NONLINEAR DIFFUSION EQUATION WITH A PERTURBED CONVECTIONTERM: POTENTIAL SYMMETRIES WITH RESPECT TO THE SECOND CONSERVATION LAW

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ABSTRACT. We consider the nonlinear diffusion equation with a perturbed convection term. The potential symmetries for the exact equation with respect to the second conservation law are classified. It is found that these exist only in the linear case. It is further shown that no nontrivial approximate potential symmetries of order one exists for the perturbed equation with respect to the other conservation law.

1. INTRODUCTION

Certain nonlinear PDEs that arise in applications depend on a small parameter. Thus it is of importance to find approximate solutions. Baikov *et al* ([1], [2], [3]) developed the theory and applications of the approximate symmetry group method to find approximate invariant solutions of DEs amongst other things.

We consider the nonlinear diffusion equation with a perturbed convection term, viz.,

$$(1.1) \qquad u_t = (k(u)u_x)_x + \varepsilon(f(u))_x,$$

where k and f are as yet arbitrary functions of u. This equation arises in several applications, e.g. in the modelling of the evolution of thermal waves in plasma [7].

The approximate potential symmetries with respect to the usual conservation law of equation (1.1) are obtained in Kara *et al* [6] and the corresponding approximate group-invariant solutions are also derived in the reference just cited.

Firstly, we investigate the conservation laws of equation (1.1) and this in turn is utilised to study the approximate potential symmetries of equation (1.1) with respect to the second conservation law.

2. CONSERVATION LAWS

We use the direct method (see e.g. Kara and Mahomed [5]) to derive the conservation laws of the perturbed equation (1.1). To that end, equation (1.1) has an obvious conserved vector T with components

$$T^1 = u, \quad T^2 = -k(u)u_x - \varepsilon f(u).$$

The other first-order in the derivatives conservation laws are determined from $D_t T^1 + D_x T^2 = 0$ which upon expansion and separation gives

$$T^1 = A(t,x)u + B(t,x),$$
$$T^2 = -k(u)u_x A(t,x) - \varepsilon f(u)A(t,x) + A_x \int k(u)du + C(t,x), \text{ where}$$

A, B and C satisfy

$$A_t u - \varepsilon f(u)A_x + A_{xx}\int k(u)du + B_t + C_x = 0.$$

An additional conservation law arises in the following cases:

(a) $k(u) \neq$ const., $f(u) = f_1 u + f_2$,
$$T^1 = (x + \varepsilon f_1 t)u,$$
$$T^2 = (-k(u)u_x - \varepsilon f_1 u - \varepsilon f_2)(x + \varepsilon f_1 t) + \int k(u)du + \varepsilon f_2 x,$$

(b) $k(u) \neq$ const., $f(u) = f_1 \int k(u)du + f_2 u + f_3$, $f_1 \neq 0$
$$T^1 = u\exp(\varepsilon f_1 x + \varepsilon^2 f_1 f_2 t),$$
$$T^2 = (-k(u)u_x - \varepsilon f_2 u)\exp(\varepsilon f_1 x + \varepsilon^2 f_1 f_2 t),$$

(c) linear case, $k(u) = k_0$, $f(u) = f_1 u + f_2$

$$T^1 = Au + B,$$
$$T^2 = -k_0 u_x A - \varepsilon(f_1 u + f_2)A + k_0 A_x u + C,$$

where A, B and C are constrained by

$$A_t - \varepsilon f_1 A_x + k_0 A_{xx} = 0,$$
$$-\varepsilon f_2 A_x + B_t + C_x = 0.$$

3. POTENTIAL SYMMETRIES WITH RESPECT TO THE SECOND CONSERVATION LAW

We calculate the potential symmetries with respect to the second conservation law of the unperturbed equation of equation (1.1), viz.,

(3.1) $$u_t = (k(u)u_x)_x.$$

The potential symmetries of equation (3.1) with respect to the usual conservation law are given in [4]. We use the results of the previous section. Equation (3.1) written in terms of the Case (a) conservation law of Section 2 is (take $\varepsilon = 0$) $D_t T^1 + D_x T^2 = 0$, where

$$T^1 = xu,$$
$$T^2 = -k(u)xu_x + \int k(u)du.$$

The associated auxiliary system $S\{x, t, u, v\}$ is given by

$$v_x = xu,$$
(3.2) $$v_t = xk(u)u_x - \int k(u)du.$$

Suppose $S\{x, t, u, v\}$ admits a local Lie group of transformations with infinitesimal generator

$$X = \xi^1(t, x, u, v)\frac{\partial}{\partial t} + \xi^2(t, x, u, v)\frac{\partial}{\partial x} + \eta^1(t, x, u, v)\frac{\partial}{\partial u} + \eta^2(t, x, u, v)\frac{\partial}{\partial v}$$

and in the extended form

$$\begin{aligned} X^{[1]} &= X + \varsigma_t^1 \frac{\partial}{\partial u_t} + \varsigma_x^1 \frac{\partial}{\partial u_x} + \varsigma_t^2 \frac{\partial}{\partial v_t} + \varsigma_x^2 \frac{\partial}{\partial v_x} \\ &= \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \varsigma_t^1 \frac{\partial}{\partial u_t} + \varsigma_x^1 \frac{\partial}{\partial u_x} + \varsigma_t^2 \frac{\partial}{\partial v_t} + \varsigma_x^2 \frac{\partial}{\partial v_x}, \end{aligned}$$

where the coefficients are given by

$$
\begin{aligned}
\varsigma_t^1 &= D_t\eta^1 - u_t D_t\xi^1 - u_x D_t\xi^2 \\
&= \eta_t^1 + u_t\eta_u^1 + v_t\eta_v^1 - u_t\xi_t^1 - (u_t)^2\xi_u^1 - u_t v_t\xi_v^1 - u_x\xi_t^2 \\
&\quad - u_x u_t\xi_u^2 - u_x v_t\xi_v^2,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\varsigma_x^1 &= D_x\eta^1 - u_t D_x\xi^1 - u_x D_x\xi^2 \\
&= \eta_x^1 + u_x\eta_u^1 + v_x\eta_v^1 - u_t\xi_x^1 - u_t u_x\xi_u^1 - u_t v_x\xi_v^1 - u_x\xi_x^2 \\
&\quad - (u_x)^2\xi_u^2 - u_x v_x\xi_v^2,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\varsigma_t^2 &= D_t\eta^2 - v_t D_t\xi^1 - v_x D_t\xi^2 \\
&= \eta_t^2 + u_t\eta_u^2 + v_t\eta_v^2 - v_t\xi_t^1 - v_t u_t\xi_u^1 - (v_t)^2\xi_v^1 - v_x\xi_t^2 \\
&\quad - v_x u_t\xi_u^2 - v_x v_t\xi_v^2,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\varsigma_x^2 &= D_x\eta^2 - v_t D_x\xi^1 - v_x D_x\xi^2 \\
&= \eta_x^2 + u_x\eta_u^2 + v_x\eta_v^2 - v_t\xi_x^1 - v_t u_x\xi_u^1 - v_t v_x\xi_v^1 - v_x\xi_x^2 \\
&\quad - v_x u_x\xi_u^2 - (v_x)^2\xi_v^2.
\end{aligned}
\tag{3.6}
$$

We have the following invariance criterion

$$
X^{[1]}\left(v_x - xu\right)\Big|_{\left\{v_x = xu,\, v_t = xk(u)u_x - \int k(u)du\right\}} = 0,
$$

$$
X^{[1]}\left(v_t - xk(u)u_x + \int k(u)du\right)\Big|_{\left\{v_x = xu,\, v_t = xk(u)u_x - \int k(u)du\right\}} = 0.
\tag{3.7}
$$

Equations (3.7) give rise to the system of determining equations

$$
\varsigma_x^2 - \xi^2 u - x\eta^1 = 0,
\tag{3.8}
$$

and

$$
\varsigma_t^2 - \xi^2 k(u)u_x - x\eta^1 k'(u)u_x + k(u)\eta^1 - xk(u)\varsigma_x^1 = 0,
\tag{3.9}
$$

on the solutions of (3.1). These equations (3.8) and (3.9) give rise to the system

$$
\xi_u^1 = 0,
$$
$$
\xi_u^2 - xk(u)\xi_v^1 = 0,
$$
$$
\eta_u^2 - xk(u)\xi_x^1 - x^2 uk(u)\xi_v^1 - xu\xi_u^2 = 0,
$$
$$
\eta_u^2 + xk(u)\xi_x^1 + x^2 uk(u)\xi_v^1 - xu\xi_u^2 = 0,
$$

$$\eta_x^2 + xu\eta_v^2 + \xi_x^1 \int k(u)du + xu\xi_v^1 \int k(u)du - xu\xi_x^2$$
$$-x^2u^2\xi_v^2 - \xi^2 u - x\eta^1 = 0,$$

$$xk(u)\eta_v^2 - xk(u)\xi_t^1 + 2xk(u)\xi_v^1 \int k(u)du - k(u)\xi^2$$
$$-xk'(u)\eta^1 - xk(u)\eta_u^1 + xk(u)\xi_x^2 = 0,$$

$$\eta_t^2 - \eta_v^2 \int k(u)du + \xi_t^1 \int k(u)du - \xi_v^1 \left(\int k(u)du\right)^2 - xu\xi_t^2$$

(3.10)
$$+xu\xi_v^2 \int k(u)du + k(u)\eta^1 - xk(u)\eta_x^1 - x^2uk(u)\eta_v^2 = 0.$$

The solution of the first four equations of (3.10) is straightforward and yields

$$\xi^1 = \alpha(t),$$
$$\xi^2 = \beta(t, x, v),$$
(3.11)
$$\eta^2 = \gamma(t, x, v).$$

The substitution of (3.11) into the fifth equation of (3.10) results in

(3.12)
$$\eta^1 = x^{-1}\gamma_x + u\gamma_v - u\beta_x - xu^2\beta_v - x^{-1}\beta u.$$

The sixth and seventh equations of (3.10), taking into account (3.11) and (3.12), then gives

$$k(u)[2x\beta_x - x\dot{\alpha} + 2x^2u\beta_v]$$
(3.13)
$$+k'(u)[-\gamma_x - xu\gamma_v + xu\beta_x + u\beta + x^2u^2\beta_v] = 0,$$

$$\gamma_t - \gamma_v \int k(u)du + \dot{\alpha} \int k(u)du - xu\beta_t + xu\beta_v \int k(u)du$$
(3.14)
$$+k(u)[2x^{-1}\gamma_x + u\gamma_v - \gamma_{xx} - 2ux\gamma_{xv} + xu\beta_{xx} - 2x^{-1}u\beta$$
$$+2x^2u^2\beta_{xv} - x^2u^2\gamma_{vv} + x^3u^3\beta_{vv} + xu^2\beta_v] = 0.$$

For arbitrary $k(u)$, the principal Lie algebra of the potential system (3.2) is obtained by analysing equations (3.13) and (3.14). Equation (3.13), for arbitrary $k(u)$, gives

$$\beta = \frac{\dot{\alpha}}{2}x, \quad \gamma = \dot{\alpha}v + b_1(t)$$

and (3.14) further results in

$$
\begin{aligned}
\alpha &= a_1 t + a_2, \\
\beta &= \frac{a_1}{2} x, \\
\gamma &= a_1 v + a_3,
\end{aligned}
$$

where the a_is are constants. Thus, the principal algebra is spanned by

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial v}, \\
(3.15) \qquad X_3 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}
\end{aligned}
$$

and is hence three-dimensional. We now investigate when the principal algebra extends. An extension of this algebra occurs if $k(u)$ satisfies

$$
(3.16) \qquad k(u)(a + 2bu) + k'(u)(c + du + bu^2) = 0,
$$

where a to d are constants not all zero. In order to obtain further simplification of (3.16) we look at equivalence transformations of equation (3.1).

4. Equivalence transformations

We write equation (3.1) as the system

$$
(4.1) \qquad u_t = k_u u_x^2 + k u_{xx},
$$

$$
(4.2) \qquad k_t = k_x = 0,
$$

in which u and k are dependent variables in the (t, x) and (t, x, u) spaces respectively. The generator of the group of equivalence transformations is

$$
\begin{aligned}
Y &= \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \varsigma_t \frac{\partial}{\partial u_t} + \varsigma_x \frac{\partial}{\partial u_x} \\
(4.3) \qquad &+ \mu \frac{\partial}{\partial k} + \mu^u \frac{\partial}{\partial k_u} + \mu^t \frac{\partial}{\partial k_t} + \mu^x \frac{\partial}{\partial k_x},
\end{aligned}
$$

where ς_t and ς_x are the usual prolongations as before and (we have set $k_t = k_x = 0$)

$$\mu^t = \tilde{D}_t\mu - k_u\tilde{D}_t\eta,$$
$$\mu^x = \tilde{D}_x\mu - k_u\tilde{D}_x\eta,$$
(4.4)
$$\mu^u = \tilde{D}_u\mu - k_u\tilde{D}_u\eta,$$

with the total derivatives given by

$$\tilde{D}_t = \frac{\partial}{\partial t} + k_t\frac{\partial}{\partial k} + \cdots,$$
$$\tilde{D}_x = \frac{\partial}{\partial x} + k_x\frac{\partial}{\partial k} + \cdots,$$
(4.5)
$$\tilde{D}_u = \frac{\partial}{\partial u} + k_u\frac{\partial}{\partial k} + \cdots.$$

The application of Y on (4.2) (subject to (4.1) and (4.2)) yields

$$\mu_t - k_u\eta_t = 0,$$
$$\mu_x - k_u\eta_x = 0.$$

These relations imply that

(4.6) $$\mu = \mu(u, k), \quad \eta = \eta(u).$$

The action of Y on equation (4.1), taking into account conditions (4.6), gives rise to the determining equation

(4.7)
$$\eta_u u_t - \xi_t^1 u_t - \xi_u^1 u_t^2 - \xi_t^2 u_x - \xi_u^2 u_t u_x - u_x^2(\mu_u + k_u\mu_k - k_u\eta_u)$$
$$2k_u u_x(\eta_u u_x - \xi_x^1 u_t - \xi_u^1 u_t u_x - \xi_x^2 u_x - \xi_u^2 u_x^2) - \mu u_{xx}$$
$$- \; k[\eta_{uu}u_x^2 + \eta_u u_{xx} - \xi_{xx}^1 u_t - 2\xi_{xu}^1 u_t u_x - \xi_{uu}^1 u_t u_x^2 - \xi_u^1 u_t u_{xx} - \xi_{xx}^2 u_x - 2\xi_{xu}^2 u_x^2$$
$$- \; \xi_{uu}^2 u_x^3 - \xi_u^2 u_x u_{xx} - 2\xi_x^1 u_{tx} - 2\xi_u^1 u_x u_{tx} - 2\xi_x^2 u_{xx} - 2\xi_u^2 u_x u_{xx}] = 0.$$

The substitution of equation (4.1) into equation (4.7) and separation results in the following system of equations

$$\xi_x^1 = \xi_u^1 = 0,$$
$$\xi_u^2 = 0,$$
$$\mu + k\xi_t^1 - 2k\xi_x^2 = 0,$$
$$2\xi_x^2 - \xi_t^1 - \mu_k = 0,$$
$$\mu_u + k\eta_{uu} = 0,$$
$$-\xi_t^2 + k\xi_{xx}^2 = 0.$$

(4.8)

The solution of system (4.8) is

$$
\begin{aligned}
\xi^1 &= c_1 + c_2 t, \\
\xi^2 &= c_3 + c_4 x, \\
\eta &= c_5 + c_6 u, \\
\mu &= k(2c_4 - c_2),
\end{aligned}
$$

(4.9)

where the c_is are constants. Hence, the generators of the equivalence group are

$$
\begin{aligned}
Y_1 &= \frac{\partial}{\partial t}, \\
Y_2 &= \frac{\partial}{\partial x}, \\
Y_3 &= t\frac{\partial}{\partial t} - k\frac{\partial}{\partial k}, \\
Y_4 &= x\frac{\partial}{\partial x} + 2k\frac{\partial}{\partial k}, \\
Y_5 &= \frac{\partial}{\partial u}, \\
Y_6 &= u\frac{\partial}{\partial u}.
\end{aligned}
$$

(4.10)

The one-parameter groups corresponding to each Y_i are (a_is are the group parameters)

$$\bar{t} = t + a_1, \bar{x} = x, \bar{u} = u, \bar{k} = k,$$
$$\bar{t} = t, \bar{x} = x + a_2, \bar{u} = u, \bar{k} = k,$$
$$\bar{t} = t, \bar{x} = x, \bar{u} = u + a_3, \bar{k} = k,$$
$$\bar{t} = t, \bar{x} = x, \bar{u} = u \exp a_4, \bar{k} = k,$$
$$\bar{t} = t \exp a_5, \bar{x} = x, \bar{u} = u, \bar{k} = k \exp(-a_5),$$

(4.11)
$$\bar{t} = t, \bar{x} = x \exp a_6, \bar{u} = u, \bar{k} = k \exp(2a_6).$$

The composition of the one-parameter groups (4.11) is the six-parameter equivalence group:

$$\bar{t} = t \exp a_5 + a_1, \bar{x} = x \exp a_6 + a_2,$$

(4.12)
$$\bar{u} = u \exp a_4 + a_3, \bar{k} = k \exp(2a_6 - a_5).$$

5. CLASSIFICATION

Under the equivalence transformations (4.12), equation (3.16) has the same differential structure, where the coefficients $\bar{a}$ to $\bar{d}$ are connected with a to d by the relations

$$
\begin{aligned}
\bar{a} &= (a + 2ba_3) \exp a_4, \\
\bar{b} &= b \exp(2a_4), \\
\bar{c} &= c + da_3 + ba_3^2, \\
\bar{d} &= (d + 2ba_3) \exp a_4.
\end{aligned}
$$

(5.1)

Relations (5.1) are used to obtain non-equivalent forms of k. Two cases arise. We consider each in turn.

1) $b \neq 0$

(i) If $d - a = 0$, then equation (3.16) takes the form

(5.2)
$$2ku + k_u(\delta + u^2) = 0, \quad \delta = \pm 1.$$

(ii) If $d - a \neq 0$, then equation (3.16) has the form

(5.3)
$$2k + k_u(\delta + u) = 0, \quad \delta = \pm 1,$$

or

(5.4) $2ku + k_u(\nu + \delta u + u^2) = 0, \quad \delta = \pm 1, \quad \nu = \text{const.} \neq 0.$

(2) $b = 0$

(i) If $d \neq 0$, then equation (3.16) has the form

(5.5) $$uk_u - \nu k = 0, \quad \nu = \text{const.}$$

(ii) If $d = 0$, $c \neq 0$, then equation (3.16) takes the form

(5.6) $$k_u - \delta k = 0, \quad \delta = \pm 1.$$

Thus, there are five forms for k each obtained by solving equations (5.2) to (5.6). In fact, we have the following:

(1) $k(u) = k_1(\delta + u^2)^{-1},$

(2) $k(u) = k_1(\delta + u)^{-2},$

(3) $k(u) = k_1 \exp[-\int \dfrac{2u}{\nu + \delta u + u^2} du], \quad \nu = \text{const.} \neq 0,$

(4) $k(u) = k_1 u^\nu, \quad \nu = \text{const.},$

(5) $k(u) = k_1 \exp(\delta u),$

where k_1 is a nonzero constant and $\delta = \pm 1$. We substitute each of the ks above into equations (3.13) and (3.14) to obtain an extension of the three-dimensional principal Lie algebra.

For Case (1), we do not obtain an extension of the principal algebra. Indeed if we substitute $k(u)$ of Case (1) into (3.13) we get

$$(\delta + u^2)[2x\beta_x - x\dot{\alpha} + 2x^2 u\beta_v] - 2u[-\gamma_x - xu\gamma_v + xu\beta_x + u\beta + x^2u^2\beta_v] = 0.$$

Separation and solution of the resultant equations give

$$\beta = \frac{\dot{\alpha}}{2}x, \quad \gamma = \dot{\alpha}v + a(t).$$

Then equation (3.14) constrains α and a to be

$$\alpha = c_1 t + c_2, \quad a = c_3.$$

Hence,

$$\begin{aligned}
\xi^1 &= c_1 t + c_2, \\
\xi^2 &= \tfrac{c_1}{2} x, \\
\eta^1 &= 0, \\
\eta^2 &= c_1 v + c_3,
\end{aligned}$$

which yields the operators for the principal algebra. Likewise Cases (2) and (3) also yield the principal algebra.

Extensions of the principal algebra occurs for Cases (4) and (5). We provide the extensions.

(4)

   (i) $k = k_1$

$$X_4 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$

$$X_5 = 4k_1 t^2 \frac{\partial}{\partial t} + 4txk_1 \frac{\partial}{\partial x} - (2v + x^2 u)\frac{\partial}{\partial u} - (x^2 - 2tk_1)\frac{\partial}{\partial v},$$

$$X_c = x^{-1}c_x \frac{\partial}{\partial u} + c\frac{\partial}{\partial v},$$

$$c_t + 2x^{-1}k_1 c_x - k_1 c_{xx} = 0.$$

   (ii) $k = k_1 u^{-2}$

$$X_4 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

   (iii) $k = k_1 u^{-1}$

$$X_4 = \frac{-x}{2}\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - k_1 t\frac{\partial}{\partial v}.$$

   (iv) $k = k_1 u^{-4/3}$

$$X_4 = 4t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u} + 3v\frac{\partial}{\partial v},$$

$$X_5 = x^2 \frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.$$

   (v) $k = k_1 u^\nu$, $\nu \neq 0, -1, -2, -4/3$

$$X_4 = -\nu t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

(5) $k = k_1 \exp(\delta u)$

$$X_4 = \delta t \frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{x^2}{2} \frac{\partial}{\partial v}.$$

Only for Case 4(i) which corresponds to the linear case do we obtain a non-trivial potential symmetry, viz., X_5. All the other symmetries in all the other cases give trivial potential symmetries.

In view of the preceding discussion, we have the following theorem.

Theorem 5.1. *Equation (3.1), with respect to the second conservation law, has nontrivial potential symmetry only for the linear case.*

The above results are quite different from those in [4].

6. APPROXIMATE POTENTIAL SYMMETRY WITH RESPECT TO THE SECOND CONSERVATION LAW

We now investigate the approximate potential symmetries of (1.1) with respect to the second conservation law given in Section 2. There are three cases to consider since there are three cases of conservation laws (see Section 2) for different ks and fs. We write the corresponding auxiliary system for each case.

(a) $k(u) \neq \text{const.}$, $f(u) = f_1 u + f_2$,

$$\begin{aligned}
v_x &= (x + \varepsilon f_1 t)u, \\
v_t &= (k(u)u_x + \varepsilon f_1 u + \varepsilon f_2)(x + \varepsilon f_1 t) - \int k(u)du - \varepsilon f_2 x.
\end{aligned}$$

(b) $k(u) \neq \text{const.}$, $f(u) = f_1 \int k(u)du + f_2 u + f_3$, $f_1 \neq 0$

$$\begin{aligned}
v_x &= u \exp(\varepsilon f_1 x + \varepsilon^2 f_1 f_2 t), \\
v_t &= (k(u)u_x + \varepsilon f_2 u) \exp(\varepsilon f_1 x + \varepsilon^2 f_1 f_2 t),
\end{aligned}$$

(c) linear case, $k(u) = k_0$, $f(u) = f_1 u + f_2$

$$\begin{aligned}
v_x &= Au + B, \\
v_t &= k_0 u_x A + \varepsilon(f_1 u + f_2)A - k_0 A_x u - C,
\end{aligned}$$

where A and B are constrained by

$$A_t - \varepsilon f_1 A_x + k_0 A_{xx} = 0,$$
$$-\varepsilon f_2 A_x + B_t + C_x = 0.$$

In Cases (a) and (b), it is clear that there is no nontrivial first-order approximate symmetry as the potential systems in each of these cases is of higher order than ε.

In Case (c), we choose the simplest system, viz. we set $k_0 = 1$, $f_1 = 1$, $f_2 = 0$ and $A = A(x)$. We find that

$$v_x = \exp(\varepsilon x) u,$$
$$v_t = (u_x + \varepsilon u) \exp(\varepsilon x) + \varepsilon u \exp(\varepsilon x).$$

This potential system also has higher than one ε terms. In fact, in general, Case (c) gives potential systems which are of higher order than ε. Hence, in Case (c) there is no nontrivial first-order approximate potential symmetry.

The results obtained here are distinct from those that appear in [6].

In view of our discussions, we can state the following result.

Theorem 6.1. *The perturbed equation (1.1), with respect to the second conservation law, has no nontrivial first-order approximate potential symmetry.*

7. CONCLUDING REMARKS

We have shown that the exact equation (3.1) with respect to the second conservation law has no nontrivial potential symmetry except in the linear case. This is quite distinct from the potential symmetries obtained for the same equation in [4] with respect to the usual conservation law. Then finally, we showed that no first-order approximate symmetry for equation (1.1) with respect to the second conservation law exists. This can easily be contrasted with the results obtained in [6] - they are distinct.

References

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