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ON THE LIMITS OF SOME *p*-ADIC SCHNEIDER CONTINUED FRACTIONS

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ABSTRACT. In the present paper, we first generalize some convergence results for continued fractions given in real domain and p-adic domain. However, we prove the transcendence of a p-adic number given by it's Schneider continued fractions, such that the sequence of partial quotients is a Thue-Morse sequence.

1. INTRODUCTION AND STATEMENTS

Schneider continued fractions are defined as sequences of the shape:

$$\frac{p_n}{q_n} = a_0 + \cfrac{p^{\alpha_0}}{a_1 + \cfrac{p^{\alpha_1}}{a_2 + \cdots \cfrac{p^{\alpha_{n-1}}}{a_n}}} \tag{1.1}$$

where $(\alpha_i)_{i\in\mathbb{N}}$ is a sequence of positive integers and $(a_i)_{i\in\mathbb{N}}$ is a sequence of integers in $\{1,\ldots,p-1\}$.

In [7], M. Kojima proved the convergence of (1.1), both in $\mathbb{Q}_p$ and in $\mathbb{R}$ when $\alpha_i = 1$ for all $i \in \mathbb{N}$ (he proved indeed slightly more, considering that the coefficients $(a_i)_{i\in\mathbb{N}}$ could be any positive integers).

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Moreover, he proved the convergence in $\mathbb{Q}_p$ for all sequences of $(\alpha_i)_{i\in\mathbb{N}}$. However, he failed to prove a similar theorem for the convergence in $\mathbb{R}$.

In our paper, we first generalize its convergence result in $\mathbb{R}$, easily to bounded sequences of $(\alpha_i)_{i\in\mathbb{N}}$ (see theorem 2.1) and with much more tedious method to more general sequences (see theorem 2.2 up to theorem 2.5). Furthermore, we prove transcendence results concerning the limits in $\mathbb{Q}_p$ and in $\mathbb{R}$, when the sequence $(a_i)_{i\in\mathbb{N}}$ is a Thue-Morse sequence (see Main Theorem 1.1). The method is based on Schlickewei theorem [8] providing a suffisant condition of transcendence.

To state our results, we will recall some definitions and basic facts from p-adic numbers and words. Throughout, p is a prime number, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{Q}^*$ is the field of nonzero rational numbers and $\mathbb{R}$ is the field of real numbers. We use $|.|$ to denote the ordinary absolute value, v_p the p-adic valuation, $|.|_p$ the p-adic absolute value. The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value.

Let us introduce the combinatorics to be used in the sequel: Let the word $W = w_1w_2\ldots w_{m-1}w_m$ be on the alphabet A, we denote $|W|$ the length m of W. The mirror of W is the word $\overline{W} = w_mw_{m-1}\ldots w_2w_1$. We say that W is a palindrome if $\overline{W} = W$.

A Thue-Morse sequence $(t_n)_{n\in\mathbb{N}}$ with values in a two elements set $\{\alpha;\beta\}$ is defined by $t_n = \alpha$ (resp. β) if the binary expansion of n has an even (resp. odd) number of digits 1. We shall identify a sequence $(a_k)_{k\in\mathbb{N}}$ of elements of a given set A with an infinite word $a_0a_1\ldots a_k\ldots$ in A^*. A Thue-Morse sequence has numerous properties (see [2]). In the sequel, the following are used:

Theorem 1.1. *Let $(t_n)_{n\in\mathbb{N}}$ be a Thue-Morse sequence. Then the word $t_0t_1\ldots t_{4^k-1}$ is a palindrome and the two letters of the alphabet have the same number of occurrences.*

The transcendence method of Adamczewski and Bugeaud [1] is based on the Schmidt's subspace theorem. We make use the p-adic version of this theorem (see [8]). Let $\nu \geq 2$ be an integer, $\mathbf{x} = (x_1,\ldots,x_\nu)$ a $\nu-$tuple of rational numbers. We put $|\mathbf{x}|_\infty = \max\{|x_i|\,; 1 \leq i \leq \nu\}$ and $|\mathbf{x}|_p = Max\{|x_i|_p; 1 \leq i \leq \nu\}$.

Theorem 1.2 (Schlickewei)**.** *Let p be a prime number, $L_{1,\infty},\ldots,L_{\nu,\infty}$ be ν linearly independent forms with variable $\mathbf{x}$ and algebraic real coefficients, $L_{1,p},\ldots,L_{\nu,p}$ be ν linearly independent forms with algebraic p-adic coefficients and same variables* *and $\delta > 0$ a real number. Then, the set of solutions x in $\mathbb{Z}^\nu/\{0\}$ of the inequality*

$$\prod_{i=1}^{\nu}\left(|L_{i,\infty}(\mathbf{x})|\,.\,|L_{i,p}(\mathbf{x})|_p\right) \leq |\mathbf{x}|_\infty^{-\delta}$$

is contained in the union of a finite number of proper subspaces of $\mathbb{Q}^\nu$.

In [3], we have studied the periodicity of rational number given by its p-adic expansion. So, in [4], we have studied the transcendence of a p-adic number given by its Ruban continued fractions, such that the sequence of partial quetients is of Thue-Morse.

Theorem 1.3. *Let p be a prime odd positive integer. Let $\alpha = \frac{\alpha_1}{\alpha_2}$ and $\beta = \frac{\beta_1}{\beta_2}$ be two rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0;p)$ such that $v_p(\alpha_1) = v_p(\beta_1) = 0$ and $v_p(\alpha_2) \geq v_p(\beta_2) \geq 1$. Let θ be defined in $\mathbb{Q}_p$ as the limit of $[0; a_1, a_2, \ldots]$ where $a_i \in \{\alpha, \beta\}$. Suppose that the sequence of partial quotients $(a_i)_{i\geq 1}$ is a Thue-Morse word. Let us denote $\Xi = Max\{\alpha;\beta\}$. If*

$$p^{\frac{5v_p(\beta_2)-v_p(\alpha_2)}{6}} > Max\{\alpha_2;\beta_2\} \times \frac{\Xi + \sqrt{\Xi^2+4}}{2},$$

then, the p-adic number θ is either transcendental or quadratic.

Using the p-adic version of the subspace theorem, we give sufficient conditions for a number defined through a Schneider continued fraction to be quadratic or transcendental.

Main Theorem 1.1. *Let p be a prime odd positive integer. Let $((a_i, \alpha_i))_{i\in\mathbb{N}}$ with values in $\{1,\ldots,p-1\} \times \mathbb{N}^*$. We suppose the sequence $((\alpha_i))_{i\in\mathbb{N}}$ is bounded, and let $A = \max\{\alpha_i; i \in \mathbb{N}\}$. Let θ be defined in $\mathbb{Q}_p$ as the limit of*

$$a_0 + \cfrac{p^{\alpha_0}}{a_1 + \cfrac{p^{\alpha_1}}{a_2 + \cdots \cfrac{p^{\alpha_{n-1}}}{a_n \cdots}}}.$$

Suppose that the sequence of partial quotients $(a_i)_{i\geq 1}$ is a Thue-Morse word. then θ is either transcendental or quadratic.

2. Continued fractions

Definitions and results of this section are well known (see [6] for the real case and [5, 9] for the p-adic case), so we just sketch the proofs.

Definition 2.1. *From a sequence $((a_i, \alpha_i))_{i\in\mathbb{N}}$ with values in $\{1, \dots, p-1\} \times \mathbb{N}^*$, we define a sequence of homographic functions of a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}_p$ by*

$$[(a, \alpha); x] = a + \frac{p^{\alpha}}{x}$$

and

$$[(a_0, \alpha_0), (a_1, \alpha_1), \dots, (a_n, \alpha_n); x] = [(a_0, \alpha_0), \dots, (a_{n-1}, \alpha_{n-1}); [(a_n, \alpha_n); x]] .$$

We call $[(a_0, \alpha_0), (a_1, \alpha_1), \dots, (a_{n-1}, \alpha_{n-1}); a_n]$ the $n-th$ convergent of this sequence.

A matrix of the homographic function $a_k + \dfrac{p^{\alpha_k}}{x}$ is $\begin{pmatrix} a_k & p^{\alpha_k} \\ 1 & 0 \end{pmatrix}$. Hence, a matrix of the homographic function $[(a_0, \alpha_0), (a_1, \alpha_1), \dots, (a_n, \alpha_n); x]$ is

$$\begin{pmatrix} a_0 & p^{\alpha_0} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & p^{\alpha_1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & p^{\alpha_n} \\ 1 & 0 \end{pmatrix}.$$

Let us denote it $\begin{pmatrix} p_n & p'_n \\ q_n & q'_n \end{pmatrix}$. We have

$$a_0 + \cfrac{p^{\alpha_0}}{a_1 + \cfrac{p^{\alpha_1}}{a_2 + \cdots \cfrac{p^{\alpha_{n-1}}}{a_n + \cfrac{p^{\alpha_n}}{x}}}} = [(a_0, \alpha_0), (a_1, \alpha_1), \dots, (a_n, \alpha_n); x] = \frac{p_n x + p'_n}{q_n x + q'_n}.$$

The sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ satisfy both the following recurrence relation:

$$u_{n+2} = a_{n+2} u_{n+1} + u_n p^{\alpha_{n+1}},$$

with $p_{-1} = 1$, $p_0 = a_0$ and $q_{-1} = 0$, $q_0 = 1$. Moreover, we have $p'_{n+1} = p_n p^{\alpha_{n+1}}$ and $q'_{n+1} = q_n p^{\alpha_{n+1}}$. Hence we have:

$$a_0 + \cfrac{p^{\alpha_0}}{a_1 + \cfrac{p^{\alpha_1}}{a_2 + \cdots \cfrac{p^{\alpha_{n-1}}}{a_n}}} = \frac{p_{n-1} a_n + p'_{n-1}}{q_{n-1} a_n + q'_{n-1}} = \frac{p_n}{q_n}.$$

Given a p-adic number α, a question is to find a sequence $((a_i,\alpha_i))_{i\in\mathbb{N}}$ with values in $\{1,\ldots,p-1\}\times\mathbb{N}^*$ such that the sequence $\big([(a_0,\alpha_0),(a_1,\alpha_1),\ldots,(a_{n-1},\alpha_{n-1});a_n]\big)_{n\in\mathbb{N}}$ converges to α in $\mathbb{Q}_p$, with a unique solution. We shall concurently consider the convergence of this sequence in $\mathbb{R}$.

Let us consider the series $\sum_{i=1}^{i=n}\left(\frac{p_i}{q_i}-\frac{p_{i-1}}{q_{i-1}}\right)$. Using the easy to check property

$$p_iq_{i-1}-q_ip_{i-1}=(-1)^{i+1}\prod_{k=0}^{k=i-1}p^{\alpha_k}$$

it comes

$$\frac{p_n}{q_n}=\frac{p_0}{q_0}+\sum_{i=1}^{i=n}\frac{(-1)^{i+1}\prod_{k=0}^{k=i-1}p^{\alpha_k}}{q_iq_{i-1}}.$$

Lemma 2.1. *For all positive n, we have: $q_n\geq p^{\frac{n}{2}}$, et $p_n\geq p^{\frac{n}{2}}$*

Proof. Easy recursion on n. □

2.1. Convergence in $\mathbb{R}$.

Theorem 2.1. *If the set $\{\alpha_i; i\in\mathbb{N}\}$ is bounded, the sequence defined by*

$$[(a_0,\alpha_0),(a_1,\alpha_1),\ldots,(a_{i-1},\alpha_{i-1});a_i]$$

converges in $\mathbb{R}$.

Proof. We shall use the following notation

$$u_i=\frac{\prod_{k=0}^{k=i-1}p^{\alpha_k}}{q_iq_{i-1}}.$$

Let us use Leibniz criterium for alternating series. We have

$$\frac{u_i}{u_{i+1}}-1=\frac{a_{i+1}q_i}{p^{\alpha_i}q_{i-1}}>0.$$

Suppose that $A=\max\{\alpha_i; i\in\mathbb{N}\}$. Then $\frac{u_i}{u_{i+1}}-1\geq\frac{1}{p^A}>0$ and the proof is complete. □

In the sequel, we need the following lemma, this subsection we provide some insight convergence in other uses:

Lemma 2.2. *Let the sequence $(v_n)_{n\in\mathbb{N}}$ be defined by $v_{n+1}=ka^nv_n+v_{n-1}$ with $0<a\leq 1$ and $0<k\leq 1$. If $a<\frac{2}{k+\sqrt{k^2+4}}$, then the sequences v_{2n} and v_{2n+1} converge, and their limits are different.*

Proof. To show the convergence, let the sequence $(F_n)_{n\in\mathbb{N}}$ defined by

$$F_{n+1} = kF_n + F_{n-1}$$

with $F_0 = v_0$ et $F_1 = v_1$. Then, F_n is given by the formula

$$F_n = \frac{1}{2}(v_0 + \frac{2v_1 - kv_0}{\sqrt{k^2+4}}){\Phi_k}^n + \frac{1}{2}(v_0 - \frac{2v_1 - kv_0}{\sqrt{k^2+4}})(\frac{-1}{\Phi_k})^n,$$

with $\Phi_k = \frac{k+\sqrt{k^2+4}}{2}$. It is easy to show by induction that $v_n \leq F_n$. In the other hand, we have $v_{n-1} \leq v_{n+1} \leq ka^n F_n + v_{n-1}$ and

$$ka^n F_n \sim \frac{k}{2}(u_0 + \frac{2u_1 - ku_0}{\sqrt{k^2+4}})(a\Phi_k)^n.$$

The proof is complete where $\Phi_k < \frac{1}{a}$.

To show that the two limits are different, suppose that v_{2n} converg to ℓ and v_{2n+1} converg to ℓ', we have

$$F_{2n} = \frac{1}{2}(v_0 + \frac{2v_1 - kv_0}{\sqrt{k^2+4}}){\Phi_k}^{2n} + \frac{1}{2}(v_0 - \frac{2v_1 - kv_0}{\sqrt{k^2+4}})(\frac{1}{\Phi_k})^{2n}$$

and

$$F_{2n+1} = \frac{1}{2}(v_0 + \frac{2v_1 - kv_0}{\sqrt{k^2+4}}){\Phi_k}^{2n+1} - \frac{1}{2}(v_0 - \frac{2v_1 - kv_0}{\sqrt{k^2+4}})(\frac{1}{\Phi_k})^{2n+1}.$$

Then, for $\lambda = \frac{2v_1 - kv_0}{\sqrt{k^2+4}}$ we have

$$v_{2n} < \frac{1}{2}(v_0 + \lambda){\Phi_k}^{2n} + \frac{1}{2}(v_0 - \lambda)(\frac{1}{\Phi_k})^{2n}$$

and

$$v_{2n+1} < \frac{1}{2}(v_0 + \lambda){\Phi_k}^{2n+1} - \frac{1}{2}(v_0 - \lambda)(\frac{1}{\Phi_k})^{2n+1}.$$

It follows that

$$\frac{v_{2n}}{v_{2n+1}} < \Phi_k \frac{(v_0 + \lambda){\Phi_k}^{4n} + (v_0 - \lambda)}{(v_0 + \lambda){\Phi_k}^{4n+2} + (v_0 - \lambda)}.$$

Passing to the limit, we obtain $\ell < \frac{1}{\Phi_k}\ell' < \ell'$. $\square$

Example 1. *Suppose that for all* i*,* $a_i = 1$ *and* $\alpha_i = i$*. Then the sequence defined by* $[(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i]$ *does not converge in* $\mathbb{R}$*.*

Proof. We put $q'_i = \frac{q_i}{p^{\frac{i^2}{4}}}$, therefore $q'_{i+1} = ka^i q'_i + q'_{i-1}$, with $a = \frac{1}{p^{\frac{1}{2}}}$ and $k = \frac{1}{p^{\frac{3}{4}}}$, so with the previous lemma the sequence $(q'_{2i})_i$ converge to the limit ℓ and the sequence $(q'_{2i-1})_i$ converge to the limits ℓ', that is to say that for all given $\epsilon > 0$ from a certain rank we have

$$q_{2i} \leq (\ell+\epsilon)p^{\frac{(2i)^2}{4}} \textit{ and } q_{2i-1} \leq (\ell'+\epsilon)p^{\frac{(2i-1)^2}{4}}$$

Then

$$u_{2i} = \frac{p^{\frac{2i(2i-1)}{2}}}{q_{2i}q_{2i-1}} \geq \frac{p^{\frac{1}{4}}}{(\ell+\epsilon)(\ell'+\epsilon)}$$

thus u_{2i} do not converge to 0. $\square$

In the sequel, we shall use this notation:

$$A_{2n} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1}$$
$$A_{2n+1} = \alpha_0 + \alpha_2 + \cdots + \alpha_{2n}$$
$$q'_i = \frac{q_i}{p^{A_i}}.$$

Hence we have obviously $u_i = \frac{1}{q'_i q'_{i-1}}$, $A_{i+2} - A_i = \alpha_{i+1}$, $q'_0 = 1$, $q'_1 = a_1$, and

$$q'_{i+2} = a_{i+2}\frac{q'_{i+1}}{p^{A_{i+2}-A_{i+1}}} + q'_i.$$

Theorem 2.2. *Suppose that for all i, a_i in $\{1, \ldots, p-1\}$ and $\alpha_i = i$. Then the sequence $[(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i]$ does not converges in $\mathbb{R}$.*

Proof. We have $A_{2n} = 1 + 3 + \cdots + (2n-1) = n^2$, $A_{2n+1} = 2 + 4 + \cdots + (2n) = n(n+1)$, $A_{i+2} - A_i = i + 1$ and we have $A_{i+2} - A_{i+1} = \frac{i}{2} + 1$ if i is even, and $]A_{i+2} - A_{i+1} = \frac{i+1}{2} + 1$ if i is odd. It is easy to prove that $q'_i \leq F_i$ where F_i is the sequence defined by: $F_{n+2} = \frac{p-1}{p}F_{n+1} + F_n$ with $F_0 = 1$ and $F_1 = a_1$.

In the other hand we have

$$0 \leq q'_{i+2} - q'_i \leq \frac{p-1}{p^{A_{i+2}-A_{i+1}+1}}F_{i+1}.$$

So, for i even, we have

$$0 \leq q'_{i+2} - q'_i \leq \frac{p-1}{p^{\frac{i}{2}+1}}F_{i+1} \sim \frac{p-1}{\sqrt{p}}\Big(\frac{1}{\sqrt{p}}\Phi_{\frac{p-1}{p}}\Big)^n.$$

$\square$

Theorem 2.3. *Suppose that for all i, $a_i \in \{1, \ldots, p-1\}$, $\alpha_{2j} = 1$ for $i = 2j$, and $\alpha_{2j+1} = (j+1)^2$ for $i = 2j+1$. Then the sequence $\big[(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{i-1}, \alpha_{i-1}); a_i\big]$ converges in $\mathbb{R}$.*

Proof. We have

$$A_{2j} = 1 + 4 + \cdots + (2j)^2 = \frac{1}{6}j(j+1)(2j+1), A_{2j+1} = j + 1.$$

It is easy to prove that $A_{2j+1} - A_{2j} = \frac{j+1}{6}(-2j^2 - j + 6)$. Then, we have

$$q'_{2j+1} \leq p^{\frac{j+1}{6}(-2j^2-j+6)}.$$

Finally, we makes j tend to infinity. □

More generally, it is possible to prove in the same way the following results.

Theorem 2.4. *If for same $\epsilon > 0$ we have*

$$A_{i+2} - A_{i+1} > i(\epsilon + \log_p(\Phi_{\frac{p-1}{p}}))$$

Then the sequence $(\frac{p_i}{q_i})_{i\in\mathbb{N}}$ does not converge in $\mathbb{R}$.

Theorem 2.5. *If we have $A_{i+2} - A_{i+1} \leq \log_p(i)$ Then the sequence $(\frac{p_i}{q_i})_{i\in\mathbb{N}}$ converge in $\mathbb{R}$.*

Conjecture 2.1. *We conjecture that: For $\alpha_i = \lfloor\sqrt{i}\rfloor$, the sequence $(\frac{p_i}{q_i})_{i\in\mathbb{N}}$ does not converge in $\mathbb{R}$. For $\alpha_i = k\lfloor\log_p(i)\rfloor$, with $k \geq 1$, the sequence $(\frac{p_i}{q_i})_{i\in\mathbb{N}}$ converge in $\mathbb{R}$.*

In all the sequel, we denote $\theta_{\mathbb{R}}$ the limit in $\mathbb{R}$. In the proof of the transcendence theorem, we shall need the following lemma.

Lemma 2.3. *Under the hypothesis of the previous theorem, we have:*

$$|q_n\theta_{\mathbb{R}} - p_n| \leq \frac{\prod_{j=0}^{j=n-1} p^{\alpha_j}}{q_{n-1}}.$$

Proof. From the Leibniz criterium for alternating series, we have

$$\left|\theta_{\mathbb{R}} - \frac{p_n}{q_n}\right| \leq \left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}\right|.$$

□

2.2. Convergence in $\mathbb{Q}_p$.

Theorem 2.6. *The sequence $[(a_0, \alpha_0), (a_1, \alpha_1), \ldots, (a_{n-1}, \alpha_{n-1}); a_n]$ converges in $\mathbb{Q}_p$.*

Proof. An easy recursion on n shows that $val_p(q_n) = 0$. Hence, the series

$$\sum_{i=1}^{i=k} \frac{(-1)^{i+1} \prod_{k=0}^{k=i-1} p^{\alpha_k}}{q_i q_{i-1}},$$

have a limit in $\mathbb{Q}_p$. □

In all the sequel, we denote $\theta_{\mathbb{Q}_p}$ the limit in $\mathbb{Q}_p$. In the proof of the transcendence theorem, we shall need the following lemmas:

Lemma 2.4. *Under the hypothesis of the previous proposition, we have:*

$$\left|q_k\theta_{\mathbb{Q}_p} - p_k\right|_p = \frac{1}{p^{\alpha_0+\cdots+\alpha_k}}.$$

Proof. Suppose $k < n$, we have

$$\frac{p_n}{q_n} - \frac{p_k}{q_k} = \sum_{i=k}^{i=n-1} \frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} = \sum_{i=k}^{i=n-1} \frac{(-1)^{i+2} \prod_{j=0}^{j=i} p^{\alpha_j}}{q_i q_{i+1}}.$$

Hence we have

$$\left|\frac{p_n}{q_n} - \frac{p_k}{q_k}\right|_p = \frac{1}{p^{\alpha_0+\cdots+\alpha_k}} \leq \frac{1}{p^{k+1}}.$$

Then take the limit when n goes to infinity to get

$$\left|\theta - \frac{p_k}{q_k}\right|_p = |q_k\theta - p_k|_p = \frac{1}{p^{\alpha_0+\cdots+\alpha_k}}.$$

□

3. Proof of main theorem

Suppose θ is algebraic. Consider now the following linear forms with variable $\mathbf{x} = (x_1, x_2, x_3)$ and algebraic coefficients.

$$L_{1,\infty}(\mathbf{x}) = \theta_{\mathbb{R}}.x_1 - x_3, \quad L_{2,\infty}(\mathbf{x}) = \theta_{\mathbb{R}}.x_3 - x_2, \quad L_{3,\infty}(\mathbf{x}) = x_3,$$
$$L_{1,p}(\mathbf{x}) = \theta_{\mathbb{Q}_p}.x_1 - x_3, \quad L_{2,p}(\mathbf{x}) = \theta_{\mathbb{Q}_p}.x_3 - x_2, \quad L_{3,p}(\mathbf{x}) = x_1.$$

Evaluating them on the triple $\mathbf{x_n} = (q_n, p_{n-1}, p_n)$, with $n = 4^k - 1$. We get from Lemma 2.3, the inequality

$$|L_{1,\infty}(\mathbf{x_n})| < \frac{p^{\sum_{j=0}^{j=n-1} \alpha_j}}{q_{n-1}},$$

and from Theorem 1.3 and Lemma 2.3, the inequality

$$|L_{2,\infty}(\mathbf{x_n})| < \frac{p^{\sum_{j=0}^{j=n-2}\alpha_j}}{q_{n-2}}.$$

For the p-adic forms, we have from Lemma 2.4

$$|L_{1,p}(\mathbf{x_n})|_p = \frac{1}{p^{\sum_{j=0}^{j=n}\alpha_j}}$$

and

$$|L_{2,p}(\mathbf{x_n})|_p = \frac{1}{p^{\sum_{j=0}^{j=n-1}\alpha_j}}.$$

From the hypothesis of the theorem, wa have $A < n$ then $p_n < p^{n+1}$, for $n = 4^k - 1$ large enough.

$$|\mathbf{x_n}|^{\delta} \prod_{i=1}^{i=3} |L_{i,\infty}(\mathbf{x_n})| \prod_{i=1}^{i=3} |L_{i,p}(\mathbf{x_n})|_p < \frac{p_n^{\delta}}{p^{\alpha_{n-1}+\alpha_n} q_{n-2}} < \frac{p_n^{\delta}}{p^2 q_{n-2}} < \frac{(p^{n+1})^{\delta}}{p^{2+\frac{n}{2}}}$$

converges to 0 for $\delta < \frac{1}{2}$.

Schlickewei's theorem confirms the existence of non-zero rational integers y_1, y_2, y_3, such that, for an infinite set of n, we have

$$y_1 q_n + y_2 p_{n-1} + y_3 p_n = 0, \quad \text{i.e.,} \quad y_1 + y_2 \frac{p_{n-1}}{q_n} + y_3 \frac{p_n}{q_n} = 0.$$

So, $y_1 + y_2 \frac{p_{n-1}}{q_{n-1}} \frac{p_n}{q_n} + y_3 \frac{p_n}{q_n} = 0.$

Passing to the limit as $n \longrightarrow +\infty$ (in $\mathbb{Q}_p$), we obtain $y_2\theta_{\mathbb{Q}_p}^2 + y_3\theta_{\mathbb{Q}_p} + y_1 = 0.$ So, $\theta_{\mathbb{Q}_p}$ is quadratic.

Passing to the limit as $n \longrightarrow +\infty$ (in $\mathbb{R}$), we obtain $y_2\theta_{\mathbb{R}}^2 + y_3\theta_{\mathbb{R}} + y_1 = 0.$ So, $\theta_{\mathbb{R}}$ is quadratic.

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