

## SUBSET BASED NON-ZERO COMPONENT UNION GRAPHS OF VECTOR SPACES

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**ABSTRACT.** In this paper, we introduce a graph structure, called a subset based non-zero component union graph on finite-dimensional vector spaces. We show that the graph is connected and find its girth, diameter, clique number and chromatic number. Further, we characterize the finite-dimensional vector spaces whose subset based non-zero component union graphs are planar, unicyclic, claw-free and bipartite.

### 1. INTRODUCTION

Algebraic structures are studying by using the graph theory properties is play the important role in last twenty years. Especially different kind of graphs from vector space are done namely, nonzero component graph of a finite dimensional vector space and non-zero component union graph of a finite-dimensional vector space are introduced and studied by A.Das in [1, 2]. Also, vector space semi-Cayley graphs was introduced by B. Tölue in [5]. Some of the authors are studied the graph theoretical properties of graphs from vector spaces, like T. Tamizh Chelvam, K. Prabha Ananthi found the genus of graphs associated with

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vector spaces in [4] and U. Ali, S. A. Bokhary studied the resolvability of a graph associated to a finite vector space in [6].

## 2. SUBSET BASED NON-ZERO COMPONENT UNION GRAPH ON FINITE-DIMENSIONAL VECTOR SPACES

Let  $\mathbb{V}$  be a vector space over a field  $\mathbb{F}$  with the basis  $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_n\}$ . Every elements  $v$  in  $\mathbb{V}$  can be written as linear combination of  $\beta_i$  where  $1 \leq i \leq n$ . (i.e.,  $v = a_1\beta_1 + a_2\beta_2 + \dots + a_n\beta_n$  here  $a_i$  are from the field  $\mathbb{F}$ .) Now the skeleton of  $v$  with respect to  $\mathcal{B}$  as  $K_{\mathcal{B}}(v) = \{\beta_i : a_i \neq 0, 1 \leq i \leq n\}$ . Let  $\mathcal{S}$  be the subset of  $\mathcal{B}$ . Also, We define skeleton of  $v$  with respect to  $\mathcal{S}$  as  $K_{\mathcal{S}}(v) = \{\beta_i \in \mathcal{S} : a_i \neq 0, 1 \leq i \leq n\}$ . Let  $0$  is the zero vector in  $\mathbb{V}$ . Now we define the graph called subset based non-zero component union graphs of vector spaces with vertex set is  $V = \mathbb{V} \setminus 0$  and two vertices are adjacent if and only if union of skeleton of thus two vertices with respect to  $\mathcal{S}$  is equal to  $\mathcal{S}$ , equivalently we say that the vertices  $u$  and  $v$  are adjacent if and only if  $K_{\mathcal{S}}(u) \cup K_{\mathcal{S}}(v) = \mathcal{S}$  and it is denoted by  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}}^{\mathbb{F}})$  simply  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ .

Now, let us recall basic definitions and notations about graphs. By a graph  $G = (V, E)$ , we mean a simple graph with non-empty vertex set  $V$  and edge set  $E$ . The number of elements in  $V$  is called order of  $G$  and the number of elements in  $E$  is called the size of  $G$ . A graph  $G$  is said to be complete if any pair of distinct vertices is adjacent in  $G$ . We denote the complete graph of order  $n$  by  $K_n$ . A graph  $G$  is bipartite if the vertex  $V$  can be partitioned into two disjoint subsets with no pair of vertices in one subset is adjacent. A star graph is a bipartite graph with any one of the partition containing a single vertex and the same is called as the center of the star. A graph  $G$  is connected if there exists a path between every pair of distinct vertices in  $G$ . The degree of the vertex  $v \in V$ , denoted by  $d(v)$ , is the number of edges in  $G$  which are incident with  $v$ . A graph  $G$  is said to be  $r$ -regular if the degree of all the vertices in  $G$  is  $r$ . The diameter of a connected graph is supremum of shortest distances between vertices in  $G$  and is denoted by  $\text{diam}(G)$ . The girth of  $G$  is defined as length of the shortest cycle in  $G$  and is denoted by  $\text{gr}(G)$ . We take  $\text{gr}(G) = \infty$  if  $G$  contains no cycles. For undefined terms in graph theory, A planar graph is a graph that can be embedded in the plane and the genus of planar graphs is zero. we refer [3].

We list out certain existing results which will be referred in this paper.

**Theorem 2.1.** ([2, Theorem 4.2]) *Let  $\mathbb{V}$  be an  $n$ -dimensional vector space over a finite field  $\mathbb{F}$  with  $q$  elements. Then  $\Gamma(\mathbb{V}_{\mathfrak{B}})$  is complete if and only if  $\mathbb{V}$  is one-dimensional or  $\mathbb{V}$  is two-dimensional and  $|\mathbb{F}| = 2$ .*

**Theorem 2.2.** ([3, Corollary 9.5.4])  *$K_5$  is nonplanar.*

**Theorem 2.3.** ([3, Lemma 9.10]) *A graph is planar if and only if it contains no subdivision of  $K_5$  or  $K_{3,3}$ .*

**Theorem 2.4.** ([7, Theorem 4.1]) *Let  $n \geq 1$  and  $q \geq 2$  be integers. Let  $\mathbb{V}$  be an  $n$ -dimensional vector space over the field  $\mathbb{F}$  with  $q$  elements. Then  $\Gamma(\mathbb{V}_{\mathfrak{B}})$  is planar if and only if either  $(n = 1 \text{ and } q \leq 5)$  or  $(n = 2 \text{ and } q = 2)$  or  $(n = 3 \text{ and } q = 2)$ .*

**Theorem 2.5.** ([7, Theorem 4.1]) *Let  $n \geq 1$  and  $q \geq 2$  be integers. Let  $\mathbb{V}$  be an  $n$ -dimensional vector space over the field  $\mathbb{F}$  with  $q$  elements. Then  $\Gamma(\mathbb{V}_{\mathfrak{B}})$  is claw-free if and only if either  $(n = 1)$  or  $(n = 2 \text{ and } q \leq 3)$ .*

### 3. GRAPH PROPERTIES OF $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathfrak{B}})$

In this section, we show that the graph  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathfrak{B}})$  is connected and found the basic properties like, diameter, grith, completeness and domination number.

**Theorem 3.1.**  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathfrak{B}})$  is connected for any subset  $\mathcal{S}$  of  $\mathfrak{B}$ .

*Proof.* The vertex set of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathfrak{B}})$  is  $\mathbb{V} \setminus \{0\}$ . If  $u$  and  $v$  are not adjacent then it is connected by the path  $u - w - v$  where  $w = \beta_1 + \beta_2 + \dots + \beta_n$ . Hence  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathfrak{B}})$  is connected.  $\square$

**Theorem 3.2.** If  $|\mathcal{S}_1| = |\mathcal{S}_2|$ , then  $\Gamma_{\mathcal{S}_1}(\mathbb{V}_{\mathfrak{B}}) \cong \Gamma_{\mathcal{S}_2}(\mathbb{V}_{\mathfrak{B}})$ .

*Proof.* Let  $\mathbb{V}$  be a vector space over a field  $\mathbb{F}$  with the basis  $\mathfrak{B} = \{\beta_1, \beta_2, \dots, \beta_n\}$ . Consider  $\mathcal{S}_1 = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  and  $\mathcal{S}_2 = \{\alpha'_1, \alpha'_2, \dots, \alpha'_r\}$  two subsets of  $\mathfrak{B}$ . Now we define the mapping  $F : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  such that  $f(\alpha_i) = \alpha'_i$  for  $i = 1, 2, \dots, r$ . One can see that function  $f$  is one to one and onto. we have to prove that  $\Gamma_{\mathcal{S}_1}(\mathbb{V}_{\mathfrak{B}}) \cong \Gamma_{\mathcal{S}_2}(\mathbb{V}_{\mathfrak{B}})$ . Let  $u = a_1\alpha_1 + a_2\alpha_2 + \dots + a_t\alpha_t + a_{t+1}\beta_{t+1} + \dots + a_n\beta_n$  and  $v = b_1\alpha_1 + b_2\alpha_2 + \dots + a'_t\alpha'_t + a'_{t+1}\beta_{t+1} + \dots + a_n\beta_n$  are adjacent in  $\Gamma_{\mathcal{S}_1}(\mathbb{V}_{\mathfrak{B}})$  i.e.,  $K_{\mathcal{S}_1}(u) \cup K_{\mathcal{S}_1}(v) = \mathcal{S}_1$ . To apply  $f$  to  $u$  and  $v$  we get,  $f(u) = a_1\alpha'_1 + a_2\alpha'_2 + \dots + a_t\alpha'_t + a_{t+1}\beta_{t+1} + \dots + a_n\beta_n$  and  $f(v) = b_1\alpha'_1 + b_2\alpha'_2 + \dots + a'_{t'}\alpha'_{t'} + a'_{t'+1}\beta_{t'+1} + \dots + a_n\beta_n$ . Hence,  $K_{\mathcal{S}_2}(u) \cup K_{\mathcal{S}_2}(v) = \mathcal{S}_2$  and  $f(u)$  and  $f(v)$  are adjacent in  $\Gamma_{\mathcal{S}_2}(\mathbb{V}_{\mathfrak{B}})$ .  $\square$

**Corollary 3.1.** *If  $|\mathcal{S}_1| < |\mathcal{S}_2|$ , then  $\Gamma_{\mathcal{S}_2}(\mathbb{V}_{\mathcal{B}})$  is spanning subgraph of  $\Gamma_{\mathcal{S}_1}(\mathbb{V}_{\mathcal{B}})$ .*

**Corollary 3.2.** *Let  $\mathbb{V}_1$  and  $\mathbb{V}_2$  be two finite-dimensional vector spaces over the same field  $\mathbb{F}$  having basis  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . The vector spaces  $\mathbb{V}_1$  and  $\mathbb{V}_2$  are isomorphic if and only if  $\Gamma_{\mathcal{S}_1}(\mathbb{V}_{\mathcal{B}_1})$  and  $\Gamma_{\mathcal{S}_2}(\mathbb{V}_{\mathcal{B}_2})$  are isomorphic where  $\mathcal{S}_1 \subseteq \mathcal{B}_1, \mathcal{S}_2 \subseteq \mathcal{B}_2$  and  $|\mathcal{S}_1| = |\mathcal{S}_2|$ .*

**Theorem 3.3.**  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is complete if, and only if, the following one of them holds:

- (1) every  $n$  with  $\mathcal{S}$  is empty;
- (2)  $n = 2$  and  $q = 2$ ;
- (3)  $n = 1$ .

*Proof.* Let  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is complete.

**Case i** Suppose  $n \geq 3$ . If  $|\mathcal{S}| \geq 3$ , then the vertices  $u = \beta_1$  and  $v = \beta_2$  are not adjacent, which is a contradiction. If  $|\mathcal{S}| = 2$ , there is an vertices  $u = \beta_i \in \mathcal{B} \setminus \mathcal{S}$  and  $v = \beta_j \in \mathcal{S}$  are not adjacent, which is a contradiction. If  $|\mathcal{S}| = 1$  there is an vertices  $u = \beta_i, v = \beta_j \in \mathcal{B} \setminus \mathcal{S}$  are not adjacent, which is a contradiction. If  $\mathcal{S}$  is empty, then  $K_{\mathcal{S}}(u) \cup K_{\mathcal{S}}(v) = \mathcal{S}$  for every  $u, v \in \Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ . Therefore for every  $n \geq 3$ ,  $\mathcal{S}$  must be empty.

**Case ii** Let  $n = 2$ , Suppose,  $\mathcal{S} = \mathcal{B}$  then by Theorem 2.1 the proof is hold.

Suppose,  $\mathcal{S} \neq \mathcal{B}$ .

If  $n = 2, |\mathcal{S}| = 1$  and  $q \geq 3$  consider the vertices  $u = \beta \notin \mathcal{S}$  and  $u = a\beta$  for some non zero  $a \neq 1 \in \mathbb{V}$ . One can seen that  $u$  and  $v$  are not adjacent for any subset  $\mathcal{S}$  of  $\mathcal{B}$ , which is a contradiction. If  $n = 2, |\mathcal{S}| = 1$  and  $q = 2$  in this case elements are  $u = \beta_1, v = \beta_2, w = \beta_1 + \beta_2$ . Hence  $u, v, w$  are adjacent for any subset  $\mathcal{S}$  of  $\mathcal{B}$ .

**Case iii** Suppose  $n = 1$  (i.e.,  $\mathcal{B} = \{\beta\}$ .) The possibility for  $|\mathcal{S}|$  is 1 and 0. If  $|\mathcal{S}| = 1$  by Theorem 2.1 the proof is hold.

Conversely,  $\mathcal{S}$  is empty then skeleton of all the elements in  $\mathbb{V}$  with respect to  $\mathcal{S}$  are empty. Hence the graph  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is complete.  $\square$

#### 4. MAXIMAL CLIQUE NUMBER OF $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$

In this section, we find out the maximal clique number of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ . Let  $\mathcal{S}$  arbitrary subset of  $\mathcal{B}$  with cardinality  $r$ . We define the set  $V_{\mathcal{S}} = \{v \in \mathbb{V} \setminus \{0\} \mid K_{\mathcal{S}}(v) = \mathcal{S}\}$  also, the cordiality of  $V_{\mathcal{S}}$  is  $q^n - q^r - 2$ . Now consider,  $U_i =$

$(\alpha_1 + \alpha_2 + \dots + \alpha_r) - \alpha_i$  where  $\alpha_i \in \mathcal{S}$  with  $1 \leq i \leq r$ . Let  $V'_{\mathcal{S}} = \{U_1, U_2, \dots, U_r\}$ . The sub graphs induced by  $V_{\mathcal{S}}$  and  $V'_{\mathcal{S}}$  are complete subgraph of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ .

**Theorem 4.1.** *If  $|\mathcal{S}| = r$  then, clique number of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is  $q^n - q^r + r - 2$ .*

*Proof.* Let  $\mathcal{M} = V_{\mathcal{S}} \cup V'_{\mathcal{S}}$ . Here,  $V_{\mathcal{S}}$  and  $V'_{\mathcal{S}}$  are complete subgraphs in  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ . Let  $u \in \mathcal{M}$  if  $u \in V_{\mathcal{S}}$  by definition  $u$  is adjacent to all the elements of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ , if  $u \in V'_{\mathcal{S}}$  by definition  $u$  is adjacent to all the elements of  $V_{\mathcal{S}}$ . This shows that  $\mathcal{M}$  is clique in  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ . Next we have to show that  $\mathcal{M}$  is maximum clique. Let any arbitrary  $v \in \Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}}) \setminus \mathcal{M}$ . Then  $|K_{\mathcal{S}}(v)| \leq r - 1$  and hence there exist  $U_i \in V'_{\mathcal{S}}$  such that  $K_{\mathcal{S}}(u) \subseteq K_{\mathcal{S}}(U_i)$  this implies that the two vertices  $v$  and  $U_i$  are not adjacent. Hence  $\mathcal{M}$  is the maximal clique and clique number of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is  $q^n - q^r + r - 2$ .  $\square$

**Theorem 4.2.** *Domination number of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is 1.*

*Proof.* Let  $\mathcal{S}$  arbitrary subset of  $\mathcal{B}$  with cardinality  $r$ . Hence the vertex  $v = \sum_{i=1}^r \alpha_i$  where  $\alpha_i \in \mathcal{S}$  is adjacent to all the elements of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$ .  $\square$

**Theorem 4.3.**  $\text{grith}(\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})) = \begin{cases} \infty, & \text{if } n = 1 \text{ and } q = 2 \text{ or } 3, \\ 3, & \text{otherwise.} \end{cases}$

*Proof.* Let  $n = 1$  then  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is complete of order  $q - 1$ . Suppose  $q = 2$  or  $3$  the graph  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is  $K_1$  or  $K_2$  respectively, thus there is no cycle and grith is infinity. If  $q \geq 4$  then the graph  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  have the subgraph  $K_3$ . Hence the grith of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is 3.

Let  $n \geq 2$ . Consider the three vertices  $v = \beta_1 + \beta_2 + \dots + \beta_n$ ,  $U_1 = \beta_2 + \beta_3 + \dots + \beta_n$  and  $U_2 = \beta_1 + \beta_3 + \dots + \beta_n$ . One can see that the above vertices are form a cycle of length 3 for any subset  $\mathcal{S}$  of  $\mathcal{B}$ .  $\square$

**Theorem 4.4.**  $\text{diam}(\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})) = \begin{cases} 1, & \text{if } \mathcal{S} = \emptyset \text{ (or) } n = 1 \text{ (or) } n = 2 \text{ and } q = 2, \\ 2, & \text{otherwise.} \end{cases}$

*Proof.* By Theorem 3.3 first case is hold. For  $n = 2$  and  $q \geq 3$  in this case, for every non-empty set  $\mathcal{S}$ , the vertices  $u = \beta_i \notin \mathcal{S}$  and  $v = a\beta_i$  are not adjacent for any nonzero  $a \neq 1 \in \mathbb{F}$ . But it is connected by the path  $u - w - v$  of length 2. Where,  $w = \beta_1 + \beta_2$ .

For  $n \geq 3$ . If  $|\mathcal{S}| \geq 3$ , for every non-empty set  $\mathcal{S}$  the vertices  $u = \beta_1$  and  $v = 2\beta_1$  are not adjacent but it is connected by the path  $u - w - v$  of length 2. Where,  $w = \beta_1 + \beta_2 \dots + \beta_n$ .

If  $|\mathcal{S}| = 2$ , there exist an elements  $u = \beta_i \in \mathcal{B} \setminus \mathcal{S}$  and  $v = \beta_j \in \mathcal{B}$  are not adjacent in  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  but it is connected by the path  $u - w - v$  of length 2. Where,  $w = \beta_1 + \beta_2 \dots + \beta_n$ .

If  $|\mathcal{S}| = 1$ , consider the vertices  $u = \beta_i \notin \mathcal{S}$  and  $v = a\beta_i$  are not adjacent for any nonzero  $a \neq 1 \in \mathbb{F}$ . But it is connected by the path  $u - w - v$  of length 2. Where,  $w = \beta_1 + \beta_2 \dots + \beta_n$ .  $\square$

## 5. PLANARITY

**Theorem 5.1.**  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is planar if and only if  $(n = 1 \text{ and } q \leq 5)$  or  $(n = 2 \text{ and } q = 2)$  or  $(n = 3, q = 2 \text{ and } \mathcal{S} = \mathcal{B})$ .

*Proof.* Let  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is planar then we have to prove either  $(n = 1 \text{ and } q \leq 5)$  or  $(n = 2 \text{ and } q = 2)$  or  $(n = 3, q = 2 \text{ and } \mathcal{S} = \mathcal{B})$ . Suppose,  $n \geq 4$  consider the vertices  $u = \beta_1 + \beta_2 + \beta_3 + \dots + \beta_n, v = \beta_2 + \beta_3 + \beta_4 + \dots + \beta_n, w = \beta_1 + \beta_3 + \beta_4 + \dots + \beta_n, x = \beta_1 + \beta_2 + \beta_4 + \dots + \beta_n$  and  $y = \beta_1 + \beta_2 + \beta_3 + \beta_5 + \dots + \beta_n$ . For any non-empty subset  $\mathcal{S}$  of  $\mathcal{B}$  the subgraph  $H$  induced by  $\Omega = \{u, v, w, x, y\}$  is  $K_5$ . By Theorem 2.3 gives the contradiction. Hence  $n \leq 3$

If  $n = 3, q \geq 2$  and  $\mathcal{S} = \mathcal{B}$  then by Theorem 2.4 gives the contradiction. Hence  $q$  must be 2 and  $|\mathcal{S}| \neq 3$ . If  $n = 3, q \geq 2$  and  $\mathcal{S} \neq \mathcal{B}$ . Suppose,  $|\mathcal{S}| = 2$  without loss of generality  $\beta_1$  and  $\beta_2$  in  $\mathcal{S}$ . Consider the vertices  $u = \beta_1, v = \beta_2, w = \beta_1 + \beta_2, x = \beta_1 + \beta_2 + \dots + \beta_n, y = \beta_1 + \beta_3, z = \beta_2 + \beta_3$ .  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  has a subgraph  $H$  induced by  $\Omega = \{u, v, w, x\}$  and  $H$  is isomorphic to  $K_4$  which is not an outer planar. But the vertices  $y, z$  are adjacent and  $N(y) \cup N(z) = \Omega$ , which is a contradiction.

Conversely, let  $n = 1$  and  $q \leq 5$ . Since by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is complete graph and order of the graph is  $q - 1 \leq 4$ . Hence  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is planar.

Let  $n = 2$  and  $q = 2$  Since by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is a complete graph and  $|\mathbb{V}| = q^n = 2^2 = 4$ . Hence, order of the graph is 3. Therefore,  $K_3 \cong \Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is planar.

Let  $n = 3, q = 2$  and  $\mathcal{S} = \mathcal{B}$ . By Theorem 2.4 gives the proof.  $\square$

**Theorem 5.2.**  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is unicyclic if and only if the following one of them holds:

- (1)  $n = 1$  and  $q = 4$ ;
- (2)  $n = 2$  and  $q = 2$ .

*Proof.* Let,  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is unicyclic. Suppose,  $n \geq 3$  then by Theorem 4.1 clique number of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}}) \geq 4$  for all possibilities of  $\mathcal{S}$ . Since,  $|V_{\mathcal{S}}| \geq 1$  and  $|V'_{\mathcal{S}}| \geq 3$ . Which gives the contradiction. Hence  $n \leq 2$ .

Suppose,  $n = 2$  and  $q \geq 3$  then  $|V_{\mathcal{S}}| \geq 3$  and  $|V'_{\mathcal{S}}| = 2$ . by Theorem 4.1 clique number of  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}}) \geq 5$  for all possibilities of  $\mathcal{S}$ . Which is a contradiction. Hence,  $q$  must be 2.

Suppose,  $n = 1$  and if  $q \geq 5$  then by Theorem 3.3 the graph  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is complete of order greater than 4. Which is a contradiction. If  $q = 2$  or 3 then  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is  $K_1, K_2$  respectively. Which is a contradiction. Since, there is no cycle in  $K_1$  and  $K_2$ .

Conversely, By Theorem 3.3 for the both case ( $n = 1$  and  $q = 4$ ) or ( $n = 2$  and  $q = 2$ ) the graph  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is a complete graph and its order is 3. This proves the theorem.  $\square$

**Theorem 5.3.**  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is claw-free if and only if the following one of them holds:

- (1)  $n = 1$ ;
- (2)  $n = 2$  and  $q \leq 3$ .

*Proof.* Let  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is claw-free. First we have to prove  $n \leq 2$ . Suppose  $n \geq 3$ . If  $|\mathcal{S}| = n$  then by Theorem 2.5 gives the contradiction. If  $0 < |\mathcal{S}| < n$  then there exist atleast one element  $\beta_i \in \mathcal{B} \setminus \mathcal{S}$  and  $\beta_j \in \mathcal{S}$  where  $i \neq j$ . Consider the set  $\Omega = \{u = \beta_i, v = \beta_j, w = \beta_i + \beta_j, x = \beta_1 + \beta_2, \dots, \beta_n\}$ . The graph induced by  $\Omega$  is isomorphic to  $K_{1,3}$  which is a contradiction. Hence,  $n \leq 2$ .

**Case i.** Let  $n = 2$ . Suppose  $q > 3$ , there exist atleast three non zero elements in  $\mathbb{F}$  namely  $a, b$  and  $c$ , we have the following cases

**Subcase i.** If  $|\mathcal{S}| = 1$ . Without loss of generality let  $\beta_1 \in \mathcal{S}$ . Consider the set  $\Omega = \{u = \beta_1, v = a\beta_2, w = b\beta_2, x = c\beta_2\}$ . The graph induced by  $\Omega$  is isomorphic to  $K_{1,3}$  which is a contradiction.

**Subcase ii.** If  $|\mathcal{S}| = 2$ . Consider the set  $\Omega = \{u = \beta_1, v = a\beta_2, w = b\beta_2, x = c\beta_2\}$ . The graph induced by  $\Omega$  is isomorphic to  $K_{1,3}$  which is a contradiction.

Hence,  $q \leq 3$ .

**Case ii.** Let  $n = 1$ , by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is a complete graph, hence it has no  $K_{1,3}$ .

Conversely, Let  $n = 1$  by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is a complete graph it is claw free.

Let  $n = 2$  and  $q \leq 3$ . Only possibilities for  $|\mathcal{S}|$  is 0 or 1 or 2. The following figure one can check that  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is claw-free.

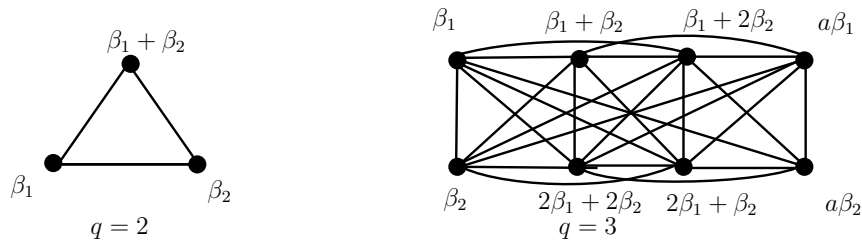


Fig A.  $n = 2$  and  $|\mathcal{S}| = 2$

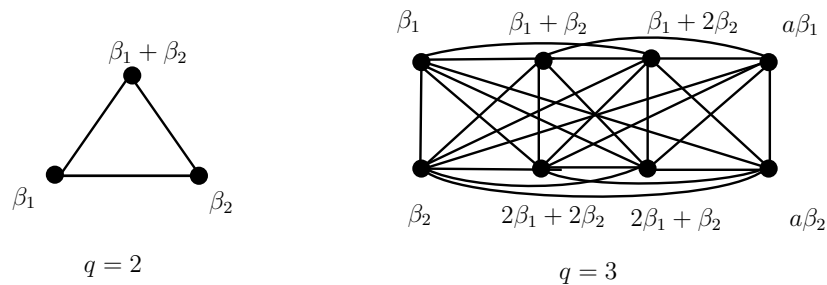


Fig B.  $n = 2$  and  $|\mathcal{S}| = 1$

□

**Theorem 5.4.**  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is bipartite if and only if  $n = 1$  and  $q = 3$ .

*Proof.* Let,  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is bipartite. Suppose  $n \geq 2$  then consider the vertices  $u = \beta_1 + \beta_2 + \cdots + \beta_{n-1}$ ,  $v = \beta_1 + \beta_2 + \cdots + \beta_{n-2} + \beta_n$  and  $w = \beta_1 + \beta_2 + \cdots + \beta_n$  form a cycle of length 3. Which is a contradiction.

Suppose,  $n = 1$  and  $q \geq 4$  then by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is a complete with order greater than 2. which is contradiction. Hence,  $q \leq 3$ . If  $q = 2$  then by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is  $K_1$ . Therefore  $q = 3$ .

Conversely, Let  $n = 1$  and  $q = 3$ . Then by Theorem 3.3  $\Gamma_{\mathcal{S}}(\mathbb{V}_{\mathcal{B}})$  is  $K_2$ .

□

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