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DYNAMICAL ANALYSIS OF FRACTIONAL ORDER ZIKV MODEL

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ABSTRACT. The ZIKV model presented in this article is developed by modifying [1]'s model. The classical order is changed into fractional order model. The equilibrium points of the model are determined and the stability conditions of each equilibrium point have been done using Routh-Hurwitz conditions. Numerical simulation is presented to verify the result of stability analysis result. Numerical simulation is also used to shows the effect of the order α to the stability of the model's equilibrium point.

1. INTRODUCTION

ZIKV is a member of Flaviviridae family, family of pathogenic viruses as causes of Dengue fever, Yellow fever, Japanese encephalitis, and West Nile fever. Main vector of ZIKV is the *Aedes aegypti* and *Aedes albopictus* [2]. ZIKV infection causes Zika fever with mild symptoms such as fever, red eyes, joint pain, headache, and rash on the skin that will be felt 2 to 7 days after incubation [3]. ZIKV can be transferred through *Aedes aegypti* and *Aedes albopictus* bite, sexual contact, blood transfusions, and can be transferred from mother to fetus in her womb during pregnancy.

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Mathematical models are used widely to represent the transmission of disease in a particular area. Various mathematical epidemic models which describe the transmission of ZIKV have been discussed in [1–9]. Most of ZIKV mathematical model is constructed by considering transmission routes between human and mosquito. One of simple ZIKV mathematical model in [1] is given by

$$\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h - (1-\mu_1)\,\beta_h S_h\,(I_v + \delta I_h) - \mu_h S_h,\\
\frac{dI_h}{dt} &= (1-\mu_1)\,\beta_h S_h\,(I_v + \delta I_h) - (\mu_h + \gamma + \eta_h\mu_2)\,I_h,\\
\frac{dR_h}{dt} &= (\gamma + \eta_h\mu_2)\,I_h - \mu_h R_h,\\
\frac{dS_v}{dt} &= \Lambda_v - (1-\mu_1)\,\beta_v S_v I_h - (\mu_v + \eta_v\mu_3)\,S_v,\\
\frac{dI_v}{dt} &= (1-\mu_1)\,\beta_v S_v I_h - (\mu_v + \eta_v\mu_3)\,I_v.
\end{aligned} \tag{1.1}$$

S_h, I_h, R_h, S_v, and I_v represent the number of suspectible human, infected human, recovered human, suspectible mosquito, and infected mosquito respectively. Λ_h and Λ_v is the recruitment rate of susceptible human and susceptible mosquito, μ_h and μ_v is the natural mortality rates for human and mosquito, β_h is the rate of transmission from human to mosquito, β_v is the rate of transmission from mosquito to human, η_h is the rate of recovery of human from infection with treatment, γ is the rate of recovery, μ_1, μ_2, and μ_3 are control parameters.

In this article, the ZIKV model is developed by modifying model in [1]. The order of ordinary differential equation fractional is changed into fractional order as demonstrated in [8] to considering memory effect. In addition, we eliminate control parameters from the model. The dynamical analysis of the fractional order have been done by determining the equilibrium point and analyzing the local stability of equilibrium points. Numerical simulation using Predictor-Corrector method for fractional order equation developed in [10] is presented to verify the result of stability analysis.

2. Preliminaries

In fractional calculus, order of derivatives and integrals α notated by $D_t^\alpha = \frac{d^\alpha}{dt^\alpha}$. In general α is any whole number, fraction, or any complex number

$\alpha = p + iq$, with $p, q \in \mathbb{R}$. Calculus Fractional is the development of calculus which includes fractional derivatives and fractional integrals. Compared to classical derivatives, fractional derivatives are considered to be able describe real system better. The memory effect on fractional derivatives means that the solution of the fractional derivative depends on the entire function value from lower bound to upper bound. The following definition is definition of Caputo fractional derivative [11].

Definition 2.1. *Suppose $\alpha \in \mathbb{R}^+$ and $n = \min\{z \in \mathbb{Z} : z \geq \alpha\}$. Caputo fractional derivative of $y(t)$ in order α, $n-1 < \alpha < n$ on $t > a$ is defined as*

$$^{C}D_a^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\xi)^{n-\alpha-1} y^{(n)}(\xi) d\xi.$$

Γ is Gamma function. The fractional differential equation (FDE) is an equation that contains the fractional derivative defined in $\alpha \in \mathbb{R}^+$ [12].

Definition 2.2. *Equation*

$$^{C}D_t^{\alpha} y(t) = f\left(t, y(t)\right)$$

referred as Caputo FDE, initial condition is written below,

$$^{C}D_t^{k} y(0) = b_k \quad k = 1, 2, \ldots, n-1. \tag{2.1}$$

The stability of the nonlinear system of FDE's equilibrium point $y^* = (y_1^*,\ y_2^*,\ \ldots, y_n^*)$ can be determined using Theorem 2.1 (See [13]).

Theorem 2.1. *The equilibrium point $y^* = (y_1^*,\ y_2^*,\ \ldots,\ y_n^*)$ locally asymptotically stable if any eigenvalues $\lambda_i, \quad i = 1, 2, \ldots n$ of the Jacobi matrix $J = \frac{\partial f_i}{\partial y}$ which is evaluated at equilibrium point satisfies*

$$\left|\arg\left(\lambda_i\right)\right| > \frac{\alpha\pi}{2}. \tag{2.2}$$

Theorem 2.2. *Let the characteristics equation of the Jacobian matrix evaluated at the equlibrium points is given by $P(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n = 0$. The polynomial equation $P(\lambda)$ has n roots that satisfy* (2.2) *if and only if the Routh-Hurwitz conditions for fractional order system are satisfied [14].*

3. Main Result

In this article, ZIKV model is presented as system of fractional differential equation. The model is the result of modifying ordinary differential equation model presented in [1] into fractional order and ignoring the control parameters. The modified model can be formulated as follows:

$$\begin{aligned}
{}^{C}D_t^{\alpha} S_h &= \Lambda_h - \beta_h S_h \left(I_v + \delta I_h\right) - \mu_h S_h, \\
{}^{C}D_t^{\alpha} I_h &= \beta_h S_h \left(I_v + \delta I_h\right) - \left(\mu_h + \gamma + \eta_h\right) I_h, \\
{}^{C}D_t^{\alpha} R_h &= \left(\gamma + \eta_h\right) I_h - \mu_h R_h, \\
{}^{C}D_t^{\alpha} S_v &= \Lambda_v - \beta_v S_v I_h - \mu_v S_v, \\
{}^{C}D_t^{\alpha} I_v &= \beta_v S_v I_h - \mu_v I_v.
\end{aligned} \tag{3.1}$$

The equilibrium points of (3.1) is obtained when

$$D_t^{\alpha} S_h = D_t^{\alpha} I_h = D_t^{\alpha} R_h = D_t^{\alpha} S_v = D_t^{\alpha} I_v = 0,$$

from system (3.1), we have

$$S_h = \frac{\Lambda_h \mu_v \left(\beta_v I_h + \mu_v\right)}{\beta_h \beta_v \Lambda_v I_h + \left(\beta_h \delta I_h + \mu_h\right)\left(\mu_v \left(\beta_v I_h + \mu_v\right)\right)}, \tag{3.2}$$

$$R_h = \frac{\left(\gamma + \eta_h\right) I_h}{\mu_h}, \tag{3.3}$$

$$S_v = \frac{\Lambda_v}{\left(\beta_v I_h + \mu_v\right)}, \tag{3.4}$$

$$I_v = \frac{\beta_v \Lambda_v I_h}{\mu_v \left(\beta_v I_h + \mu_v\right)}, \tag{3.5}$$

and

$$\begin{aligned}
I_h^3 \left(-A_4 \mu_v \beta_h \beta_v \delta\right) + I_h^2 \left(\mu_v \beta_h \beta_v \delta \Lambda_h - A_4 \beta_h \beta_v \Lambda_v - A_4 \mu_h \mu_v \beta_v - A_4 \mu_v^2 \beta_h \delta\right) \\
+ I_h \left(\beta_h \beta_v \Lambda_h \Lambda_v + \mu_v^2 \beta_h \delta \Lambda_h - A_4 \mu_h \mu_v^2\right) = 0.
\end{aligned}$$

with $A_4 = \mu_h + \gamma + \eta_h$. If $I_h = 0$, obtained the first equilibrium point E_0 and called as disease-free equilibrium (DFE) written below:

$$E_0 = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, \frac{\Lambda_v}{\mu_v}, 0\right).$$

If $I_h \neq 0$, then I_h satisfy the quadratic equation below:

$$A I_h^2 + B I_h + C = 0, \tag{3.6}$$

with

$$\begin{aligned} A &= -A_4\mu_v\beta_h\beta_v\delta, \\ B &= \mu_v\beta_h\beta_v\delta\Lambda_h - A_4\beta_h\beta_v\Lambda_v - A_4\mu_h\mu_v\beta_v - A_4\mu_v^2\beta_h\delta, \\ C &= \beta_h\beta_v\Lambda_h\Lambda_v + \mu_v^2\beta_h\delta\Lambda_h - A_4\mu_h\mu_v^2. \end{aligned}$$

Equation (3.6) has real and unique positive root if only if $C < 0$. The real and unique positive root of equation (3.6) is referred as point I_h^*. Substitute point I_h^* into equations (3.2), (3.3), (3.4) and (3.5), then obtained the second equilibrium point or referred as endemic equilibrium (EE) written as $E_1 = (S_h^*, I_h^*, R_h^*, S_v^*, I_v^*)$, with

$$\begin{aligned} S_h^* &= \frac{\Lambda_h\mu_v\left(\beta_v I_h^* + \mu_v\right)}{\beta_h\beta_v\Lambda_v I_h^* + \left(\beta_h\delta I_h^* + \mu_h\right)\left(\mu_v\left(\beta_v I_h^* + \mu_v\right)\right)}, \\ R_h^* &= \frac{\left(\gamma + \eta_h\right) I_h^*}{\mu_h}, \\ S_v^* &= \frac{\Lambda_v}{\left(\beta_v I_h^* + \mu_v\right)}, \\ I_v^* &= \frac{\beta_v\Lambda_v I_h^*}{\mu_v\left(\beta_v I_h^* + \mu_v\right)}. \end{aligned}$$

To analyze the stability of the equilibrium point, we perform linearization in the form of a Jacobi matrix at point $\hat{E} = \left(\hat{S}_h, \hat{I}_h, \hat{R}_h, \hat{S}_v, \hat{I}_v\right)$, written as:

(3.7)

$$J = \begin{bmatrix} -\beta_h\left(\hat{I}_h + \delta\hat{I}_h\right) - \mu_h & -\beta_h\delta\hat{S}_h & 0 & 0 & -\beta_h\hat{S}_h \\ \beta_h\left(\hat{I}_v + \delta\hat{I}_h\right) & \beta_h\delta\hat{S}_h - (\mu_h + \gamma + \eta_h) & 0 & 0 & \beta_h\hat{S}_h \\ 0 & \gamma + \eta_h & -\mu_h & 0 & 0 \\ 0 & -\beta_v\hat{S}_v & 0 & -\beta_v\hat{I}_h - \mu_v & 0 \\ 0 & \beta_v\hat{S}_v & 0 & \beta_v\hat{I}_h & -\mu_v \end{bmatrix}.$$

The Jacobi matrix (3.7) is evaluated at E_0, obtained the following charasteristic equation:

$$(-\mu_h - \lambda)^2\left(-\mu_v - \lambda\right)\left(\lambda^2 + d_1\lambda + d_2 = 0\right) = 0, \tag{3.8}$$

with

$$\begin{aligned} d_1 &= -A_2 + A_4 + \mu_v, \\ d_2 &= \mu_v(-A_2 + A_4) - A_1A_3. \end{aligned}$$

Based on equation (3.8), we have 5 eigenvalues, i.e. $\lambda_{1,2} = -\mu_h,\ \lambda_3 = -\mu_v$. It is clear that $\lambda_j < 0,$ for $j = 1, 2, 3$ and $\arg(\lambda_j) = \pi$ so that $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$. Eigenvalues λ_4 and λ_5 are the roots of quadratic equation below:

$$\lambda_{4,5} = \frac{-d_1 \pm \sqrt{\triangle_1}}{2},$$

which $\triangle_1 = d_1^2 - 4d_2$. Disease-free equilibrium point E_0 is locally asymptotically stable if and only if it satisfy any of the following condition:

(i) If $d_1 > 0$ and satisfy one of the following conditions,
  (1) $\triangle_1 = 0$, or
  (2) $\triangle_1 > 0$, and $d_2 > 0$, or
  (3) $\triangle_1 < 0.$

(ii) If $d_1 < 0$ and $|\arg(\lambda_{4,5})| > \frac{\alpha\pi}{2}$.

Next to determine the stability of E_1, Jacobi matrix (3.7) is evaluated at E_1, so that it is obtained the characteristic equation below:

$$(\lambda + \mu_v)(\lambda + \mu_h)\left(\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3\right) = 0, \tag{3.9}$$

with

$$\begin{aligned}
k_1 &= B_{11} - B_{12} + B_{44} + \mu_h + \mu_v + A_4,\\
k_2 &= B_{11}B_{44} + B_{11}\mu_v + B_{11}A_4 - B_{12}B_{44} - B_{12}\mu_h - B_{12}\mu_v - 2 * B_{15}B_{42}\\
&\quad + B_{44}\mu_h + B_{44}A_4 + \mu_h\mu_v + \mu_h A_4 + \mu_v A_4, \text{ and}\\
k_3 &= B_{11}B_{15}B_{42} + B_{11}B_{44}A_4 + B_{11}\mu_v A_4 - B_{12}B_{44}\mu_h - B_{12}\mu_h\mu_v - B_{15}B_{42}B_{44}\\
&\quad - 2B_{15}B_{42}\mu_h + B_{44}\mu_h A_4 + \mu_h\mu_v A_4.
\end{aligned}$$

Based on equation (3.9) obtained $\lambda_1 = -\mu_h$ and $\lambda_2 = -\mu_v$. Its clear that $\lambda_J < 0,$ for $J = 1, 2$ and $\arg(\lambda_J) = \pi$ so that it satisfy stability condition $|\arg(\lambda_J)| > \frac{\alpha\pi}{2}$. $\lambda_{3,4,5}$ are the roots from the cubic equation which has a discriminant written as follows:

$$\begin{aligned}
\triangle_2 &= -\begin{vmatrix} 1 & k_1 & k_2 & k_3 & 0\\ 0 & 1 & k_1 & k_2 & k_3\\ 3 & 2k_1 & k_2 & 0 & 0\\ 0 & 3 & 2k_1 & k_2 & 0\\ 0 & 0 & 3 & 2k_1 & k_2 \end{vmatrix},\\
&= 18k_1k_2k_3 + (k_1k_2)^2 - 4k_3k_1^3 - 4k_2^3 - 27k_3^2.
\end{aligned}$$

Based on Routh-Hurwitz condition, equilibrium point E_1 is locally asymptotically stable if and only if one of the following condition is satisfied:

(i) If $\triangle_2 > 0$ and satisfy the following condition,
  (1) $k_1 > 0$,
  (2) $k_3 > 0$, and
  (3) $k_1 k_2 > k_3$.

(ii) If $\triangle_2 < 0$, $\alpha < \frac{2}{3}$, and satisfy the following condition.
  (1) $k_1 \geq 0$,
  (2) $k_2 \geq 0$, and
  (3) $k_3 > 0$.

(iii) If $\triangle_2 < 0$, $\alpha > \frac{2}{3}$, and satisfy the following condition.
  (1) $k_1 < 0$ and
  (2) $k_2 < 0$.

(iv) $\triangle_2 < 0$ and satisfy the following condition.
  (1) $k_1 < 0$,
  (2) $k_2 < 0$, and
  (3) $k_1 k_2 = k_3$.

4. NUMERICAL SIMULATION

The numerical simulation have been done using Predictor-Corrector method, the parameter values used are presented in Table 1 below:

TABLE 1. Parameter value

Parameter	Paremeter value	Source
Λ_h	10	Assumption
Λ_v	100	Assumption
δ	0.05	[9]
μ_h	$\frac{1}{365\times 60}$	[1]
μ_v	$\frac{1}{14}$	[1]
η_h	0.01	[1]
γ	0.05	[15]

(i) Simulation 1

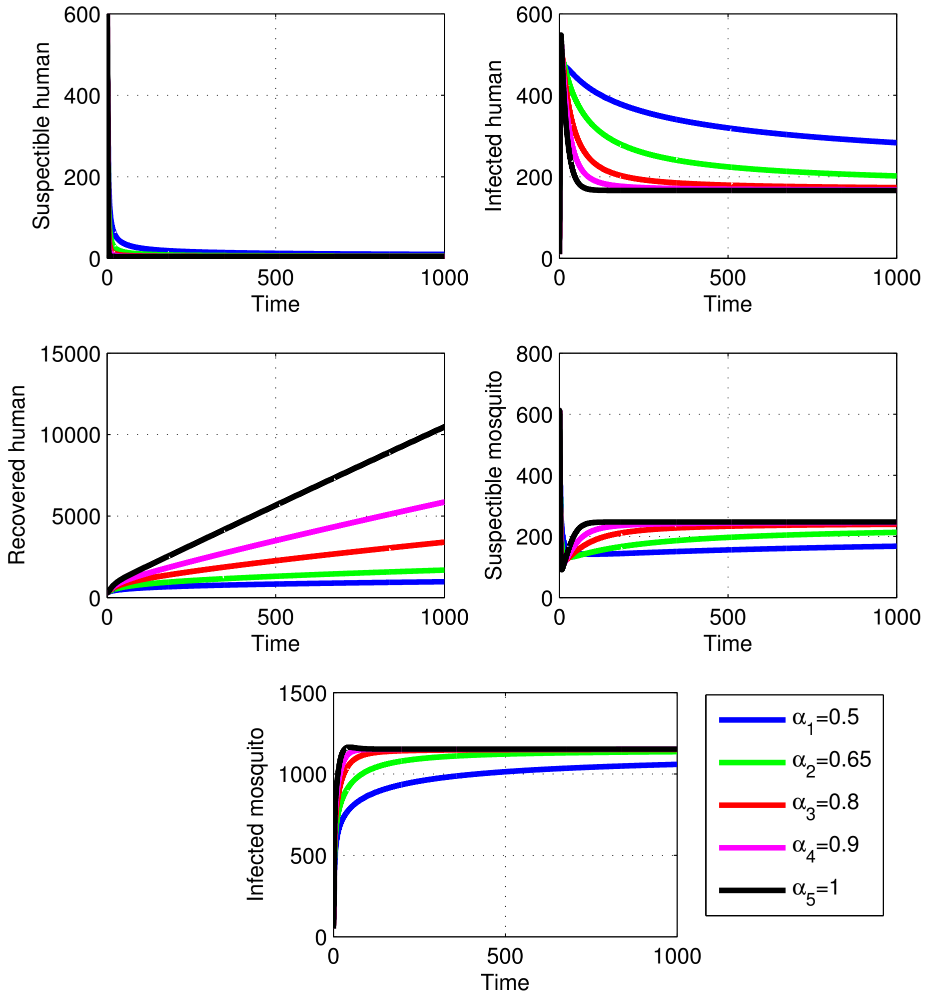

FIGURE 1. The growth of (a) suspectible human, (b) infected human, (c) recovered human, (d) suspectible mosquito, and (e) infected mosquito in Simulation 1.

In the first simulation, selected parameters value are $\beta_h = 2 \times 10^{-7}$, $\beta_v = 2 \times 10^{-8}$. Selected parameters value satisfy the local stability of E_0. The results of the simulation verify the results of stability analysis. The stability of E_0 is not affected by α value. The α value affects the speed of convergence of the system to E_0. The system converge faster to E_0 as the closer α value to 1.

(ii) Simulation 2

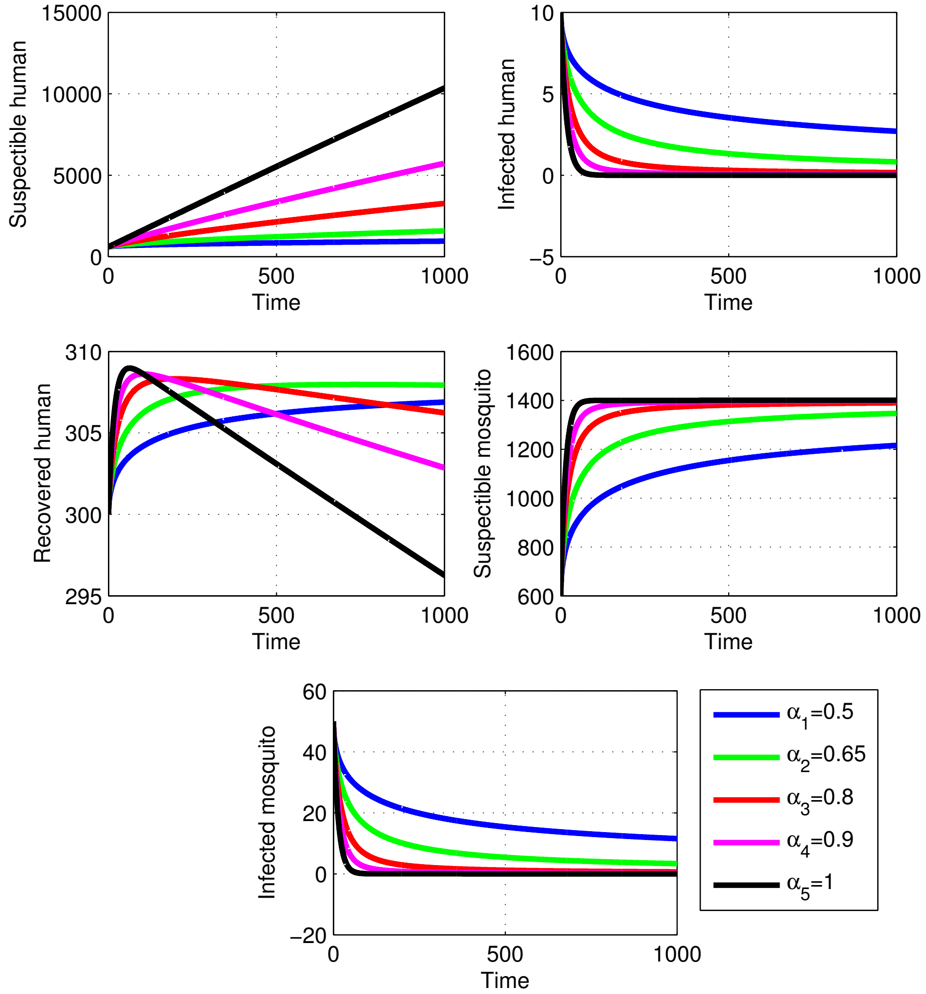

FIGURE 2. The growth of (a) suspectible human, (b) infected human, (c) recovered human, (d) suspectible mosquito, and (e) infected mosquito in Simulation 2.

The selected parameter value are presented in Table 1 and $\beta_h = 2 \times 10^{-3}$ and $\beta_v = 2\times10^{-3}$. The first local stability condition of E_1 is satisfied and those result are supported by the simulation result in Figure 2. The stability of E_1 is not affected by α value. The α value affects the speed of convergence of the system to E_1. The system converge faster to E_1 as the closer α value to 1.

(iii) Simulation 3

Parameters value used in the third simulation are same as second simulation, except $\gamma = 5$ and $\eta_h = 5$. Based on second stability condition of E_1, E_1 is locally asymptotically stable if and only if $\alpha < \frac{2}{3}$. The simulation results of infected human and infected mosquito using various α value are presented in Figure 3 and 4. As shown in Figure 3 and 4, E_1 stable when $\alpha < \frac{2}{3}$ and unstable when $\alpha > \frac{2}{3}$. The simulation result with $\alpha = 0.5 < \frac{2}{3}$ is presented in Figure 5. In Figure 5, E_1 is locally asimptotically stable as the second stability condition is satisfied.

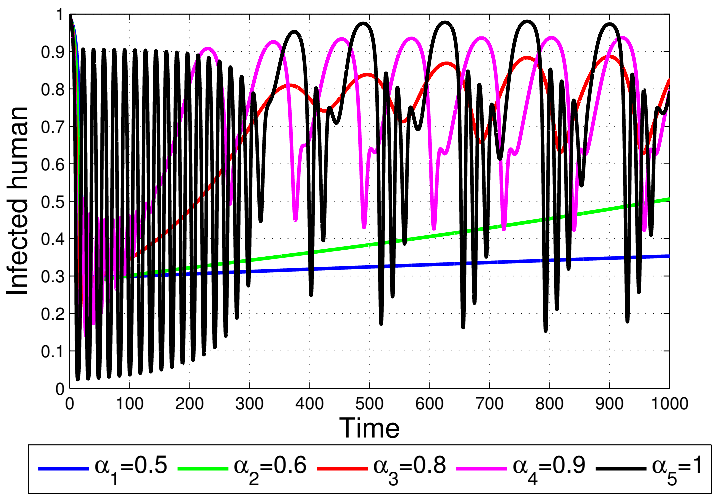

FIGURE 3. The growth of the infected human in Simulation 3.

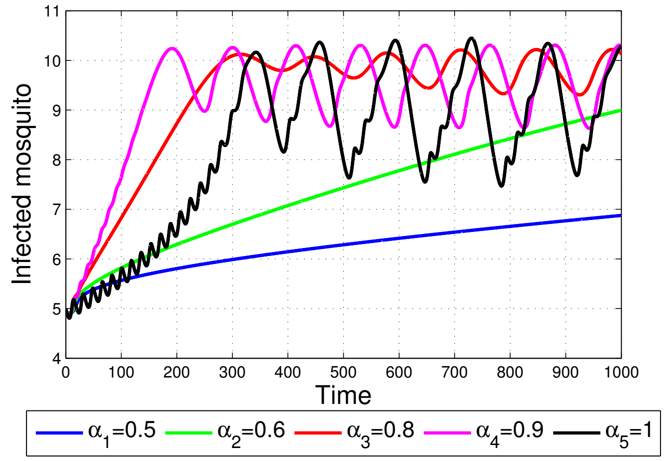

FIGURE 4. The growth of the infected mosquito in Simulation 3.

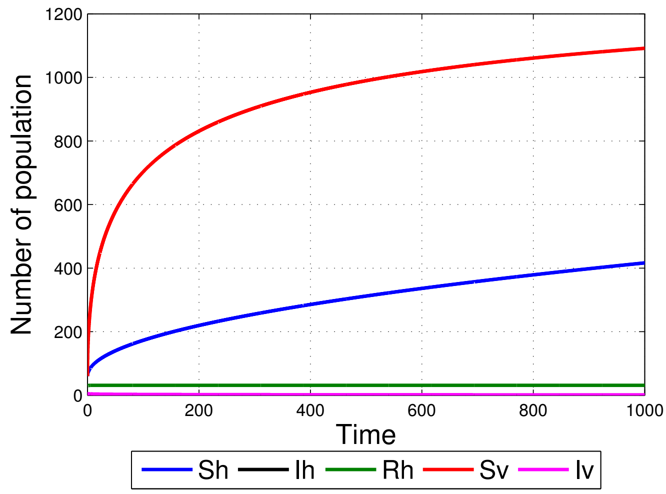

FIGURE 5. The growth of the suspectible human, infected human, recovered human, suspectible mosquito, and infected mosquito in Simulation 3.

5. CONCLUSION

The ZIKV model presented as system of fractional differential equations which is formulated by considering two populations and divided into 5 compartments, namely suspectible human (S_h), infected human (I_h), recovered humand (R_h), susceptible mosquito (S_v), and infected mosquito (I_v). The model has two points of equilibrium, i.e. the disease free equilibrium (E_0) and endemic equilibrium (E_1). Both of the equilibrium points are locally asymptotically stable under different certain conditions. Numerical simulation verify the stability analysis result of each equilibrium point and show the effect of α to equilibrium point's stability. The stability of E_0 is not affected by α, and the stability of E_1 can be affected by the order size α depend on the which conditions are satisfied by the parameter values used.

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