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SAFE ETERNAL 1-SECURE SETS IN GRAPHS

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ABSTRACT. An eternal 1-secure set, in a graph $G = (V, E)$ is a set $D \subset V$ having the property that for any finite sequence of vertices $r_1, r_2, \dots, r_k$ there exists a sequence of vertices $v_1, v_2, \dots, v_k$ and a sequence $D = D_0, D_1, D_2, \dots, D_k$ of dominating sets of G, such that for each i, $1 \leq i \leq k$, $D_i = (D_{i-1} - \{v_i\}) \cup \{r_i\}$, where $v_i \in D_{i-1}$ and $r_i \in N[v_i]$. Here $r_i = v_i$ is possible. The cardinality of the smallest eternal 1-secure set in a graph G is called the eternal 1-security number of G. In this paper we study a variations of eternal 1-secure sets named safe eternal 1-secure sets. A vertex v is safe with respect to an eternal 1-secure set S if $N[v] \bigcap S = 1$. An eternal 1 secure set S is a safe eternal 1 secure set if at least one vertex in G is safe with respect to the set S. We characterize the class of graphs having safe eternal 1-secure sets for which all vertices - excluding those in the safe 1-secure sets - are safe. Also we introduce a new kind of directed graphs which represent the transformation from one safe 1 - secure set to another safe 1-secure set of a given graph and study its properties.

1. INTRODUCTION

Throughout this paper by a graph $G = (V, E)$ we mean a finite, connected, undirected graph without loops or multiple edges and by $D = (V, A)$ we mean

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a finite and simple directed graph without multiple arcs. For graph theoretic terminology we refer to West [15]. A *dominating set* in a graph is a set of vertices D having the property that every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set D is a minimal dominating set if no proper subset of D is a dominating set. The minimum number of elements in a minimal dominating set is called the domination number and it is denoted by $\gamma(G)$. The concepts related to domination are well explained in the books by Haynes, Hedetniemi and Slater [4].

Police forces are deployed to guard the important places in a city on the occasion of riots. Deploying the policemen at all places is not practically possible. Instead group of policemen can be deployed at the key places and the remaining places can be brought under their surveillance. In case of emergency, the police party could move to the place where a need arises. The policemen can guard all the places only if they either have a camp there or they can reach the place directly from the place of their camp. This problem can be modeled as follows.

All important places in a city are represented by nodes in a graph and the direct road connectivity between two places by edges connecting the corresponding nodes. The police party must be stationed at the places which correspond to a dominating set in the graph for effective surveillance of the places. If a place where no policemen stationed is attacked by the terrorists then the police from an adjacent place will come forward to defend the attack and will remain in the new place until there is no attack in the neighboring places. Set of the places where the police force is stationed is denoted by D and it must be domination set in the underlying graph. The position of police force may change after each attack. This situation is mathematically modeled below.

If a place represented by the vertex $w \in V - D$ is attacked by the enemy, then the police force from one of the places represented by a vertex v in the dominating set D moves to defend the attack. But the resulting set $D - \{v\} \cup \{w\}$ may not be a dominating set. Motivated by this fact, the eternal 1-secure set is defined. An *eternal* 1*-secure set* is a dominating set $D \subset V$ having the property that for any finite sequence of vertices $r_1, r_2, \ldots, r_k$, there exists a sequence of vertices $v_1, v_2, \ldots, v_k$ and a sequence $D = D_0, D_1, D_2, \ldots, D_k$ of dominating sets of G, such that for each i, $1 \leq i \leq k$, $D_i = (D_{i-1} - \{v_i\}) \cup \{r_i\}$, where $v_i \in D_{i-1}$ and $r_i \in N[v_i]$. It may be noted that $r_i = v_i$ is possible. The number of elements of a smallest eternal 1-secure set is called the *eternal* 1*-security number*. This parameter was introduced by Burger et al. [1, 2]. They used the notation γ_∞, which was later replaced by the notation $\sigma_1(G)$ by Goddard et al. [3].

Goddard et al. [3] proved the following inequality, which is valid for all graphs G,

$$\alpha(G) \leq \sigma_1(G) \leq \theta(G).$$

Here $\alpha(G)$ is the independence number and $\theta(G)$ is the clique covering number of G. Also we have $\sigma_1(G) \leq \alpha_1(G)$, where $\alpha_1(G)$ equals the *edge covering number* of a graph G. It is well known that for any graph G of order n, $\alpha_1(G) + \beta_1(G) = n$. The following upper bound in terms of independence number is given by Klostermeyer et al.

Theorem 1.1. [5] *For any graph G with independence number $\alpha(G) \geq 1$,*

$$\sigma_1(G) \leq \binom{\alpha(G)+1}{2}.$$

Fractional versions of domination was studied by Reji et al. in a sequence of papers [7–14] Connected eternal domination was defined by Reji et al. [6] by adding the condition that the eternal dominating set must induce a connected sub-graph of the underlying graph. In this paper we study another variation of the eternal domination problem. We restate the problem as a two player game. One player is the defender and the other player is the attacker. The defender will select a dominating set D_i in the i^{th} step of the play, such that there exists at least one vertex v satisfying $|N[v] \bigcap D_i| = 1$. Let $A_{D_i} = \{v \in V | N[v] \bigcap D_i = 1\}$. In each step an attacking player selects a vertex $r_i \in A_{D_i}$. Then the first player modifies his set D_i to obtain $D_{i+1} = (D_i - \{v_i\}) \bigcup \{r_i\}$, where $v_i \in N[r_i]$. While playing the game, if the set A_{D_i} becomes empty, then the game stops. The main question is, whether for a given graph G, the game terminates after some time or goes indefinitely? We try to find answer to these questions in the following section. The vertices in the set A_{D_i}, are called *safe vertices*. If a vertex $x \in A_{D_i}$ is in D_i also, then x is a *safe dominating vertex*. The dominating set D_i is called a *safe 1-secure set* if $A_{D_i} \neq \emptyset$.

If the second player selects a safe dominating vertex, then we get $D_i = D_{i+1}$ and we call it a *trivial case*. An initial dominating set D_0 can be called a *failed dominating set*, if responding to a sequence of attacks at the vertices

$r_1, r_2, r_3, \ldots, r_i$, the set obtained D_{i+1} is not a dominating set. If the dominating set is not a failed dominating set , then it is a *winner dominating set*.

In the next section we prove that every graph has at least one safe 1 - secure set and give a characterization of the graphs in which all vertices excluding one is a safe vertex. We then classify the whole family of graphs into three classes based on the existence of safe 1 - secure set.

In the third section, we define a new kind of directed graphs which represent the transformation from one safe 1 - secure set to another safe 1 - secure set of a given graph.

2. Safe eternal 1 - security in graphs

If for any finite sequence of safe vertices $r_1, r_2, \ldots, r_k$, there exists a sequence of vertices $v_1, v_2, \ldots, v_k$ and a sequence $D = D_0, D_1, D_2, \ldots, D_k$ of safe 1-secure sets of G, which satisfy the conditions given above, then the same is true for any infinite sequence of safe vertices $r_1, r_2, \ldots,$ and vice versa. Next we proceed to prove that every graph has at least one safe 1 - secure set.

Theorem 2.1. *Every graph has at least one safe 1 - secure set.*

Proof. Let $x \in V$ be any leaf in a graph G. Then the set $D = V - \{x\}$ is a safe 1-secure set. If the graph G does not have any leaf, then we have the following cases.

Case 1: There is a vertex $v \in V$ such that all vertices in $N[v] - \{v\}$ are adjacent to a vertex in $V - N[v]$. Make D using exactly one vertex, which is in $N[v] - \{v\}$ and all the vertices in $V - N[v]$.

Case 2: There is no vertex $v \in V$ such that all vertices in $N[v] - \{v\}$ are adjacent to a vertex in $V - N[v]$. Since there is no leaf in G, for any $x \in V$, there exist some $y \in N(x)$ such that $N[y] \subseteq N[x]$. If there exists only one such vertex in $N(x)$ for some x, say w, then the set $D = (V - N[w]) \bigcup \{x\}$ is a safe 1-secure set.

Next assume that G is a graph such that for any $v \in V$, there exists two vertices $x_{v1}, x_{v2} \in N(v)$ and $N[x_{v1}], N[x_{v2}] \subseteq N[v]$. If x_{v1} and x_{v2} are adjacent, then both $(V - N[x_{v1}]) \bigcup \{x_{v1}\}$ and $(V - N[x_{v2}]) \bigcup \{x_{v2}\}$ are safe 1-secure sets.

Finally assume that G is a graph such that for any $v \in V$, there exists two vertices $x_{v1}, x_{v2} \in N(v)$, $N[x_{v1}], N[x_{v2}] \subseteq N[v]$ and x_{v1} and x_{v2} are not adjacent.

We claim that this case is impossible. Otherwise the inclusion described above is proper and this implies the existence of a proper chain of subsets of V; $V \supset N[x] \supset N[y] \supset N[z] \supset \ldots$, where $y \in N(x)$, $z \in N(y)$ etc. This contradicts the fact that the graph is finite. $\square$

Next theorem is a characterization of the graphs such that all safe 1 - secure sets are safe eternal 1 - secure sets and all vertices in $V - D$ are safe with respect to D, for any safe eternal 1 - secure set D.

Theorem 2.2. *Let D_0 be an eternal 1 - secure set of a graph G and the subsequent safe 1 - secure sets are $D_1, D_2, \ldots$. All vertices in $V - D_i$ are safe with respect to D_i, where $i = 0, 1, 2, \ldots$ if and only if G is a disjoint union of complete graphs.*

Proof. Let G be a disjoint union of complete graphs. We claim that any safe eternal 1 - secure set of G contains exactly one vertex from each component. Suppose, two vertices in a component are present in a safe eternal 1 - secure set. None of the vertices in that component is a safe vertex. So it is clear that any safe eternal 1 - secure set of G contains exactly one vertex from each component and all other vertices in each component are safe vertices.

To prove the converse, first we shall prove that if u, v and w are any three vertices in the graph such that the edges (u, v) and (v, w) are present in it, then (u, w) is also present. Suppose that (u, w) is not present in G. Now we have to consider the following two cases.

Case 1: The vertex v is a member of the safe eternal 1 - secure set D_0. Then u and w cannot be in D_0. Suppose that the vertex u is selected by the second player. The fist player modifies D_0 by replacing v by u to get the eternal 1 - secure set D_1. If the second player selects the vertex w subsequently, then $D_2 = (D_1 - \{u\}) \cup \{w\}$ must be an eternal 1 - secure set. Otherwise $|D_0 \cap N[w]| \geq 2$, which is a contradiction.

Case 2: The vertex $v \notin D_0$. Then there exists exactly one vertex $x \in N(v)$, such that $x \in D_0$. The possibility that x is either u or w is not ruled out. All vertices in $N(v)$ must be adjacent to x. If $x = u$ or $x = w$, we are done. Otherwise suppose that the second player selects the vertex v and consequently the first player selects a new dominating set which contains v. Since u and w are not in the new dominating set, by the steps of case 1, we can show that u and w are adjacent.

Thus whenever there are three vertices u, v and w, such that the edges (u, v) and (v, w) are present in the graph, then the edge (u, w) is also present in G. Thus if G is connected, then it is a complete graph. Otherwise it is a disjoint union of complete graphs. □

Now we define three different types of graphs, which form the basis of the classification of the family of all graphs. A graph G is an α_1 - *graph* if it has no safe eternal 1 - secure set. A graph is a β_1 - *graph* if it has at least one safe eternal 1 - secure set. A graph is a γ_1 - *graph* if every safe 1 - secure set of the graph is a safe eternal 1 - secure set. The following are examples of graphs for each type. Let G be a path having length two in which the vertex v is adjacent to u and w. The set $\{u, v, w\}$ is not a safe 1 - secure set. The set $D = \{u, v\}$ is a safe 1 - secure set. But after defending an attack at w, the new set obtained, that is $\{u, w\}$ not a safe 1 - secure set. The set $D = \{v, w\}$ gives a similar result. Next we have to consider all one vertex cases. The sets $\{u\}$ and $\{v\}$ are not safe 1 - secure sets. Even though $D = \{v\}$ is a safe 1 - secure set, after one attack, the resulting set is not a safe 1 - secure set.

Theorem 2.3. *Complete multipartite graphs $K_{r_1,r_2,\ldots r_n}$ where $r_i \geq 2$ for some i are in α_1 category.*

Proof. Let $G = K_{r_1,r_2,\ldots r_n}$ be a complete multipartite graph whose vertex set is partitioned in to V_1, $V_2, \ldots V_n$. Let D_0 be a safe 1 - secure set chosen by the first player and let $u \in V(G)$ be a safe vertex. Then $|N(u) \cap D_0| = 1$. We can relabel the partition so that $u \in V_1$ and $v \in V_2$. Since u is a safe vertex, $D_0 \bigcap (V - V_1 - \{v\}) = \phi$.

Now we have to consider two cases.

Case 1 : $|V_1| > 1$. We claim that, $x \in D_0$ for all $x \in V_1 - \{u\}$. Suppose not. Let $x \in V_1$ and $x \notin D_0$. The second player chooses the vertex u and subsequently the first player selects the set $D_1 = D_0 - \{v\} \cup \{u\}$. But this set is not a dominating set as the vertex x is not defended by any guard in D_1. Hence the claim is proved. Next we proceed by assuming that $|D_0 \bigcap (V_1 - \{u\})| = 1$. After u being chosen by the second player, $D_1 \bigcap V_1 = V_1$ and $D_1 \bigcap (V - V_1) = \phi$. There are only safe dominating vertices left in the graph. So the graph is in the α_1 category.

Case 2 : $|V_1| = 1$. We claim that $|V_i| = 1$ for all i. Suppose $|V_i| > 1$ for some $i > 1$. After the vertex u being selected by the second player, $D_1 \bigcap V_i = \phi$ for all $i > 1$. Subsequently if the second player chooses a vertex $x \in V_i$, from V_i for which $|V_i| > 1$. Then $D_2 = D_1 - \{u\} \cup \{x\}$ is not a dominating set. So the graph is in α_1 category. □

For a complete graph any dominating set containing exactly one vertex is a safe eternal 1- secure set and no other dominating set is a safe 1- secure set. So all complete graphs are in γ_1 - category.

Next we proceed to find a class of graphs which fall in β_1 - category. A member of this class is constructed using the graphs $G_1, G_2, \ldots, G_r$ such that $G_i = K_n$ where $n \geq 3$. Let $\mathfrak{T}$ be the class of all graphs obtained by using G_is, either fusing some vertices in $V(G_i)$ and $V(G_j)$ or adding edges between a vertex $V(G_i)$ and a vertex in $V(G_j)$ or applying both operations, such that there exists at least one vertex $v_i \in V(G_i)$ satisfying $N[v_i] \subseteq V(G_i)$, for each i.

Theorem 2.4. *If $G \in \mathfrak{T}$, then G is a β_1 - graph. Also, $\sigma_{s1}(G) = r$ where r is the number of complete graphs used to construct G.*

Proof. First player can choose $D_0 = \{v_1, v_2, ..., v_r\}$, where $v_i \in V(G_j)$ if $i = j$ and $v_i \notin V(G_i)$ if $i \neq j$. Second player can select any vertex which is not in D_0, say $r \in G_i$. Then $D_1 = (D_0 - \{v_i\}) \cup \{r\}$. Renaming r by v_i, we can repeat the game any number of times. Next suppose that there exists a safe eternal dominating set with $|D_0| = r - 1$. Let $D_0 = \{v_1, v_2, ..., v_{(r-1)}\}$. Then at least one vertex is common to G_i and G_j, where $i \neq j$.

We can place one guard exactly at one vertex in each G_i. This guard can move to defend any attack with in G_i. The presence of at least one vertex in every complete graph $v_i \in V(G_i)$ with the property $N[v_i] \subseteq V(G_i)$ ensures minimum one safe vertex in each G_i. This arrangement is the smallest possible. So we get $\sigma_{s1}(G) = r$. □

Another class of graphs $\mathfrak{G}$ is obtained by joining to some vertices in $G \in \mathfrak{T}$, the vertex having degree n in $K_{1,n}$, where $n \geq 2$ using an edge.

Theorem 2.5. *Let $G \in \mathfrak{G}$ be constructed using r complete graphs and the stars K_{1,n_1}, K_{1,n_2}, ... K_{1,n_t}. Then G is a β_1 - graph and $\sigma_{s1}(G) = r + \sum_i n_i$.*

Proof. Arrangement of guards in the clique subgraphs in G is done as in the pervious proof. In addition, we have to arrange guards at leaves in K_{1,n_i}, for all i. This arrangement is a safe eternal 1 security set. Thus the graph is in β_1 category. The number of guards required is $r + \sum_i n_i$. Next suppose that the graph has an arrangement of guards in its safe eternal 1 security set which contains less vertices than $r + \sum_i n_i$. If the guard absent is in a clique, then two connected cliques C_1 and C_2 must be guarded by a guard. Let v_1 and v_2 be the vertices in C_1 and C_2 respectively, where no other clique is joined. An attack either at v_1 or v_2 results into a non secure set. If a guard is absent in a star subgraph (say in K_{1,n_i}) of G, then one guard must be present at the vertex with degree n_i and two leaves must be vacant. If the enemy attacks one of the leaves, the guard at the center vertex must defend it. This makes the second vacant leaf un-dominated. So at least $r + \sum_i n_i$ guards are required. □

We have proved that one can easily find a safe one secure set in any graph. But for a safe one secure set to be a safe eternal one secure set is extremely difficult. This difficulty naturally reduces the possibility to exist graphs having the property that all safe one secure sets are safe eternal one secure sets. So we conjecture that the only class of graphs which is in the γ_1 category is K_n. Next we proceed to describe the three categories of graphs in an elegant way. The relationship of the three classes of graphs is given in the Figure 2.1.

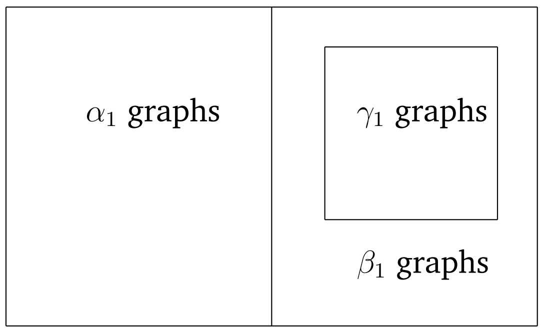

Figure 2.1

We denote the set of all subsets of V, excluding the empty set, by $\mathfrak{P}(G)$. The set of all safe one secure sets of G is denoted by $\mathfrak{S}(G)$. Clearly $\mathfrak{S}(G) \subseteq \mathfrak{P}(G)$. We can construct a directed graph with vertices equivalent to the elements in $\mathfrak{P}(G)$ and a directed edge from $S \in \mathfrak{S}(G)$ to $P \in \mathfrak{P}(G)$ if and only if it is possible to get P as the new arrangement of guards when a safe vertex in the set S is attacked. The graph thus obtained is named the *safe graph*. We denote the safe graph of a graph G by G_s. In the following section we study the general nature of the safe graph of a graph G.

3. Safe graph of a graph

The following properties of safe graphs are very clear.

- There is no directed edge from a vertex in $\mathfrak{P}(G) - \mathfrak{S}(G)$ to a vertex in $\mathfrak{S}(G)$.
- It is possible to exist directed edges in both directions between two vertices.
- For a given graph G, G_s is unique.
- The graph G_s is not in general connected because some graphs have safe 1 secure sets of different cardinality. Since rearrangements of guards do not alter their number, there cannot be a directed path connection between two vertices representing safe 1 secure sets of different cardinality. So, if a graph has safe 1 secure sets of different cardinality, then G_s is disconnected.

Lemma 3.1. *If there is a leaf S in G_s with in-degree, then $S \in \mathfrak{P}(G) - \mathfrak{S}(G)$.*

Proof. Since S a leaf in G_s with in-degree, it is clear that S is not a safe one secure set. Hence the lemma. □

Lemma 3.2. *If G has two safe 1 secure sets of different cardinality, then G_s is disconnected.*

Proof. In G_s, if S_1 and S_2 represent two safe 1 secure sets of different cardinality, then it is not possible to arrange guards in S_2 starting from S_1 or vice versa, defending a sequence of attacks at the vertices in G. Hence the result. □

Theorem 3.1. *A graph G is an α_1 graph if and only if there is no vertex s in G_s such that the induced subgraph of all the vertices reachable from s has no vertex, which is an element of $\mathfrak{P}(G) - \mathfrak{S}(G)$.*

Proof. Suppose that the graph G_s has a vertex s such that the induced subgraph of all the vertices reachable from s has no vertex, which is an element of $\mathfrak{P}(G) - \mathfrak{S}(G)$. Consider all the vertices which are reachable through directed paths from the vertex s. Name the set of all vertices on such paths by R. By our assumption,

$R \cap \mathfrak{P}(G) - \mathfrak{S}(G) = \emptyset$. Since $\mathfrak{S}(G)$ is a finite set, vertices in each directed path must repeat an infinite number of times. Thus some vertices in R make directed cycles in G_s. If two directed cycles have some vertices in common, then take the union of the cycles. Consider the biggest such subgraph C of G_s, which is the join of directed cycles. Let S be a safe one secure set corresponding to a vertex v in C. We can defend any sequence of attacks at the vertices in G arranging guards at the vertices in S. So S is an safe eternal one secure set and G is not an α_1 graph.

Conversely, if G is not an α_1 graph, then it is a β_1 graph. By definition there exists a safe eternal one secure set S. Then reversing the arguments in the first part of the proof, we get a vertex s which corresponds to S, such that such that the induced subgraph of all the vertices reachable from s has no vertex, which is an element of $\mathfrak{P}(G) - \mathfrak{S}(G)$. □

Theorem 3.2. *A graph G is a β_1 graph if and only if there is a connected induced subgraph of G_s, which has all vertices in $\mathfrak{S}(G)$.*

Proof. Suppose that the graph G_s has a safe 1 secure set. vertex s such that the induced subgraph of all the vertices reachable from s has no vertex, which is an element of $\mathfrak{P}(G) - \mathfrak{S}(G)$. Consider all the vertices which are reachable through directed paths from the vertex s. Name the set of all vertices on such paths by R. By our assumption, $R \cap \mathfrak{P}(G) - \mathfrak{S}(G) = \emptyset$. Since $\mathfrak{S}(G)$ is a finite set, vertices in each directed path must repeat infinite times. Thus some vertices in R make directed cycles in G_s. If two directed cycles have some vertices in common, then take the union of the cycles. Consider the biggest such subgraph C of G_s, which is the join of directed cycles. Let S be a safe one secure set corresponding to a vertex v in C. We can defend any sequence of attacks at the vertices in G arranging guards at the vertices in S. So S is a safe eternal one secure set and hence G is not an α_1 graph.

Conversely, if G is not an α_1 graph, then it is a β_1 graph. By definition, there exists a safe eternal one secure set S. Then reversing the arguments in the first part of the proof, we get a vertex s, which corresponds to S, such that the induced subgraph of all the vertices reachable from s has no vertex, which is an element of $\mathfrak{P}(G) - \mathfrak{S}(G)$. □

Theorem 3.3. *A graph G is a γ_1 graph if and only if there is no edge directed from a vertex in $\mathfrak{S}(G)$ to a vertex in $\mathfrak{P}(G) - \mathfrak{S}(G)$.*

Proof. Suppose that the graph G_s has an edge directed from a vertex $S \in \mathfrak{S}(G)$ to a vertex $P \in \mathfrak{P}(G) - \mathfrak{S}(G)$. Clearly S is not a safe eternal 1 secure set. Hence G is not a γ_1 graph.

Conversely, suppose that G is not a γ_1 graph. Then there exists at least one safe 1 secure set which is not safe eternal. So after defending a finite sequence of attacks we get an arrangement of guards, which does not correspond to a safe 1 secure set. □

4. Problems for further research

In this paper the idea of safe eternal 1 - security is introduced as a game of two persons. Lower and upper bounds of safe eternal 1 - security number are not known for classes of graphs. Actual values of the eternal security number is determined for only a few classes of graphs. Many classes of graphs are yet to be explored.

References

[1] A.P. Burger, E.J. Cockayne, W.R. Grundling, C.M. Mynhardt, J.H. Van Vuuren, W. Winterbatch: *Finite order dominating sets in graphs*, J. Combin. Math. Combin. Comput., **49** (2004), 159-176.

[2] A.P. Burger, E.J. Cockayne, W.R. Grundling, C.M. Mynhardt, J.H. Van Vuuren, W. Winterbatch: *Infinite order dominating sets in graphs*, J. Combin. Math. Combin. Comput., **50** (2004), 179-194.

[3] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi: *Eternal security in graphs, J. Combin. Math. Combin. Comput.,* **52** (2005), 160-180.

[4] T.W. Haynes, S.T. Hedetniemi, P.J. Slater: *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[5] W. Klostermeyer, G. MacGillivray: *Eternal security in graphs of fixed independence number*, J. Combin. Math. Combin. Comput., to appear.

[6] K. Reji Kumar, E.N. Satheesh: *On connected eternal domination*, Advances and Applications in Discrete Mathematics, **10**(2) (2012), 135-147.

[7] K. Reji Kumar: *Topological properties of the set of all minimal dominating of graphs*, Proceedings of ICDM 2006 published by Ramanujan Mathematical Society lecture notes series, 2008.

[8] K. Reji Kumar: *Convexity of minimal dominating functions and minimal edge dominating functions*, Complexity in the discrete structures, Ramanujan Mathematical Society Lecture Note Series, 2008.

[9] K. REJI KUMAR, GARY MACGILLIVRAY, R.B. BAPAT: *Topological properties of the set of all minimal total dominating functions of graphs*, J. Combin. Math. Combin. Comput. **76** (2011), 213-223.

[10] K. REJI KUMAR, GARY MACGILLIVRAY: *Structure of a set of all minimal total dominating functions of some classes of graphs*, Discuss. Math Graph Theory, **30**(3) (2010), 407-423.

[11] K. REJI KUMAR, S ARUMUGAM: *Basic minimal dominating functions*, Util. Math. **77** (2008), 235-247.

[12] K. REJI KUMAR, S ARUMUGAM, G. MACGILLVRAY: *On maximal fractional independent sets in graphs*, Ars Combin. **87** (2008), 205-212.

[13] K. REJI KUMAR: *Convexity graph of minimal dominating functions*, Bulletin of Kerala Mathematical Association, **2** (2007), 27-36.

[14] K. REJI KUMAR: *Studies in Graph Theory-Dominating functions*, Ph.D. Thesis, Manonmaniam Sundaranar University, Thirunelveli, India, 2004.

[15] D.B. WEST: *Graph Theory: An introductory course*, Prentice Hall, New York, 2002.

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