Advances in Mathematics: Scientific Journal **10** (2021), no.5, 2483–2489
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.5.14

ON SOME LOGARITHMIC INEQUALITIES

Neha Elizabeth Thomas[1], Arya Chandran, and K. Vishnu Namboothiri

ABSTRACT. We establish some inequalities involving $\log(1+x)$ using elementary techniques. Using these inequalities, we show an alternate approach to evaluate the integral $\int_1^\infty \frac{\log t}{t^2}\,dt$. This integral is later used to evaluate the asymptotic value of a logarithmic sum.

1. INTRODUCTION

There are many inequalities involving the natural logarithm function, which are useful in many areas of science and engineering in addition to being interesting in its own in pure Mathematics. The very basic inequality

$$\log(1+x) \leq x, \quad x \geq 0,$$

is such an example. Somewhat more useful is the inequality

$$1 - \frac{1}{x} \leq \log x \leq x - 1 \text{ for } x > 0.$$

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 26D15, 26D99.
Key words and phrases. Logarithmic inequalities, Improper integrals, Euler Summation Formula.
Submitted: 17.03.2021; *Accepted:* 05.04.2021; *Published:* 07.05.2021.

Deriving and using such bounds have been an active area of interest for mathematicians. For example, a number of useful logarithmic ineqalities were derived by F. Topsøe in [4]. Two of them stated there without proof are the following:

$$\frac{2x}{2+x} \leq \log(1+x) \leq \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad x \geq 0, \tag{1.1}$$

$$\frac{2x}{2+x} \geq \log(1+x) \leq \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad -1 \leq x \leq 0. \tag{1.2}$$

In [3], C. Chesneau *et. all* established a upper bound for $\log(1+x)$ for $x \geq 0$ involving arctan function as follows:

$$\log(1+x) \leq \frac{f(x)}{\sqrt{x+1}},$$

$$\text{where } f(x) = \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\text{arctan}\left(\sqrt{x+1}\right).$$

In connection with the above, M. Kostic [2] proved the following:

Theorem 1.1. *There is no rational real function which intermediates the functions* $\log(1+x)$ *and* $\frac{f(x)}{\sqrt{x+1}}$ *for* $x \in (-1, 0]$*, where* $f(x) = \pi + \frac{1}{2}(4+\pi)x - 2(x+2)\arctan\left(\sqrt{x+1}\right)$.

In this paper, we provide a proof of the second part of both the inequalities (1.1) and (1.2) claimed to be true by F. Topsøe in [4]. Precisely, we establish the following two inequalities:

$$\log(1+x) \geq \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad -1 \leq x \leq 0,$$

$$\log(1+x) \leq \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad 0 \leq x \leq \infty.$$

It will be interesting to note that these inequalities combined with the result of M. Kostic (1.1) gives the immediate consequence that

$$\left(\frac{2+x}{1+x}\right) > \frac{f(x)}{\sqrt{x+1}}$$

where f is as in (1.1). We try to prove these inequalities using some elementary techniques. We also try to assess the asymptotic behaviour of the sum $\sum\limits_{n \leq x} \frac{\log n}{n}$ using these inequalities. The importance of such logarithmic integrals and inequlities can be understood to some extent if one considers various results leading to the proof of the classical Prime Number Theorem and the role of logarithmic inequalities in the proof of the theorem.

To derive the inequalities in question, we need to prove the existence of some improper integrals. Here, we try to prove the existence of the improper integral $\int\limits_1^\infty \frac{\log t}{t^2}\,dt$ using the general method of comparison test.

2. Notations and basic results

Most of the notations, functions, and identities we use in this writeup are standard and can be found in [1]. The remaining ones we introduce in this section.

Definition 2.1 (The big oh notation)**.** *Let f, g be arithmetical functions. If $g(x) > 0$ for all $x \geq a$, we write $f(x) = O(g(x))$ to mean that the quotient $\frac{f(x)}{g(x)}$ is bounded for $x \geq a$.*

If $[x]$ denotes the greatest integer $\leq x$ for any real number x, then we have the following result.

Theorem 2.1 (Euler summation formula)**.** *[1, Theorem 3.1] If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then*

$$\sum_{y<n<x} f(n) = \int\limits_y^x f(t)\,dt + \int\limits_y^x (t-[t])\,(f'(t))\;dt + f(x)\,([x]-x)\;-f(y)\,([y]-y)\,.$$

3. Main Results and proofs

To find some reasonable bound for the integral $\int\limits_1^\infty \frac{\log t}{t^2}\,dt$ and hence to arrive at an estimate for $\sum\limits_{n\leq x} \frac{\log n}{n}$, we prove the inequalities we claimed in the first section of this paper.

Proposition 3.1. *The natural logarithm function satisfies the following inequalities:*

$$\log(1+x) \geq \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad -1 \leq x \leq 0, \tag{3.1}$$

(3.2) $$\log(1+x) \leq \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad 0 \leq x \leq \infty,$$

Proof. Define

$$L(x) = \log(1+x) - \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right), \quad -1 \leq x \leq 0.$$

The first inequality (3.1) is equivalent to the claim that $L(x) \geq 0$ for $-1 \leq x \leq 0$.

At $x = 0$, $L(x) = 0$. Now it is suffices to prove that $L(x)$ is decreasing in $(-1, 0)$. To prove the monotonic nature, we use derivative test. Consider the derivative of the function $L(x)$:

$$L'(x) = \frac{1}{1+x} - \frac{1}{2} - \frac{1}{2(x+1)^2} = \frac{1}{2}\left(\frac{2x+1}{(x+1)^2} - 1\right).$$

Since for $-1 < x < 0$, $\frac{2x+1}{(x+1)^2} < 1$ we quickly arrive at the conclusion $L'(x) \leq 0$. Hence the first equality holds.

We use a similar argument to prove the second inequality (3.2).

Define

$$H(x) = \left(\frac{x}{2}\right)\left(\frac{2+x}{1+x}\right) - \log(1+x), \quad 0 \leq x \leq \infty.$$

Clearly $H(x) = 0$ at $x = 0$. It is enough to prove H is an increasing function.

Now

$$H'(x) = \frac{1}{2}\left(1 - \frac{2x+1}{(x+1)^2}\right).$$

Since $\frac{2x+1}{(x+1)^2} \leq 1$ for $0 \leq x \leq \infty$, $H'(x) \geq 0$. This completes the proof. □

Corollary 3.1.

$$\log x \leq \frac{x-1}{\sqrt{x}}, \quad 1 \leq x \leq \infty.$$

Proof. For $1 \leq x \leq \infty$, write $\log x = \log(1+x-1)$. Note that when $1 \leq x \leq \infty$ we have $0 \leq x-1 \leq \infty$. Now use inequality (3.2) to get

$$\log x = \log(1+x-1) \leq \left(\frac{x-1}{2}\right)\left(\frac{2+x-1}{1+x-1}\right) = \frac{1}{2}\left(\frac{x^2-1}{x}\right) = \frac{1}{2}\left(x - \frac{1}{x}\right).$$

Therefore, $\log x \leq \frac{1}{2}\left(x - \frac{1}{x}\right)$ for $0 \leq x \leq \infty$.

Now let $x = x^a$ where $a > 0$.

Since $0 \leq x^a \leq \infty$, we get

$$\log(x^a) \leq \frac{1}{2}\left(x^a - x^{-a}\right) \quad \Rightarrow \quad \log x \leq \frac{1}{2a}\left(x^a - x^{-a}\right).$$

Let $a = \frac{1}{2}$. Hence

$$\log x \leq \sqrt{x} - \frac{1}{\sqrt{x}} = \frac{x-1}{\sqrt{x}}.$$

$\square$

Now we are ready to prove the existence of the promised integral.

Proposition 3.2. *The improper integral* $\int\limits_1^\infty \frac{\log t}{t^2}\, dt$ *exists.*

Proof. To prove the existence of $\int\limits_1^\infty \frac{\log t}{t^2}\, dt$, we use comparison test.

By corollary 3.1, the integrand $\frac{\log t}{t^2}$ is dominated by $\frac{t-1}{t^{\frac{5}{2}}}$. Thus,

$$0 \leq \int\limits_1^\infty \frac{\log t}{t^2}\, dt \leq \int\limits_1^\infty \frac{t-1}{t^{\frac{5}{2}}}\, dt = \int\limits_1^\infty \left(t^{\frac{-3}{2}} - t^{\frac{-5}{2}}\right)\, dt = \frac{4}{3}.$$

Hence $\int\limits_1^\infty \frac{\log t}{t^2}\, dt$ exists. $\square$

Applying Proposition 3.2, we immediately get the following result.

Corollary 3.2. *For* $x \geq 2$,

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + A + O\left(\frac{\log x}{x}\right),$$

where A *is a constant.*

Proof. By Euler's summation formula, we see that

$$\begin{aligned}
\sum_{n \leq x} \frac{\log n}{n} &= \int\limits_1^x \frac{\log t}{t}\, dt + \int\limits_1^x (t-[t])\left(\frac{1-\log t}{t^2}\right)\, dt + (t-[t])\frac{\log x}{x} \\
&= \frac{\log^2 x}{2} + \int\limits_1^x (t-[t])\left(\frac{1-\log t}{t^2}\right)\, dt + O\left(\frac{\log x}{x}\right) \\
&= \frac{\log^2 x}{2} + \int\limits_1^\infty (t-[t])\left(\frac{1-\log t}{t^2}\right)\, dt - \int\limits_x^\infty (t-[t])\left(\frac{1-\log t}{t^2}\right)\, dt \\
&\quad + O\left(\frac{\log x}{x}\right).
\end{aligned}$$

Observe that

$$\int_x^\infty (t-[t])\left(\frac{1-\log t}{t^2}\right)\,dt \le \int_x^\infty \left(\frac{1-\log t}{t^2}\right)\,dt = \int_x^\infty \frac{1}{t^2}\,dt - \int_x^\infty \frac{\log t}{t^2}\,dt$$
$$\le \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x} = O\left(\frac{1}{x}\right).$$

Now,

$$\int_1^\infty (t-[t])\left(\frac{1-\log t}{t^2}\right)\,dt \le \int_1^\infty \left(\frac{1-\log t}{t^2}\right)\,dt = \int_1^\infty \frac{1}{t^2}\,dt - \int_1^\infty \frac{\log t}{t^2}\,dt,$$

exists. Hence

$$\sum_{n\le x}\frac{\log n}{n} = \frac{1}{2}\log^2 x + \int_1^\infty (t-[t])\left(\frac{1-\log t}{t^2}\right)\,dt + O\left(\frac{\log x}{x}\right)$$
$$= \frac{1}{2}\log^2 x + A + O\left(\frac{\log x}{x}\right),$$

where $A = \int_1^\infty (t-[t])\left(\frac{1-\log t}{t^2}\right)\,dt$, a constant. $\square$

4. Further directions

It is interesting to compare our last result above with the well known result (see [1, Theorem 4.10])

$$\sum_{p \text{ prime}\le x}\frac{\log p}{p} = \log x + O(1). \tag{4.1}$$

To get (4.1), powerful tools like Shapiro's theorem [1, Theorem 4.8] was used, where as we used only elementary results to establish our claim. We hope to use this technique to get better bounds for other logarithmic polynomials without much difficulty.

ACKNOWLEDGMENT

The first author thanks the Kerala State Council for Science,Technology and Environment, Thiruvananthapuram, Kerala, India for providing financial support for carrying out research work. The second author thanks the University Grants Commission of India for providing financial support for carrying out research work through their Junior Research Fellowship (JRF) scheme.

REFERENCES

[1] T. APOSTOL: *An Introduction to Numerical Analysis*, Springer, 1976.

[2] M. KOSTIC: *New inequalities for the function $y = t \ln t$*, Electronic Journal of Mathematical Analysis and Applications, **8**(2) (2020), 291–296.

[3] C. CHESNEAU, Y.J. BAGUL: *New sharp bounds for the logarithmic function*, 2019.

[4] F. TOPSØE: *Some bounds for the logarithmic function in Inequality theory and applications*, Nova Sci. Publ. **4**(01) (2007), 137–151.

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE
THIRUVANANTHAPURAM, KERALA - 695034
INDIA.
Email address: nehathomas2009@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE
THIRUVANANTHAPURAM, KERALA - 695034
INDIA.
Email address: aryavinayachandran@gmail.com

DEPARTMENT OF MATHEMATICS
GOVERNMENT COLLEGE, AMBALAPUZHA, KERALA - 688561, INDIA
DEPARTMENT OF COLLEGIATE EDUCATION, GOVERNMENT OF KERALA, INDIA.
Email address: kvnamboothiri@gmail.com