n\}$, $y = \{y_n\}$. Then d is a b-metric space with coefficient $s = 2^{1/p}$.

Example 2. *[20] Let (X, d) be a metric space and $\sigma_d : X \times X \to \mathbb{R}^+$ defined by $\sigma_d(x, y) = [d(x, y)]^p$ for all $x, y \in X$, where $p > 1$ is a fixed real number. Then σ_d is a $b - metric$ with $s = 2^{p-1}$.*

Remark 1.1. *If $\theta(x, y) = s$ for $s \geq 1$, then we obtain the definition of a b-metric space.*

Example 3. *Let X =$\{1, 2, 3\}$ Define $\theta : X \times X \to \mathbb{R}^+$ and $d_\theta : X \times X \to \mathbb{R}^+$as:*

$$\begin{aligned}
&\theta(x, y) = x + y, \text{ and} \\
&d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0, \\
&d_\theta(1, 2) = d_\theta(2, 1) = 2, d_\theta(1, 3) = d_\theta(3, 1) = 3, d_\theta(2, 3) = d_\theta(3, 2) = 4.
\end{aligned}$$

Definition 1.3. *[7] Let (X, d) be a b-metric space and $f : X \to X$ be a mapping. An orbit of f at a point x of X is the set $O(x, f) = \{x, fx, f^2x, \ldots\}$.*

Definition 1.4. *[7] Let (X,d) be a $b-metric$ space and $f:X\to X$ be a mapping. X is said to be $f-$orbitally complete if every cauchy sequance in $O(x,f)$, $x\in X$, converges to a point in X.*

Definition 1.5. *[11] A single-valued mapping $f:X\to X$ is called a Ćirić strong almost contraction if there exist a constant $\alpha\in[0,1]$ and some $L\geq 0$ such that*

$$d(fx,fy)\leq \alpha M(x,y)+Ld(y,fx)$$

for all $x,y\in X$, where

$$M(x,y)=\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}.$$

Definition 1.6. *[7] Let $f:X\to B(X)$ be a multivalued mapping. Then $x\in X$ is said to be a fixed point of f if $x\in fx$.*

Definition 1.7. *[7] Let (X,d) be a $b-metric$ space and $f:X\to B(X)$ be a mapping. Then we say that X is f-multivalued orbitally complete (or multivalued orbitally complete) if any Cauchy subsequence (x_{n_i}) of $\{x=x_0,x_1\in fx_0,x_2\in fx_2,\ldots\},x\in X$ converges in X.*

Theorem 1.1. *[12] Let (X,d) be an orbitally complete metric space and let $f:X\to X$ be a given mapping. Suppose that there exist nonnegative real numbers δ, γ and L with $\delta+\gamma<1$ and $L\geq 0$ such that*

$$\int_0^{d(fx,fy)}\phi(s)ds\leq\delta\int_0^{m(x,y)}\phi(s)ds+\gamma\int_0^{M(x,y)}\phi(s)ds+L\int_0^{N(x,y)}\phi(s)ds$$

for all $x,y\in X$, where

$$m(x,y)=d(y,fy)\frac{1+d(x,fx)}{1+d(x,y)},$$
$$M(x,y)=\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}$$

and

$$N(x,y)=\min\{d(x,fx),d(y,fx)\}.$$

Then f has a unique fixed point.

Definition 1.8. *[7] The function $\phi:[0,+\infty)\to[0,+\infty)$is called an altering distance function, if the following properties are satisfied:*

(1) *ϕ is continuous and nondecreasing;*

(2) $\phi(t) = 0$ *if and only if* $t = 0$.

For some fixed point theorems based on altering distance function, we refer the reader as example to [13–19].

In this paper, we introduce the notion of a generalized ψ-weak contraction mapping and establish some results in orbitally complete extended b-metric spaces, where ψ is an altering distance function.

2. THE MAIN RESULT

Let (X, d) be a b-metric space and $B(X)$ denotes to the class of nonempty bounded subsets of X. For $A, B \in B(X)$, let $D(A, B) = inf\{d(a, b) : a \in A, b \in B\}$, and $\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$.

Definition 2.1. *Let* (X, d) *be a b-metric space and* ψ *be an altering distance function. A mapping* $f : X \to B(X)$ *is said to be a multivalued generalized* $\psi - weak$ *contraction mapping if there are nonnegative real numbers* a, b *and* L *with* $a + \frac{b}{s} < 1, b < 1$ *and* $L \geq 0$ *such that*

$$\psi(\delta(fx, fy)) \leq a\psi(m_1(x, y)) + \frac{b}{s}\psi(M_{s1}(x, y)) + L\psi(N_1(x, y)) \tag{2.1}$$

for all $x, y \in X$, *where*

$$m_1(x, y) = D(y, fy)\frac{1 + D(x, fx)}{1 + d(x, y)},$$

$$M_{s1}(x, y) = \max\left\{d(x, y), D(x, fx), D(y, fy), \frac{D(x, fy) + D(y, fx)}{2s}\right\}$$

and

$$N_1(x, y) = \min\{D(x, fx), D(y, fx)\}.$$

Theorem 2.1. *Let* (X, d) *be a multivalued orbitally complete b-metric space, If* $f : X \to B(X)$ *is a multivalued generalized* ψ*-weak contraction mapping such that* $\psi(ax) \leq a\psi(x)$ *for all* $x \in X, a \geq 0$, *then* f *has a unique fixed point.*

Proof. Let $x_0 \in X$ and defined a sequences (x_n) in X such that $x_{n+1} \in fx_n$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n \in fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By inequality (2.1)., we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\leq \psi(\delta(fx_{n-1},f_{x_n}))\\ &\leq a\psi(m_1(x_{n-1},x_n))+\frac{b}{s}\psi(M_{s_1}(x_{n-1},x_n))\\ &\quad+L\psi(N_1(x_{n-1},x_n)),\end{aligned}\tag{2.2}$$

where

$$m_1(x_{n-1},x_n)=D(x_n,fx_n)\frac{1+D(x_{n-1},fx_{n-1})}{1+d(x_{n-1},x_n)}$$

$$\leq d(x_n,x_{n+1})\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}=d(x_n,x_{n+1}),\tag{2.3}$$

$$\begin{aligned}&M_{s_1}(x_{n-1},x_n)\\ &=\max\Big\{d(x_{n-1},x_n),D(x_{n-1},fx_{n-1}),D(x_n,fx_n),6\\ &\quad\frac{D(x_{n-1},fx_n)+D(x_n,fx_{n-1})}{2s}\Big\}\\ &\leq\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_{n+1})}{2s}\right\}\\ &\leq\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{s\,[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2s}\right\}\\ &=\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\}\end{aligned}\tag{2.1}$$

$$=\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},\tag{2.4}$$

and

$$N_1(x_{n-1},x_n)=\min\{D(x_{n-1},fx_{n-1}),D(x_n,fx_{n-1})\}$$

$$\leq\min\{d(x_{n-1},x_n),d(x_n,x_n)\}=0.\tag{2.5}$$

From (2.2)–(2.5) and the fact that ψ is an altering distance function, we get

$$\psi(d(x_n,x_{n+1}))\leq a\psi(d(x_n,x_{n+1}))+\frac{b}{s}\,\psi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}).\tag{2.6}$$

If

$$\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}=d(x_n,x_{n+1}),$$

then by (2.6) we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\leq a\psi(d(x_n,x_{n+1})) + \frac{b}{s}\,\psi(d(x_n,x_{n+1}))\\ &= \left(a+\frac{b}{s}\right)\psi(d(x_n,x_{n+1})) < \psi(d(x_n,x_{n+1})),\end{aligned}$$

a contradiction. Thus,

$$\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x_n).$$

Therefore (2.6) becomes

$$\psi(d(x_n,x_{n+1})) \leq a\psi(d(x_n,x_{n+1})) + \frac{b}{s}\psi(d(x_{n-1},x_n)),$$

this implies that,

$$\psi(d(x_n,x_{n+1})) \leq \frac{b}{s(1-a)}\,\psi(d(x_{n-1},x_n)). \tag{2.7}$$

Let $r = \frac{b}{s(1-a)}$. Then from (2.7), we have

$$\psi(d(x_n,x_{n+1})) \leq r\psi(d(x_{n-1},x_n)). \tag{2.8}$$

Repeating (2.8) n-times, we get

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\leq r\psi(d(x_{n-1},x_n))\\ &\leq r^2\psi(d(x_{n-2},x_{n-1}))\\ &\ \ \vdots\\ &\leq r^n\psi(d(x_0,x_1)). \qquad (2.9)\end{aligned}$$

Letting $n \to +\infty$ and using the fact that $r < 1$, we get

$$\lim_{n\to+\infty} \psi(d(x_n,x_{n+1})) = 0.$$

Since ψ is an altering distance function, we have

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0. \tag{2.10}$$

Next, we show that, (x_n) is a Cauchy sequence in X. Suppose to the contrary; that is, (x_n) is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences $(x_{m(i)})$ and $(x_{n(i)})$ of (x_n) such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad d(x_{m(i)},x_{n(i)}) \geq \epsilon. \tag{2.11}$$

This means that

$$d(x_{m(i)}, x_{n(i)-1}) < \epsilon. \tag{2.12}$$

From (2.11), (2.12) and (b3) inequality, we get

$$\begin{aligned}\epsilon \leq d(x_{m(i)}, x_{n(i)}) &\leq s\left[d(x_{m(i)}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{n(i)})\right] \\ &< s\epsilon + sd(x_{n(i)-1}, x_{n(i)}).\end{aligned}$$

Using (2.10) and letting $i \to +\infty$ in above inequalities, we get

$$\epsilon \leq \lim_{i\to+\infty} \sup d(x_{m(i)}, x_{n(i)}) < s\epsilon. \tag{2.13}$$

Again, by using (b3), we obtain that

$$d(x_{m(i)-1}, x_{n(i)}) \leq s\left[d(x_{m(i)-1}, x_{m(i)}) + d(x_{m(i)}, x_{n(i)})\right]. \tag{2.14}$$

Taking limit supremum as $i \to \infty$ in (2.14), from (2.10) and (2.13), we get

$$\lim_{i\to+\infty} \sup d(x_{m(i)-1}, x_{n(i)}) < s^2\epsilon. \tag{2.15}$$

Similarly, we can show that

$$\lim_{i\to+\infty} \sup d(x_{m(i)}, x_{n(i)-1}) < s^2\epsilon. \tag{2.16}$$

Finally, we obtain that

$$\begin{aligned}&d(x_{m(i)-1}, x_{n(i)-1}) \\ &\leq s\left[d(x_{m(i)-1}, x_{m(i)}) + d(x_{m(i)}, x_{n(i)-1})\right] \\ &\leq sd(x_{m(i)-1}, x_{m(i)}) + s^2\left[d(x_{m(i)}, x_{n(i)}) + d(x_{n(i)}, x_{n(i)-1})\right].\end{aligned} \tag{2.17}$$

Taking limit supremum as $i \to \infty$ in (2.17), from (2.10) and (2.13), we get

$$\lim_{i\to+\infty} \sup d(x_{m(i)-1}, x_{n(i)-1}) < s^3\epsilon.$$

From (2.1), we have

$$\begin{aligned}\psi(d(x_{m(i)}, x_{n(i)})) &\leq \psi(\delta(fx_{m(i)-1}, fx_{n(i)-1})) \\ &\leq a\psi(m_1(x_{m(i)-1}, x_{n(i)-1})) + \frac{b}{s}\psi(M_{s_1}(x_{m(i)-1}, x_{n(i)-1})) \\ &+ L\psi(N_1(x_{m(i)-1}, x_{n(i)-1})),\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}m_1(x_{m(i)-1}, x_{n(i)-1}) &= D(x_{n(i)-1}, fx_{n(i)-1})\frac{1+D(x_{m(i)-1}, fx_{m(i)-1})}{1+d(x_{m(i)-1}, x_{n(i)-1})}\\ &\leq d(x_{n(i)-1}, x_{n(i)})\frac{1+d(x_{m(i)-1}, x_{m(i)})}{1+d(x_{m(i)-1}, x_{n(i)-1})},\end{aligned} \tag{2.19}$$

$$\begin{aligned}&M_{s_1}(x_{m(i)-1}, x_{n(i)-1})\\ &= \max\Big\{d(x_{m(i)-1}, x_{n(i)-1}), D(x_{m(i)-1}, fx_{m(i)-1}), D(x_{n(i)-1}, fx_{n(i)-1}),\\ &\quad \frac{D(x_{m(i)-1}, fx_{n(i)-1})+D(fx_{m(i)-1}, x_{n(i)-1})}{2s}\Big\}\\ &\leq \max\Big\{d(x_{m(i)-1}, x_{n(i)-1}), d(x_{m(i)-1}, x_{m(i)}), d(x_{n(i)-1}, x_{n(i)}),\\ &\quad \frac{d(x_{m(i)-1}, x_{n(i)})+d(x_{m(i)}, x_{n(i)-1})}{2s}\Big\}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}N_1(x_{m(i)-1}, x_{n(i)-1}) &= \min\{D(x_{m(i)-1}, fx_{m(i)-1}), D(fx_{m(i)-1}, x_{n(i)-1})\}\\ &\leq \min\{d(x_{m(i)-1}, x_{m(i)}), d(x_{m(i)}, x_{n(i)-1})\}.\end{aligned} \tag{2.21}$$

Letting $i \to +\infty$ in (2.19)–(2.21). Then by using (2.10), (2.13), (2.15), (2.16) and the properties of ψ, we get

$$\psi(\lim_{i\to+\infty} m_1(x_{m(i)-1}, x_{n(i)-1})) = 0, \tag{2.22}$$

$$\psi(\lim_{i\to+\infty} \sup M_{s_1}(x_{m(i)-1}, x_{n(i)-1})) \leq \psi(s\epsilon) \tag{2.23}$$

and

$$\psi(\lim_{i\to+\infty} N_1(x_{m(i)-1}, x_{n(i)-1})) = 0. \tag{2.24}$$

Letting $i \to +\infty$ in (2.14). Then using (2.13) and (2.22)–(2.24) we have

$$\psi(\epsilon) \leq \frac{b}{s}\psi(s\epsilon) \leq b\psi(\epsilon) < \psi(\epsilon),$$

a contradiction. Thus $(x_0, x_1 \in fx_0, x_2 \in fx_1, \ldots)$ is a Cauchy sequence in X. Since X is multivalued orbitally complete b-metric space, there exists $u \in X$ such that

$$\lim_{i\to+\infty} x_n = u.$$

Now, we will show that $u \in fu$. By (2.1), we have

$$\begin{aligned} \psi(d(x_{n+1}, fu)) &\leq \psi(\delta(fx_n, fu)) \\ &\leq a\psi(m_1(x_n, u)) + \frac{b}{s}\psi(M_{s_1}(x_n, u)) + L\psi(N_1(x_n, u)), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} m_1(x_n, u) &= D(u, fu)\frac{1 + D(x_n, fx_n)}{1 + d(x_n, u)} \\ &\leq D(u, fu)\frac{1 + d(x_n, x_{n+1})}{1 + d(x_n, u)}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} M_{s_1}(x_n, u) &= \max\left\{d(x_n, u), D(x_n, fx_n), D(u, fu), \frac{D(x_n, fu) + D(fx_n, u)}{2s}\right\} \\ &\leq \max\left\{d(x_n, u), d(x_n, x_{n+1}), D(u, fu), \frac{D(x_n, fu) + d(x_{n+1}, u)}{2s}\right\}, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} N_1(x_n, u) &= \min\{D(x_n, fx_n), D(u, fx_n)\} \\ &\leq \min\{d(x_n, x_{n+1}), d(u, x_{n+1})\}. \end{aligned} \tag{2.28}$$

Letting $n \to +\infty$ in (2.26)–(2.28) and using the property of ψ,we get

$$\begin{aligned} \psi(\lim_{n\to+\infty} \sup m_1(x_n, u)) &\leq \psi(d(u, fu)), \\ \psi(\lim_{n\to+\infty} \sup M_1(x_n, u)) &\leq \psi(D(u, fu)) \end{aligned}$$

and

$$\psi(\lim_{n\to+\infty} N_1(x_n, u)) = 0.$$

Letting $n \to +\infty$ in (2.25). Then we get

$$\begin{aligned} \psi(\delta(u, fu)) &\leq a\psi(D(u, fu)) + \frac{b}{s}\psi(D(u, fu)) \\ &\leq \left(a + \frac{b}{s}\right)\psi(\delta(u, fu)). \end{aligned}$$

The above inequality happened only if $\delta(u, fu) = 0$. Thus $u \in fu$. So u is a fixed point of f. To prove the uniqueness of fixed point. Suppose that f has two fixed point u and v such that $u \neq v$. By (2.1), we have

$$\begin{aligned}
\psi(d(u,v)) &\leq \psi((\delta(fu,fv)) \leq a\psi(m_1(u,v)) + \frac{b}{s}\psi(M_{s_1}(u,v)) + L\psi(N_1(u,v)) \\
&= a\psi(D(u,fu)\frac{1+D(u,fu)}{1+d(u,v)}) \\
&\quad + \frac{b}{s}\psi\left(\max\left\{d(u,v), D(u,fu), D(v,fv), \frac{D(u,fv)+D(fu,v)}{2s}\right\}\right) \\
&\quad + L\psi(\min\{D(u,fu), D(v,fu)\}) \\
&= \frac{b}{s}\psi(d(u,v)) < \psi(d(u,v)), \qquad (2.29)
\end{aligned}$$

a contradiction. Thus $u = v$. Therefore f has a unique fixed point. □

Corollary 2.1. *Let (X,d) be a multivalued orbitally complete b-metric space, and let $f : X \to B(X)$ be a multivalued mapping. Suppose there exist $a \in [0,1)$ and $L \geq 0$ such that*

$$\psi(\delta(fx,fy)) \leq \frac{a}{s}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y))$$

for all $x,y \in X$, such that $\psi(\alpha x) \leq \alpha\psi(x)$ for all $x \in X$, $\alpha \geq 0$.Then f has a unique fixed point.

Corollary 2.2. *Let (X,d) be a multivalued orbitally complete $b-$metric space, and let $f : X \to B(X)$ be a multivalued mapping. Suppose there exist $a+\frac{b}{s} \in [0,1), b < 1$ and $L \geq 0$ such that*

$$\psi(\delta(fx,fy)) \leq a\psi(m_1(x,y)) + \frac{b}{s}\psi(M_{s1}(x,y))$$

for all $x,y \in X$, where $\psi(\alpha x) \leq \alpha\psi(x)$ for all $x \in X$, $\alpha \geq 0$. Then f has a unique fixed point.

Let $\psi = i$, the identity function, in Theorem 2.1 and Corollaries 2.1 and 2.2. Then we have the following results.

Corollary 2.3. *Let (X,d) be a multivalued orbitally complete $b-$metric space where and let $f : X \to B(X)$ be a multivalued mapping. Suppose there exist $a+\frac{b}{s} \in [0,1)$, $b < 1$ and $L \geq 0$ such that*

$$\delta(fx,fy) \leq am_1(x,y) + \frac{b}{s}M_{s1}(x,y) + LN_1(x,y)$$

for all $x,y \in X$. Then f has a unique fixed point.

Corollary 2.4. *Let (X, d) be a multivalued orbitally complete $b-$metric space where and let $f : X \to B(X)$ be a multivalued mapping. Suppose there exist $a \in [0, 1)$ and $L \geq 0$ such that*

$$\delta(fx, fy) \leq \frac{a}{s} M_{s1}(x, y) + LN_1(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point.

Corollary 2.5. *Let (X, d) be a multivalued orbitally complete $b-$metric space, where and let $f : X \to B(X)$ be a multivalued mapping. Suppose there exist $a + \frac{b}{s} \in [0, 1)$, $b < 1$ and $L \geq 0$ such that*

$$\delta(fx, fy) \leq am_1(x, y) + \frac{b}{s} M_{s1}(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point.

Let $f : X \to X$ be a single valued mapping. In the rest of this paper, we set

$$m(x, y) = d(y, fy) \frac{1 + d(x, fx)}{1 + d(x, y)},$$

$$M_s(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(y, fx) \}.$$

Now, we introduce the following definition.

Definition 2.2. *Let (X, d) be an extended b-metric space and ψ be an altering distance function. We say that a mapping $f : X \to X$ is a generalized ψ-weak contraction mapping if there are nonnegative real numbers a, b and L with $a + \frac{b}{s} < 1$, $b < 1$ and $L \geq 0$ such that*

$$\psi(d(fx, fy)) \leq a\psi(m(x, y)) + \frac{b}{s} \psi(M_s(x, y)) + L\psi(N(x, y))$$

for all $x, y \in X$.

As a consequence result of Theorem 2.1, we have the following result.

Corollary 2.6. *Let (X, d) be an orbitally complete extended b-metric space. If $f : X \to X$ is a generalized ψ-weak contraction mapping, such that $\psi(\alpha x) \leq \alpha\psi(x)$ for all $x \in X, \alpha \geq 0$, then f has a unique fixed point.*

Corollary 2.7. *Let* (X,d) *be an orbitally complete b-metric space, and let* $f : X \to X$ *be a mapping. Suppose there exist* $a \in [0,1)$ *and* $L \geq 0$ *such that*

$$\psi(d(fx,fy)) \leq \frac{a}{s}\psi(M_s(x,y)) + L\psi(N(x,y))$$

for all $x, y \in X$, *such that* $\psi(\alpha x) \leq \alpha\psi(x)$ *for all* $x \in X$, $\alpha \geq 0$. *Then* f *has a unique fixed point.*

Corollary 2.8. *Let* (X,d) *be an orbitally complete* $b-$*metric spaceand let* $f : X \to X$ *be a mapping. Suppose there exist* $a + \frac{b}{s} \in [0,1)$ *and* $L \geq 0$ *such that*

$$\psi(d(fx,fy)) \leq a\psi(m(x,y)) + \frac{b}{s}\psi(M_s(x,y))$$

for all $x, y \in X$, *such that* $\psi(\alpha x) \leq \alpha\psi(x)$ *for all* $x \in X$, $\alpha \geq 0$. *Then* f *has a unique fixed point.*

Let $\psi = i$, the identity function, in Corollaries (2.6)–(2.8). Then we have the following results.

Corollary 2.9. *Let* (X,d) *be an orbitally complete* $b-$*metric space, let* $f : X \to X$ *be a mapping. Suppose there exist* $a + \frac{b}{s} \in [0,1)$, $b < 1$ *and* $L \geq 0$ *such that*

$$d(fx,fy) \leq am(x,y) + \frac{b}{s}M_s(x,y) + LN(x,y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

Corollary 2.10. *2Let* (X,d) *be an orbitally complete* $b-$*metric space, let* $f : X \to X$ *be a mapping. Suppose there exist* $a \in [0,1)$ *and* $L \geq 0$ *such that*

$$d(fx,fy) \leq \frac{a}{s}M_s(x,y) + LN(x,y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

Corollary 2.11. *Let* (X,d) *be an orbitally complete* $b-$*metric space, let* $f : X \to X$ *be a mapping. Suppose there exist* $a + \frac{b}{s} \in [0,1)$, $b < 1$ *and* $L \geq 0$ *such that*

$$d(fx,fy) \leq am(x,y) + \frac{b}{s}M_s(x,y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

3. APPLICATIONS

Denote by Φ the set of functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

1. ϕ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$,
2. for every $x > 0$, we have $\int\limits_0^x \phi(t)dt > 0$,
3. $\phi(at) \leq a\phi(t)$.

Now, the maping $\psi : [0, +\infty) \to [0, +\infty)$ defined by

$$\psi(x) = \int\limits_0^x \phi(t)dt.$$

is an altering distance function with $\psi(ax) \leq a\psi(x)$.

Now, we have the following result.

Corollary 3.1. *Let (X, d) be a multivalued orbitally complete b-metric space and $f : X \to B(X)$ be a multivalued mapping. Suppose that there exist nonnegative real numbers a, b and L with $a + \frac{b}{s} < 1, b < 1,$ and $L \leq 0$ such that*

$$\int\limits_0^{\delta(fx,fy)} \phi(t)dt \leq a \int\limits_0^{m_1(x,y)} \phi(t)dt + \frac{b}{s} \int\limits_0^{M_{s_1}(x,y)} \phi(t)dt + L \int\limits_0^{N_1(x,y)} \phi(t)dt$$

for all $x, y \in X$. Then f has a unique fixed point.

Proof. Follows from Theorem 2.1 by taking $\psi(x) = \int\limits_0^x \phi(t)dt$. ☐

Corollary 3.2. *Let (X, d) be an orbitally complete b-metric space and let $f : X \to X$ be a given mapping. Suppose that there exist nonnegative real numbers a, b and L with $a + \frac{b}{s} < 1, b < 1,$ and $L \geq 0$ such that*

$$\int\limits_0^{d(fx,fy)} \phi(t)dt \leq a \int\limits_0^{m(x,y)} \phi(t)dt + \frac{b}{s} \int\limits_0^{M_s(x,y)} \phi(t)dt + L \int\limits_0^{N(x,y)} \phi(t)dt$$

for all $x, y \in X$. Then f has a unique fixed point.

Proof. Follows from Corollary 3.1 by noting that every single valued mapping can be considered as a multivalued mapping from X into $B(X)$. ☐

4. EXAMPLES

In this section, we present some examples to support the useability of our results.

Example 4. *Let* $X = [0,4]$. *Define* $f : X \to X$ *by:*

$$fx = \sinh^{-1}\frac{x}{c}, \qquad x \in [0,4], c > 2.$$

Let $d : X \times\ X \to [0,+\infty)$ *be given by*

$$d(x,y) = |x-y|^2$$

for all $x, y \in X$. *Then* (X,d) *is a complete* $b-$*metric space with coefficient* $s = 2$. *Define an altering distance functions* $\psi : [0,+\infty) \to [0,+\infty)$ *by* $\psi(t) = rt$, $r \geq 0$.

Proof. Let $x, y \in X$, then

$$M_s(x,y) = \max\left\{ |x-y|^2, |x - \sinh^{-1}\frac{x}{c}|^2, |y - \sinh^{-1}\frac{y}{c}|^2, \right.$$
$$\left. \frac{|x - \sinh^{-1}\frac{y}{c}|^2 + |y - \sinh^{-1}\frac{x}{c}|^2}{4} \right\}.$$

By using the mean value theorem simultaneously for the inverse hyperbolic sine function we get,

$$r\left|fx - fy\right|^2 = r\left|\sinh^{-1}\frac{x}{c} - \sinh^{-1}\frac{y}{c}\right|^2 \leq \frac{r}{c^2}\left|x-y\right|^2 \leq \frac{r}{c^2}M_s(x,y),$$

we have

$$\psi(d(fx,fy)) \leq \frac{\psi(M_s(x,y))}{c^2} + L\psi(N(x,y)) \text{ for any } L \geq 0.$$

So f satisfies all the hypotheses of Theorem 2.1 Therefore f has a unique fixed point. Here 0 is the unique fixed point of f. ☐

Example 5. *Let* $X = \{0,1,2,\ldots\}$. *Define* $f : X \to B(X)$ *by:*

$$fx = \begin{cases} \{1\}, & x = 0; \\ \{0\}, & x = 1; \\ \{x-1, x-2\}, & x \geq 2. \end{cases}$$

Let $d : X \times X \to \mathbb{R}^+$ *be given by*

$$d(x,y) = \begin{cases} 0, & x = y; \\ (x+y)^2, & x \neq y. \end{cases}$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ *by* $\psi(t) = te^t$. *Then*

1. (X, d) *is a complete b-metric space with* $s = 2$.
2. f *hase no fixed point.*
3. $\psi(at) > a\psi(t), \forall a > 1$.
4. $\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x, y)) + \psi(N_1(x, y))$ *for any* $x, y \in X$.

Proof. The proof of (1, 2, 3) are clear. The proof (4) is divided to the following cases:

case 1: If $x > y \geq 2$, then $fx = \{x - 1, x - 2\}$ and $fy = \{y - 1, y - 2\}$. Thus

$$\delta(fx, fy) = (x + y - 2)^2$$

and

$$M_{s1}(x, y) = \max\left\{(x + y)^2, (2x - 2)^2, (2y - 2)^2, \frac{(x + y - 2)^2}{4} \text{ or } \frac{(y + x - 2)^2}{2}\right\}.$$

Since

$$(x + y - 2)^2 e^{(x+y-2)^2} \leq (x + y)^2 e^{(x+y)^2 - 4(x+y)+4} \leq e^{-12}(x + y)^2 e^{(x+y)^2} \leq e^{-1}(x + y)^2 e^{(x+y)^2} \leq e^{-1} M_{s1}(x, y) e^{M_{s1}(x,y)},$$

we have

$$\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x, y)) + \psi(N_1(x, y)).$$

case 2: If $x \geq 2$ and $y = 1$, then $fx = \{x - 1, x - 2\}$ and $fy = 0$. Thus $\delta(fx, fy) = (x - 1)^2$ and

$$M_{s1}(x, y) = \max\left\{(x + 1)^2, (2x - 2)^2, 1, \frac{x^2 + (x - 1)^2}{4} or \frac{x^2}{4}\right\}.$$

Since

$$(x - 1)^2 e^{(x-1)^2} \leq (x + 1)^2 e^{x^2 - 3} \leq e^{-3}(x + 1)^2 e^{x^2} \leq e^{-3}(x + 1)^2 e^{(x+1)^2} \leq e^{-1} M_{s1}(x, y) e^{M_{s1}(x,y)},$$

we have

$$\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x, y)) + \psi(N_1(x, y)).$$

case 3: (i) If $x > 2$ and $y = 0$, then $fx = \{x-1, x-2\}$ and $fy = 1$. Thus, $\delta(fx, fy) = (x)^2$ and

$$M_{s1}(x,y) = \max\left\{x^2, 4(x-1)^2, 1, \frac{(x+1)^2 + (x-2)^2}{4} \text{ or } \frac{(x+1)^2}{4}\right\}.$$

Since

$$\begin{aligned} x^2 e^{x^2} &\leq x^2 e^{4(x-1)^2 - 1} \leq 4(x-1)^2 e^{-1} e^{4(x-1)^2} \\ &\leq e^{-1} M_{s1}(x,y) e^{M_{s1}(x,y)}, \end{aligned}$$

we have

$$\psi(\delta(fx, fy)) \leq \frac{2e^{-3}}{2}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y)).$$

(ii) If $x = 2$ and $y = 0$, then $fx = \{1, 0\}$ and $fy = 1$. Thus, $\delta(fx, fy) = 1$ and

$$M_{s1}(x,y) = \max\left\{4, 4, 1, \frac{9}{4}\right\} = 4.$$

Since

$$e \leq 4e^{-3}e^4 \leq 4e^{-1}e^4 \leq e^{-1} M_{s1}(x,y) e^{M_{s1}(x,y)},$$

we have

$$\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x,y)) + \psi(N_1(x,y)).$$

case 4: If $x = y \geq 2$, then $fx = fy = \{x-1, x-2\}$. Thus $\delta(fx, fx) = (2x-3)^2$ and

$$M_{s1}(x,y) = \max\left\{0, (2x-2)^2, \frac{(2x-2)^2}{2}\right\} = (2x-2)^2$$

Since

$$\begin{aligned} (2x-3)^2 e^{(2x-3)^2} &\leq (2x-2)^2 e^{((2x-2)-1)^2} \leq e^{-3}(2x-2)^2 e^{(2x-2)^2} \\ &\leq e^{-1} M_{s1}(x,y) e^{M_{s1}(x,y)}, \end{aligned}$$

we have

$$\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x,y)) + \psi(N_1(x,y)).$$

case 5: If $x, y \in \{0, 1\}$

(i) $x = y$, then $fx = fy$. Thus

$$\delta(fx, fy) = 0.$$

Hence

$$\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x, y)) + \psi(N_1(x, y)).$$

(ii) $x = 1$ and $y = 0$ then $fx = 0$ and $fy = 1$. Thus

$$\delta(fx, fy) = 1 = M_{s1}(x, y)$$

and

$$N_1(x, y) = \min\{D(x, fx), D(y, fy)\} = 1.$$

Since

$$e \leq 1 + e,$$

we have

$$\psi(\delta(fx, fy)) \leq e^{-1}\psi(M_{s1}(x, y)) + \psi(N_1(x, y)).$$

So f satisfies all the hypotheses of Theorem 2.1 except the condition $\psi(at) \leq a\psi(t)$ for all $a \geq 0$, this means that this condition is necessary for existence of a fixed point of f. □

Example 6. *Let* $X = \{0, 1, 2, \ldots\}$. *Define* $f : X \to B(X)$ *by:*

$$fx = \begin{cases} \{0\}, & x \in \{0, 1\}; \\ \{x - 1, x - 2\}, & x \geq 2. \end{cases}$$

Let $d : X \times X \to \mathbb{R}^+$ *be given by*

$$d(x, y) = \begin{cases} 0, & x = y; \\ (x + y)^2, & x \neq y. \end{cases}$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ *by* $\psi(t) = te^t$. *Then*

1. (X, d) *is a complete b-metric space with* $s = 2$.
2. $\psi(\delta(fx, fy)) \leq e^{-3}\psi(M_{s1}(x, y)) + L\psi(N_1(x, y))$*for any* $x, y \in X$ *and any* $L \geq 0$.

Proof. The proof of (1) is clear. The proof (2) is divided to the following cases:

case 1: If $x > y \geq 2$, then $fx = \{x-1, x-2\}$ and $fy = \{y-1, y-2\}$. Thus

$$\delta(fx, fy) = (x+y-2)^2$$

and

$$M_{s1}(x,y) = \max\left\{(x+y)^2, (2x-2)^2, (2y-2)^2,\right.$$
$$\left.\frac{(x+y-2)^2}{4}\text{or } (y+x-2)^2\right\}$$

Since

$$(x+y-2)^2 e^{(x+y-2)^2} \leq (x+y)^2 e^{(x+y)^2-4(x+y)+4} \leq$$
$$e^{-12}(x+y)^2 e^{(x+y)^2} \leq e^{-3}(x+y)^2 e^{(x+y)^2} \leq \frac{2e^{-3}}{2} M_{s1}(x,y) e^{M_{s1}(x,y)};$$

we have

$$\psi(\delta(fx, fy)) \leq \frac{2e^{-3}}{2}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y)).$$

case 2: If $x \geq 2$ and $y = 1$, then $fx = \{x-1, x-2\}$ and $fy = 0$. Thus $\delta(fx, fy) = (x-1)^2$ and

$$M_{s1}(x,y) = \max\left\{(x+1)^2, (2x-2)^2, 1, \frac{x^2+(x-1)^2}{4} or \ \frac{x^2}{4}\right\}.$$

Since

$$(x-1)^2 e^{(x-1)^2} \leq (x+1)^2 e^{x^2-3} \leq e^{-3}(x+1)^2 e^{x^2}$$
$$\leq e^{-3}(x+1)^2 e^{(x+1)^2} \leq e^{-3} M_{s1}(x,y) e^{M_{s1}(x,y)},$$

we have

$$\psi(\delta(fx, fy)) \leq \frac{2e^{-3}}{2}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y)).$$

case 3: If $x \geq 2$ and $y = 0$, then $fx = \{x-1, x-2\}$ and $fy = 0$. Thus, $\delta(fx, fy) = (x-1)^2$ and

$$M_{s1}(x,y) = \max\left\{x^2, (2x-2)^2, 0, \frac{x^2+(x-2)^2}{4} or \ \frac{x^2}{4}\right\}.$$

Since

$$(x-1)^2 e^{(x-1)^2} \leq x^2 e^{(x-1)^2} = x^2 e^{x^2-2x+1} \leq e^{-3} x^2 e^{x^2}$$
$$\leq e^{-3} M_{s1}(x,y) e^{M_{s1}(x,y)},$$

we have

$$\psi(\delta(fx,fy)) \leq \frac{2e^{-3}}{2}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y)).$$

case 4: If $x = y \geq 2$, then $fx = fy = \{x-1, x-2\}$. Thus $\delta(fx,fx) = (2x-3)^2$ and

$$M_{s1}(x,y) = \max\left\{0, (2x-2)^2, \frac{(2x-2)^2}{2}\right\}$$

Since

$$\begin{aligned}(2x-3)^2e^{(2x-3)^2} &\leq (2x-2)^2e^{((2x-2)-1)^2} \leq e^{-3}(2x-2)^2e^{(2x-2)^2}\\ &\leq \frac{2e^{-3}}{2}M_{s1}(x,y)e^{M_{s1}(x,y)},\end{aligned}$$

we have

$$\psi(\delta(fx,fy)) \leq \frac{2e^{-3}}{2}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y)).$$

case 5: If $x, y \in \{0,1\}$, then $fx = fy = 0$. Thus $\delta(fx,fy) = 0$. Hence

$$\psi(\delta(fx,fy)) \leq \frac{2e^{-3}}{2}\psi(M_{s1}(x,y)) + L\psi(N_1(x,y)).$$

Clear, f has a unique fixed point although f does not satisfiy all the hypotheses of Theorem 2.1. ☐

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