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A NEW CLASS OF TRANSMUTED MODIFIED WEIGHTED WEIBULL DISTRIBUTION AND ITS PROPERTIES

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ABSTRACT. An additional parameter was added to Modified Weighted Weibull distribution with method of quadratic rank transmutation which led to a newly developed distribution called Transmuted Modified Weighted Weibull distribution. Two distributions that emanated from the new distribution are Transmuted Modified Rayleigh and Transmuted Modified Exponential distributions. Some properties of the distribution that were obtained include; the survival rate, hazard rate, reverse hazard rate function; and moment generating function, mean and variance. Also, parameters of the model were estimated using maximum likelihood estimation method. The model was applied to a life time data set of total milk production of the first birth of 107 cows which showed a better performance compared to some existing known distributions.

1. INTRODUCTION

Quadratic Rank Method (QRM) is one in thousand ways of adding or introducing parameter into any distribution. In this work, we added a parameter to existing distribution and it becomes Transmuted Modified Weighted Weibull (TMWW) distribution. Meanwhile, in literature, numerous authors have worked on either quadratic rank method or any other method such as: [3, 4, 7, 8, 11];

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they worked on different distributions. On the other hand, some used other methods include: generator approach, beta link function, quantile function etc, For instance, [1, 2, 5, 9, 10] and so on.

Hence, order of arrangement is as follows: section two consists the propose distribution, in section three, we have moments and parameter estimation, four contains the data analysis (application) and five has the concluding remark.

2. Transmuted Modified Weighted Weibull (TMWW) Distribution

The QRM is given as

$$K(x) = F(x)(1+\theta) - \theta F(x)^2, |\theta| \leq 1 \tag{2.1}$$

where, θ is the added parameter and $F(x)$ is the distribution function of the parent distribution. However, the density and distribution function of MWW distribution by [6] are expressed as

$$f(x) = \beta\gamma(c\lambda^\gamma + 1)x^{\gamma-1}e^{(-\beta(c\lambda^\gamma+1)x^\gamma)} \tag{2.2}$$

and

$$F(x) = 1 - e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}. \tag{2.3}$$

By setting (2.1) and (2.3) together, this results to (2.4) and

$$f(x) = \beta\gamma(c\lambda^\gamma + 1)x^{\gamma-1}[1 - e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}], \tag{2.4}$$

where, β is the scale and γ, c and λ are shape parameters.

Also, the distribution and density function of the TMWW distribution is given by

$$F_{TMWW}(x) = [1 - e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}][1 + \theta e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}], \tag{2.5}$$

then, the density function is

$$f_{TMWW}(x) = \beta\gamma(c\lambda^\gamma + 1)x^{\gamma-1}e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}[1 + \theta e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}]. \tag{2.6}$$

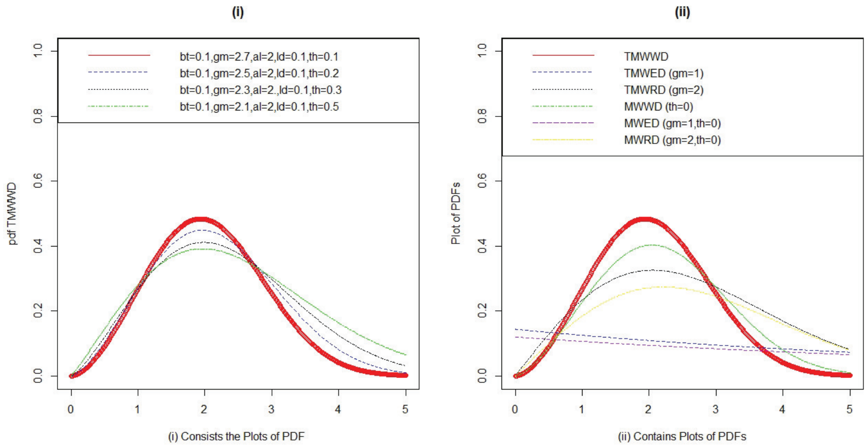

FIGURE 1. The pdf plots of TMWWD, TMWED, TMWRD, MWWD, MWED and MWRD

2.1. **Special Distributions and Properties.** Some new and known distributions were obtained from the TMWW distribution.

(i) If $\gamma = 1$ from (2.6) above, it gives TMW exponential Distribution (New):

$$f_{TMWE}(x) = \beta\gamma(c\lambda + 1)e^{(-\beta(c\lambda+1)x)}[1 + \theta e^{(-\beta(c\lambda+1)x)}]. \tag{2.7}$$

(ii) When $\gamma = 2$ it yields TMW Rayleigh distribution (New):

$$f_{TMWE}(x) = 2\beta(c\lambda^2 + 1)xe^{(-\beta(c\lambda^2+1)x^2)}[1 + \theta e^{(-\beta(c\lambda^2+1)x^2)}]. \tag{2.8}$$

(iii) Suppose $\theta = 0$, we have modified weighted Weibull distribution (known):

$$f(x) = \beta\gamma(c\lambda^\gamma + 1)x^{\gamma-1}e^{(-\beta(c\lambda^\gamma+1)x^\gamma)}. \tag{2.9}$$

(iv) If $\theta = 0$ in (2.7), we obtain modified weighted exponential distribution (known:

$$f(x) = \beta\gamma(c\lambda + 1)e^{(-\beta(c\lambda+1)x)}, \tag{2.10}$$

and

(v) When $\theta = 0$ in (2.8), we get modified weighted Rayleigh distribution (known):

$$f(x) = 2\beta(c\lambda^2 + 1)xe^{(-\beta(c\lambda^2+1)x^2)}. \tag{2.11}$$

The Survival Rate function of the TMWW distribution is given as:

$$Sur_{TMWW}(x) = 1 - I_{(F(x,\beta,\gamma,c,\lambda,\theta))},$$

where, $I_{(F(x,\beta,\gamma,c,\lambda,\theta))}$ is in (2.5) above. Therefore,

$$Sur_{TMWW}(x) = e^{(-\beta(c\lambda^{\gamma}+1)x^{\gamma})}[-\theta e^{(-\beta(c\lambda^{\gamma}+1)x^{\gamma})}] \tag{2.12}$$

The Hazard Rate Function is obtained by dividing (2.6) by (2.5) as expressed below:

$$HR_{TMWW}(x) = \frac{f_{TMWW}(x)}{1 - F_{TMWW}(x)} = \frac{f_{TMWW}(x)}{Sur_{TMWW}(x)},$$

$$HR_{TMWW}(x) = \frac{\beta\gamma(c\lambda^{\gamma}+1)x^{\gamma-1}e^{(-\beta(c\lambda^{\gamma}+1)x^{\gamma})}[1+\theta e^{(-\beta(c\lambda^{\gamma}+1)x^{\gamma})}]}{[e^{(-\beta(c\lambda^{\gamma}+1)x^{\gamma})}][1+\theta e^{(-\beta(c\lambda^{\gamma}+1)x^{\gamma})}]}. \tag{2.13}$$

2.2. **Moments.** We present here the moments for the TMWW distribution. The Sth order moments of TMWW random variable X using gamma function $\Gamma(.)$, is expressed as

$$E(X^{S}) = \beta\Gamma(1+\frac{s}{\gamma})(1+c\lambda^{\gamma})^{-\frac{s}{\gamma-1}}[(1+\theta)(1+c\lambda^{\gamma})\beta^{-\frac{s}{\gamma}}]. \tag{2.14}$$

Furthermore, the expected value and variance of TMWW random variable X are given respectively below:

$$E(X^{1}) = \beta\Gamma(1+\frac{1}{\gamma})(1+c\lambda^{\gamma})^{-\frac{1}{\gamma}}[(1+\theta)(1+c\lambda^{\gamma})\beta^{-\frac{1}{\gamma}}],$$

and its corresponding variance is

$$Var(X) = \beta^{2}\Gamma(1+\frac{2}{\gamma})(1+c\lambda^{\gamma})^{-\frac{2}{\gamma-1}}[(1+\theta)(1+c\lambda^{\gamma})\beta^{-\frac{2}{\gamma}}] \cdot \Gamma^{2}(1+\frac{1}{\gamma})(1+c\lambda^{\gamma})^{-\frac{1}{\gamma}} \cdot [(1+\theta)(1+c\lambda^{\gamma})\beta^{-\frac{1}{\gamma}}].$$

3. Parameter Estimation

The cdf of the TMWW conjunction with MLEs is used to obtain the likelihood function; and the parameters were obtained using the pdf together with MLEs as follows: suppose a sample of size n say $x_1, x_2, \ldots, x_n$ be from a TMWW then, the likelihood function is given by

$$LF_{TMWW}(x,\beta,\gamma,c,\lambda,\theta) = (\beta\gamma)^n exp[-\beta \sum_{(i=1)}^{n} (c\lambda^\gamma + 1)x_i^\gamma](c\lambda^\gamma + 1)x_i^{\gamma-1} \cdot [1 + \theta e^{(-\beta(c\lambda^\gamma+1)x_i^\gamma)}].$$

Then, the log-likelihood function $ll = \ln L$ yields:

$$ll(\varphi) = n \ln \beta\gamma - n\gamma + \gamma - 1 \sum_{(i=1)}^{n} \ln(c\lambda^\gamma + 1)x_i - \sum_{(i=1)}^{n} \beta(c\lambda^\gamma + 1)x_i^\gamma + \sum_{(i=1)}^{n} [\ln[1 + \theta e^{(-\beta(c\lambda^\gamma+1)x_i^\gamma)}]],$$

and the components of the unit score vector are

$$\frac{\partial ll(\varphi)}{\partial \varphi} = (\frac{\partial ll(\varphi)}{\partial \beta}, \frac{\partial ll(\varphi)}{\partial \gamma}, \frac{\partial ll(\varphi)}{\partial c}, \frac{\partial ll(\varphi)}{\partial \lambda}, \frac{\partial ll(\varphi)}{\partial \theta})' \tag{3.1}$$

$$\frac{\partial ll(\varphi)}{\partial \beta} = \frac{n}{\beta} - \gamma[\beta(c\lambda + 1)x_i^\gamma] + \theta \sum_{(i=1)}^{n} \frac{(\ln x_i^\gamma e^{(-\beta(c\lambda+1)x_i^\gamma)})}{(1 + \theta e^{(-\beta(c\lambda+1)x_i^\gamma)})} = 0 \tag{3.2}$$

$$\frac{\partial ll(\varphi)}{\partial \gamma} = \frac{n}{\gamma} + \frac{\ln\lambda(c\lambda^\gamma)}{(c\lambda^\gamma + 1)} - \beta \sum_{(i=1)}^{n} \frac{\gamma \ln\lambda x_i(c\lambda^\gamma)}{(c\lambda^\gamma + 1)} + \sum_{(i=1)}^{n} \frac{\gamma \ln x_i(1 + \theta)e^{(-\beta(\frac{c\lambda^\gamma}{c\lambda^\gamma + 1}))}}{1 + \theta e^{(-\beta(c\lambda^\gamma+1)x_i^\gamma)}} = 0 \tag{3.3}$$

$$\frac{\partial ll(\varphi)}{\partial c} = \ln x_i(\frac{\lambda^\gamma}{c\lambda^\gamma + 1}) - \beta \sum_{(i=1)}^{n} (\frac{\lambda^\gamma}{c\lambda^\gamma + 1}) + \theta \sum_{(i=1)}^{n} \frac{\ln x_i(1 + \theta)e^{(-\beta(\frac{c\lambda^\gamma}{c\lambda^\gamma + 1}))x_i^\gamma}}{1 + \theta e^{(-\beta(c\lambda^\gamma+1)x_i^\gamma)}} = 0 \tag{3.4}$$

$$\frac{\partial ll(\varphi)}{\partial \lambda} = \ln x_i^{\gamma} \frac{c\gamma^{(\gamma-1)}}{c\gamma\lambda^{\gamma}+1} - \beta \sum_{(i=1)}^{n} \ln x_i^{\gamma} \frac{c\gamma^{(\gamma-1)}}{c\gamma\lambda^{\gamma}+1} + \sum_{(i=1)}^{n} \frac{\ln x_i^{\gamma} e^{(-\beta(\frac{c\lambda^{\gamma-1}}{c\lambda^{\gamma}+1}))x_i^{\gamma}}}{1+\theta e^{(-\beta(c\lambda^{\gamma}+1)x_i^{\gamma})}} = 0 \tag{3.5}$$

$$\frac{\partial ll(\varphi)}{\partial \theta} = \sum_{(i=1)}^{n} \frac{\ln x_i^{\gamma} e^{(-\beta(c\lambda^{\gamma}+1)x_i^{\gamma})}}{1+\theta e^{(-\beta(c\lambda^{\gamma}+1)x_i^{\gamma})}} = 0. \tag{3.6}$$

One can easily use nonlinear optimization algorithm such as Newton Raphson algorithm to solve the nonlinear system of equation from $(3.3-3.7)$ above.
The standard error and asymptotic confidence interval according to [8], as $n \to \infty$, and asymptotic distribution of the maximum likelihood estimation $(\hat{\beta}), (\hat{\gamma}), (\hat{c}), (\hat{\lambda}), (\hat{\theta})$ is given by

$$\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\\ \hat{c}\\ \hat{\lambda}\\ \hat{\theta}\end{pmatrix} \sim N\left[\begin{pmatrix}\beta\\ \gamma\\ c\\ \lambda\\ \theta\end{pmatrix} \cdot \begin{pmatrix}\hat{M}_{11} & \hat{M}_{12} & \hat{M}_{13} & \hat{M}_{14} & \hat{M}_{15}\\ \hat{M}_{21} & \hat{M}_{22} & \hat{M}_{23} & \hat{M}_{24} & \hat{M}_{25}\\ \hat{M}_{31} & \hat{M}_{32} & \hat{M}_{33} & \hat{M}_{34} & \hat{M}_{35}\\ \hat{M}_{41} & \hat{M}_{42} & \hat{M}_{43} & \hat{M}_{44} & \hat{M}_{45}\\ \hat{M}_{51} & \hat{M}_{52} & \hat{M}_{53} & \hat{M}_{54} & \hat{M}_{55}\end{pmatrix}\right],$$

where, $\hat{M}_{ij} = M_{ij}|_{\alpha=\hat{\alpha}}$ and the matrix below is the variance and covariance matrix

$$l^{-1}(V) = \begin{pmatrix}M_{11} & M_{12} & M_{13} & M_{14} & M_{15}\\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25}\\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35}\\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45}\\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55}\end{pmatrix} = \begin{pmatrix}V_{11} & V_{12} & V_{13} & V_{14} & V_{15}\\ V_{21} & V_{22} & V_{23} & V_{24} & V_{25}\\ V_{31} & V_{32} & V_{33} & V_{34} & V_{35}\\ V_{41} & V_{42} & V_{43} & V_{44} & V_{45}\\ V_{51} & V_{52} & V_{53} & V_{54} & V_{55}\end{pmatrix},$$

and the elements are written as follows:

$$V_{11} = \frac{\partial^2 ll}{\partial \beta^2}, V_{12} = \frac{\partial^2 ll}{\partial \beta \partial \gamma}, V_{22} = \frac{\partial^2 ll}{\partial \gamma^2}, V_{23} = \frac{\partial^2 ll}{\partial \gamma \partial c}, V_{33} = \frac{\partial^2 ll}{\partial c^2},$$

$$V_{34} = \frac{\partial^2 ll}{\partial c \partial \lambda}, V_{44} = \frac{\partial^2 ll}{\partial \lambda^2}, V_{45} = \frac{\partial^2 ll}{\partial \lambda \partial \theta}, V_{55} = \frac{\partial^2 ll}{\partial \theta^2}.$$

An appropriate $(1-\alpha)100\%$ with two sided confidence intervals for $\beta, \gamma, c, \lambda$ and θ are given respectively:

$$\hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{M}_{11}}, \hat{\gamma} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{M}_{22}}, \hat{c} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{M}_{33}}, \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{M}_{44}}, \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{M}_{55}}.$$

4. Data Analysis

The data used represent the total milk production in the first birth of 107 cows which was extracted from [12].

TABLE 1. MLEs of the Parameters, Standard Error (in parentheses) and model selection criterion of the distributions.

Para/Dist	TMWW	TMWE	TMWR	MWW	MWE	MWR
β	0.49162	0.09023	0.03723	0.49126	0.50000	0.09678
	(1.40675)	(0.39310)	(0.00000)	(0.00000)	(0.03559)	(0.00000)
γ	0.56654	1.00000	2.00000	0.53376	1.00000	2.00000
	(0.00106)	(0.00040)	(0.000133)	(0.00046)	(0.00241)	(0.00000)
c	0.00019	0.00048	0.06614	0.00040	0.10000	0.00055
	(0.00010)	(0.00000)	(0.00000)	(0.00000)	(0.03502)	(0.00000)
λ	0.49439	0.50062	0.57781	0.48084	0.50000	0.69137
	(0.27349)	(0.16411)	(0.00000)	(0.00000)	(0.00696)	(0.00410)
θ	0.58522	0.63992	1.09915	0.00000	0.00000	0.54956
	(0.00811)	(0.00932)	(0.01717)	(0.00000)	(0.00526)	(0.01260)
Log-lik	18702.8	15771.6	11712.1	12350.1	74000.9	92240.4
AIC	-37400.6	-31539.1	-23420.1	-24696.2	-14798.8	-18445.9
BIC	-37382.2	-31524.8	-23405.4	-24681.5	-14787.8	-18434.8
CAIC	-37381.2	-31523.8	-23404.4	-24680.5	-14786.8	-18433.8

5. Result and Concluding Remarks

5.1. **Result.** The analysis showed in Table 1 above are the numerical values with MLEs, their corresponding standard errors (in parentheses) and the model selection criterion (i.e AIC, BIC and CAIC) of the distribution parameters for comparing TMWW with other distributions listed. The results revealed that the values of the TMWW under model selection were smaller than other competing distributions. The graphs of the fitted distributions using the milk data set are

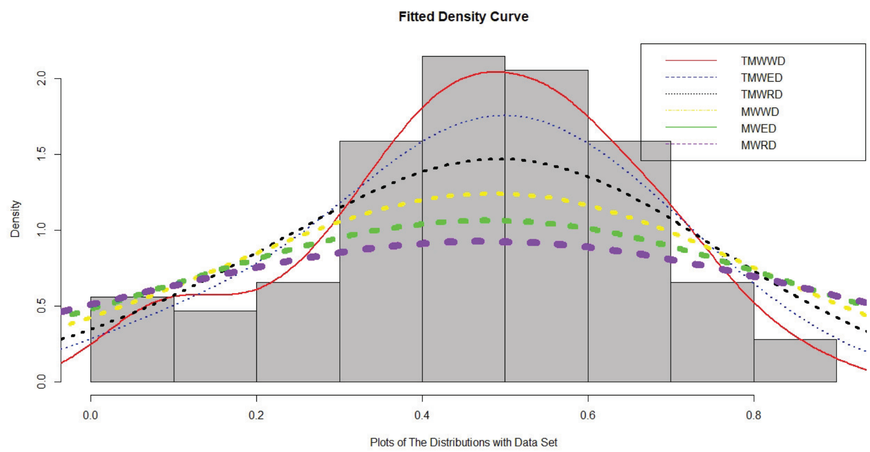

FIGURE 2. Fitted Densities of the TMWWD, TMWED, TMWRD, MWWD, MWED and MWRD using milk data set.

TABLE 2. Reflects the values of distributions using the pdf in (2.6).

$f(x)$	$\beta = 10$	$\gamma = 2$	$c = 10$	$\lambda = 0.1$	$\theta = 0.5$	
x	TMWW	TMWE	TMWR	MWW	MWE	MWR
1	0.28536	0.26512	0.18259	0.23078	0.16375	0.18097
2	0.37463	0.35977	0.26973	0.17900	0.13406	0.26813
3	0.29080	0.29337	0.24417	0.13988	0.10976	0.24394
4	0.16442	0.17649	0.16054	0.11006	0.08986	0.16152
5	0.07257	0.08410	0.08086	0.08711	0.07358	0.08209
6	0.02540	0.03237	0.03195	0.06931	0.06024	0.03279
7	0.00704	0.01006	0.01003	0.05540	0.04932	0.01043
8	0.00154	0.00252	0.00252	0.04446	0.04038	0.00266
9	0.00027	0.00051	0.00051	0.03580	0.03306	0.00054
10	0.00003	0.00008	0.00008	0.02890	002707	0.00009

shown in Figures 1 and 2. Table 2 also reflects the values under each distribution as x taken the values from 1 to 10 and the parameters are fixed as shown above. Also, its showed that as x increases, the values of the distributions became smaller but the output of the proposed distribution is smaller than others.

Therefore, this indicates that the TMWW is better than any one of the distributions considered.

5.2. **Concluding Remarks.** We propose a new class of transmuted modified weighted Weibull distribution generated quadratic rank method. We have derived important properties of the ME distribution like hazard rate function, moments, asymptotic distribution, characterizations and maximum likelihood estimation of parameters. We have illustrated the application of ME distribution to two real data sets used by researchers earlier. By comparing ME distribution with other popular generalization of exponential models we conclude that ME distribution performs better.

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