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BULK VISCOUS BIANCHI TYPE I BAROTROPIC FLUID COSMOLOGICAL MODEL WITH VARYING Λ AND FUNCTIONAL RELATION ON HUBBLE PARAMETER IN ROSEN'S BIMETRIC GRAVITY

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ABSTRACT. We have deduced that bulk viscous Bianchi type I barotropic fluid cosmological model with varying Λ and functional relation on hubble parameter by solving the field equations bimetric theory of gravitation. It is observed that our model has exponentially accelerating expansion at late time starting with decelerating expansion which agreed the observation of Perlmutter (1998), Knop (2003), Tegmark (2004) and Spergel (2006). In the beginning, our model has more than three spatial-dimensions then it switched over to three-dimensional spatial geometry at late epoch of time and it is agreed with Borkar et al. (2013). Other geometrical and physical behavior of the model have been studied.

1. INTRODUCTION

It is well known that the cosmological models based on General Relativity contain an initial singular state (the big bang) from which the universe expands. This singular state can be avoided if the behavior of matter and radiation is described by the quantum theory. Unfortunately, nobody has given a way to do this satisfactorily. A satisfactory physical theory should be free from singularities

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because the presence of a singularity means a break-down of the physical laws provided by the theory. Naturally, taking into consideration these singularities in general relativity, one looks carefully at the foundation of general relativity and thinks whether modification can be made to improve it. With this motivation, Rosen [1, 2] proposed a bimetric theory of gravitation incorporating the covariance and equivalence principles. It is based on a simple form of Lagrangian and has a simpler mathematical structure than that of the general theory of relativity. In this theory at each point of space-time, there are two metric tensors: a Riemannian metric tensor g_{ij} and the background flat space-time metric tensor γ_{ij}. The tensor g_{ij} describes the geometry of a curved space-time and the gravitational fields. Here the background metric tensor γ_{ij} refers to inertial forces. This theory also satisfies covariance and equivalence principles. It is pointed out that this theory agrees with general theory of relativity up to the accuracy of observations made up to now. The field equations of bimetric theory of gravitation proposed by Rosen [1] are

$$N_i^j - \frac{1}{2}Ng_i^j = -T_i^j + \Lambda(t)g_i^j, \tag{1.1}$$

where $N_i^j = \frac{1}{2}\gamma^{\alpha\beta}(g^{sj}g_{si|\alpha})_{|\beta}$ and $k = \sqrt{\frac{g}{\gamma}}$ together with $g = det(g_{ij}), \gamma = det(\gamma_{ij})$ and the stroke ($|$) stands for γ-covariant differentiation with respect to γ_{ij} and T_{ij} is the energy momentum tensor of the matter. Several aspects of bimetric theory of gravitation have been studied and investigated many cosmological models in it by researchers like Yilmiz [3], Isrelit [4-6], Goldman [7], Karade [8], Kryzier and Kryzier [9], Reddy and Rao [10], Mohanty and Sahoo [11-13], Reddy [14-16], Khadekar et al. [17], Katore and Rane [18], Borkar et al. [19-23], Gaikwad et al. [24] Sahoo et al. [25-28], Berg et al. [29-31] have studied several aspects of bimetric theory of gravitation. Reddy and Venkateswarlu [13] have shown the non-existence of anisotropic Bianchi type-I perfect fluid models in Rosen's bimetric theory. In particular Reddy and Rao [14], Mohanty and Sahoo [12, 13] have established the non-existence of anisotropic spatially homogeneous Bianchi type cosmological models in bimetric theory when the source of gravitation is governed by either perfect fluid or mesonic perfect fluid. Reddy [15] have discussed the non-existence of anisotropic spatially homogeneous Bianchi type-I cosmological models in bimetric theory of gravitation in case of cosmic strings. Francesco Torsello [30] classified the asymptotic structure of black holes in Bimetric theory. They presented a propositions, whose validity is not limited to black hole solutions, which establishes the relation between the curvature singularities of two metric and the invertibility of their interaction potential. M Kocic et al.[31] Presented a method for solving the constraint equations in the Hassan–Rosen theory to determine bimetric initial data by deforming the existing GR initial data also they Obtained the Lane–Emden-like equations for the bimetric initial data specifically assuming the conformally flat spatial metrics at the moment of time symmetry We have deduced that bulk viscous Bianchi type I barotropic fluid cosmological model with varying Λ and functional relation on hubble parameter by solving the field equations bimetric theory of gravitation. Other geometrical and physical behavior of the model have been studied.

2. THE MODEL AND FIELD EQUATIONS

The Bianchi type I metric is

$$ds^2 = -dt^2 + A^2dx^2 + b^2dy^2 + c^2dz^2, \tag{2.1}$$

where A,B and C are the function of t-only and corresponding flat metric is

$$d\eta^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.2}$$

The Energy-momentum tensor with Bulk viscous fluid is given by

$$T^i_j = (\rho + \bar{p})v_iv^j + \bar{p}g^j_i, \tag{2.3}$$

where

$$\bar{p} = p - \zeta\theta \tag{2.4}$$

is the effective pressure related with isotropic pressure p and scalar expansion θ. The four velocity v^i is space like with magnitude

$$v_iv^i = -1. \tag{2.5}$$

The Rosen's field equation (1.1) (in geometrized units $(8\pi k = c = 1)$)with time varying cosmological constant $\Lambda(t)$ for the metric (2.1) and (2.2) takes the form

$$\left[-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}^2}{A^2} - \frac{\dot{B}^2}{B^2} - \frac{\dot{C}^2}{C^2}\right] = -2(\bar{p} - \Lambda), \tag{2.6}$$

$$\left[\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} - \frac{\dot{C}^2}{C^2}\right] = -2(\bar{p} - \Lambda), \tag{2.7}$$

$$\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{\dot{A}^2}{A^2}-\frac{\dot{B}^2}{B^2}+\frac{\dot{C}^2}{C^2}\right]=-2(\bar{p}-\Lambda), \tag{2.8}$$

$$\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{\dot{A}^2}{A^2}-\frac{\dot{B}^2}{B^2}-\frac{\dot{C}^2}{C^2}\right]=2(\rho-\Lambda), \tag{2.9}$$

where $\dot{A}=\frac{dA}{dt}$,$\ddot{A}=\frac{d^2}{dt^2}$ etc.

3. The solution of the Field Equations

The equations (2.6),(2.7),(2.8),(2.9) are four differential equations in six unknowns A,B,C,$\bar{p}$,ρ and Λ .In order to get the solutions for the system of differential equations (2.6),(2.7),(2.8),(2.9), we assume two conditions: First condition we assume that the special law of variation for generalized Hubble's parameter that yield a constant value of deceleration parameter q. since the line element is characterized by Hubble parameter H. The Hubble parameter H is related to the average scale factor R by a relation

$$H(R)=a(R^{-n}+1), \tag{3.1}$$

where $a(>0)$ and $n(>1)$ are constants. The divergence of Rosen's field equation (1.1) lead to

$$\dot{\rho}+(\rho+\bar{p})\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right]+\Lambda=0, \tag{3.2}$$

from (2.6),(2.7) and (2.8) we write

$$\frac{\dot{A}}{A}-\frac{B}{B}=k_1, \tag{3.3}$$

$$\frac{\dot{B}}{B}-\frac{C}{C}=k_2, \tag{3.4}$$

where k_1 and k_2 are constants. The volume V and our average scale factor R are to be defined as $v^3=ABC$ and $R^3=ABC$, so that the Hubble parameter H may be defined as

$$H=\frac{\dot{R}}{R}=\frac{1}{3}\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right]. \tag{3.5}$$

The equation (3.5) together with equations (3.3),(3.4) yield the directional of Hubble parameter as

$$H_1 = \frac{\dot{A}}{A} = \frac{\dot{R}}{R} + \frac{2k_1 + k_2}{3}, \tag{3.6}$$

$$H_2 = \frac{\dot{B}}{B} = \frac{\dot{R}}{R} + \frac{k_2 - k_1}{3}, \tag{3.7}$$

$$H_3 = \frac{\dot{C}}{C} = \frac{\dot{R}}{R} - \frac{k_1 + 2k_2}{3}. \tag{3.8}$$

On integrating the above equations (3.6),(3.7) and (3.8) we arrived at

$$A = Re^{\frac{(2k_1+k_2)t}{3}}, \tag{3.9}$$

$$B = Re^{\frac{(k_2-k_1)t}{3}}, \tag{3.10}$$

$$C = Re^{\frac{-(k_1+2k_2)t}{3}}. \tag{3.11}$$

From equation (3.1) and (3.5), we have

$$R^n = e^{(nat+\alpha)-1}, \tag{3.12}$$

in which α is the constant of integration. At early stage $t = 0$, the average scale factor $R = 0$, so that $\alpha = 0$. Hence

$$R^n = e^{(nat)} - 1. \tag{3.13}$$

Using this equation (3.13), we write the values of scale factor A,B and C as

$$A = (e^{nat} - 1)^{\frac{1}{n}} e^{\frac{(2k_1+k_2)t}{3}}, \tag{3.14}$$

$$B = (e^{nat} - 1)^{\frac{1}{n}} e^{\frac{(k_2-k_1)t}{3}}, \tag{3.15}$$

$$C = (e^{nat} - 1)^{\frac{1}{n}} e^{-\frac{(k_1+2k_2)t}{3}}. \tag{3.16}$$

The equations ((3.14),(3.15),(3.16)) represents the solution of Bianchi type I bulk viscous barotropic fluid cosmological model with varying Λ and functional relation on Hubble parameter in bimetric theory of gravitation. The solutions ((3.14),(3.15),(3.16)) represent the exponential volumetric expansion of the model. At early stage of the universe, A, B and C attains zero values. So that volume V is zero and the scale factor admits zero values at early stage and then they starts increasing with increase in time t. This infers that initially the model starts with zero volume and zero scalar factor and then model expanding exponentially and has infinite volume at final epoch of time.

4. Physical Parameters in the Model

The scalar expansion θ, shear tensor σ_i^j and deceleration parameter q and are given by the formulae

$$\theta = \nu^i_{|i} = 3\frac{\dot{R}}{R} = 3H = \left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right],$$

$$\sigma_{ij} = \frac{1}{2}(\nu_{i;j} + \nu_{i;j}) - \frac{1}{2}(\nu_i\nu_j + \nu_j\nu_i) - \frac{1}{3}\theta(g_{ij} + \nu_i\nu_j),$$

$$q = -\frac{\ddot{R}}{RH^2},$$

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \tag{4.1}$$

$$\sigma_1^1 = \left(\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right) = \frac{2k_1 + k_2}{3},$$

$$\sigma_2^2 = \left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right) = \frac{k_2 + k_1}{3}, \tag{4.2}$$

$$\sigma_3^3 = \left(\frac{\dot{C}}{C} - \frac{\dot{R}}{R}\right) = \frac{-k_1 + 2k_2}{3},$$

$$\sigma_4^4 = 0,$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{(k_1^2 + k_1k_2 + k_2^2)}{3} = \frac{k^2}{3}, \tag{4.3}$$

$$q = \frac{n}{R^n + 1} - 1. \tag{4.4}$$

It is observed that when $R = 0$, $q = n - 1 > 0$, $q = 0$ for $R^n = n - 1$ and for $R^n > n - 1, q < 0$.

The mean H and the' directional H_1,H_2 and H_3 of the Hubble parameter are given by

$$H = \frac{ae^{nat}}{e^{nat} - 1},$$

$$H_1 - \frac{(2k_1 + k_2)}{3} = H_2 - \frac{k_2 - k_1}{3} = H_3 + \frac{(k_1 + 2k_2)}{3} = \frac{ae^{nat}}{e^{nat} - 1}.$$

The mean Hubble parameter H and its directional and are decreasing function of time and they infinite at $t = 0$ and they having constant values when $t \to \infty$. The directional parameters H_1 and H_2 along x and y-axes increasing from mean Hubble parameter by constant factors while parameter H_3 along z-axis decreases. Using equation (3.12), we write equation (4.4) as

$$q = \frac{n}{e^{nat}} - 1. \tag{4.5}$$

The scalar expansion θ having the value

$$\theta = 3H = \frac{3a}{1 - e^{-nat}}. \tag{4.6}$$

The magnitude of shear tensor σ_i^j will be

$$\sigma = \frac{k}{\sqrt{3}}. \tag{4.7}$$

The deceleration parameter q is negative when $R^n > n - 1$. This shows that the universe has accelerating expansion as we expect from exponential expansion. Further, from equation (4.6) of scalar expansion, it is realized that at early stage of the universe, the expansion is infinite and at late epoch of time t, expansion θ attain constant value which shows there is uniform exponential expansion of the universe at final epoch of time t From equation (4.7), it is observed that the shear has constant magnitude, which suggested that the universe has uniform shear in whole range of time t, $0 \leq t < \infty$. Equations (2.6),(2.7),(2.8),(2.9),(3.1) in terms of H, ρ,$\bar{p}$ and q are as under:

$$(q+1)H^2 = 2(\bar{p} - \Lambda), \tag{4.8}$$

$$-3(q+1)H^2 = 2(\rho + \Lambda), \tag{4.9}$$

$$\dot{\rho} + 3(\rho + \bar{p})\frac{\dot{R}}{R} + \Lambda = 0. \tag{4.10}$$

In order to find pressure p energy density ρ and cosmological constant Λ we assumed second condition that the matter obeys the equation of state $p = \omega\rho$, with $0 \leq \omega \leq 1$. The effective time dependent cosmological term Λ which is measure of the energy of empty space corresponding to a perfect fluid with energy density $\rho_{vac} = \Lambda$ and pressure $p_{vac} = -\Lambda$. From (4.8) and (4.9), we have

$$\rho + \bar{p} = \rho + p - \zeta\theta = -(q+1)H^2, \tag{4.11}$$

or

$$\rho(1+\omega) = -(q+1)H^2 + \zeta\theta.$$

Using equations (4.5) and (4.6), the above equation (4.11) yield

$$\rho = \frac{1}{(\omega+1)}\left(\frac{-na^2e^{nat}}{(e^{nat}-1)^2} + \frac{3a\zeta e^{nat}}{(e^{nat}-1)}\right) \tag{4.12}$$

and

$$p = \omega\rho = \frac{\omega}{(\omega+1)}\left(\frac{-na^2e^{nat}}{(e^{nat}-1)^2} + \frac{3a\zeta e^{nat}}{(e^{nat}-1)}\right). \tag{4.13}$$

The matter density is a function of time t. At $t = 0$, it is infinite and it is decreases with increase in time t and approaches to constant value at $t \to \infty$. This suggested that at early stage of the universe there is very high energy density of the matter i.e. in the beginning of the model there is very high density matter and universe contains the matter with uniform density at final epoch of time. In the absence of bulk viscosity ζ, the model goes over to vacuum universe at final stage. From equation (2.4), using equations (4.6) and (4.13), the effective pressure $\bar{p}$ have been calculated as

$$\bar{p} = \frac{-1}{(\omega+1)}\left(\frac{na^2e^{nat}\omega}{(e^{nat}-1)^2} + \frac{3a\zeta e^{nat}}{(e^{nat}-1)}\right). \tag{4.14}$$

Using equation (4.8), we deduced

$$\Lambda = \frac{-(3\omega+1)}{2(\omega+1)}\left(\frac{na^2e^{nat}\omega}{(e^{nat}-1)^2} - \frac{1}{(\omega+1)}\frac{3a\zeta e^{nat}}{(e^{nat}-1)}\right) \tag{4.15}$$

and

$$\Lambda \neq 0,\ since\ \zeta \neq \frac{-1}{6}\frac{na(3\omega+1)}{e^{nat}-1} \text{ never negative.} \tag{4.16}$$

The cosmological constant Λ is a function of time t and it measure of energy of empty space corresponding to a perfect fluid with energy density $\rho_{vac} = \Lambda$. It is seen that at $t = 0$ it diverges to infinity and it attain the value $\frac{3(3\omega+1)a}{2(\omega+1)}$ at $t \to \infty$. This shows that the cosmological term Λ does not gives us any information at early stage of the universe but as the time t increases, it is coming in the picture and has constant value finally at late epoch of time t. It is never zero, since the coefficient if bulk viscosity cannot negative. In the absence of bulk viscosity, the cosmological constant Λ disappeared at final stage of the universe.

5. Conclusion

We have deduced that bulk viscous Bianchi type I barotropic fluid cosmological model with varying Λ and functional relation on hubble parameter by solving the field equations bimetric theory of gravitation. It is observed that our model has exponentially accelerating expansion at late time starting with decelerating expansion which agreed the observation [32-35]. In the beginning, our model has more than three spatial-dimensions then it switched over to three-dimensional spatial geometry at late epoch of time and it is agreed with Borkar et al. [35]. The cosmological term Λ does not appear at the beginning of the universe but finally it remains constant at late epoch of time. Other geometrical and physical behavior of the model have been studied.

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