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EXCELLENCE ON NEIGHBORHOOD DISTINGUISHING COLORING

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ABSTRACT. Given a simple graph G, a Neighborhood distinguishing coloring (ND-coloring) [4] is a proper coloring of vertices such that the codes for the vertices determined by the color partition are distinct for distinct vertices. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a proper color partition of a simple graph G. Fixing this order of π, for each $u \in V(G)$, we assign a code denoted by $C(u)$ $(or\ C_\pi(u))$ as $C(u) = \{|N(u) \cap V_i|, i = 1, 2, \ldots, k\}$. Then π is called a Neighborhood Distinguishing Coloring partition (abbreviated as NDC-partition) if $C(u) \neq C(v)$ for all distinct $u, v \in V$. Also, the Graph G is called a Neighborhood Distinguishing Coloring Graph(NDC-Graph). The minimum cardinality of a neighborhood distinguishing coloring partition of a graph G is called "Neighborhood Distinguishing Coloring Number of G" and it is denoted by $\chi_{NDC}(G)$. In this papar the Excellence of Neighborhood distinguishing coloring graphs has been studied.

1. INTRODUCTION

Irregular coloring(ir-coloring) has been introduced by Mary Radcliffe and Ping Zhang in [2,3]. If $c : V(G) \to \{1, 2, \ldots, k\}$ is a proper coloring of a graph G where k is a positive integer, then the color code of vertex v of G with respect to c is the ordered $(k+1) - tuple = (a_0, a_1, a_2, \ldots, a_k)$ where a_0 is the color

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assigned to v $(ie., c(v))$ and a_i is the number of vertices adjacent to v which are colored with i, $1 \leq i \leq k$. c is called irregular coloring if distinct vertices have distinct color codes with respect to c. The irregular chromatic number denoted by $\chi_{ir}(G)$ is the minimum positive integer k for which G has an irregular $k - coloring$. For any simple graph G, ir-coloring exists but ND-coloring is not guaranteed. For example, $K_{1,n}$ does not admit ND-coloring. It has been proved in [4] that ND-coloring exists in a graph G if and only if the neighborhoods of any two non-adjacent vertices of G are distinct.

2. MAIN RESULTS

Definition 2.1. ***Excellent NDC-Graphs*** *Let G be a simple graph. A vertex $u \in V(G)$ is said to be χ_{NDC} - good if there exists a χ_{NDC} - partition π such that $\{u\} \in \pi$. Otherewise, u is said to be χ_{NDC} - bad. G is said to be χ_{NDC} - excellent if every vertex of G is χ_{NDC} - good.*

Example 1. (i) *K_n is χ_{NDC}-excellent.*

(ii) *nK_2 is $\chi_{NDC} - excellent$.*

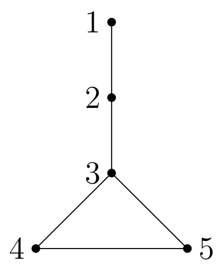
(iii) *K_{r,sK_2} is $\chi_{NDC} - excellent$.*

(iv) *Consider C_9. Let $\pi = \{\{1,3\},\{2,4\},\{5,7\},\{6,8\},\{9\}\}$. Then π is a χ_{NDC}-partition of C_9. Since C_9 is regular, we can obtain χ_{NDC}-partitions with any vertex appearing as a singleton. Hence C_9 is χ_{NDC}-excellent.*

(v) *C_5 is χ_{NDC}-excellent, since $\pi = \{\{1,3\},\{2,4\},\{5\}\}$ is a χ_{NDC}-partition of C_5 and it is regular.*

(vi) *There are values for n such that C_n is not $\chi_{NDC} - excellent$. For example, C_{16}, C_{20} are not $\chi_{NDC} - excellent$.*

(vii) *Let G =*



$\pi_1 = \{\{1,5\},\{2,3\},\{4\}\}$ and $\pi_2 = \{\{1,4\},\{3,5\},\{2\}\}$ are NDC-partitions of $V(G)$. Therefore, $\chi_{NDC}(G) \leq 3$. $\chi_{NDC}(G)$ is always greater than

or equal to 2. If $\chi_{NDC}(G) = 2$ then G is bipartite. Here, G is not bipartite as G contains a triangle. Therefore, $\chi_{NDC}(G) = 3$. There is no χ_{NDC}-partition containing $\{3\}$, since the possible partitions containing $\{3\}$ are $\{\{1,5\},\{2,4\},\{3\}\}$ and $\{\{1,4\},\{2,5\},\{3\}\}$. Both are not χ_{NDC}-partitions. Therefore, G is not χ_{NDC} - excellent.

Theorem 2.1. *Let $G(\neq K_2)$ be a graph with $\chi_{NDC}(G) = 2$. Then G is not χ_{NDC} - excellent.*

Proof. Since $\chi_{NDC}(G) = 2$, G is bipartite and either the partite sets are equal or they differ in cardinality exactly by 1.

Case (i): Let V_1 and V_2 be the partite sets of G. Let $|V_1| = |V_2|$. If $|V(G)| \geq 4$ then any χ_{NDC} - partition of $V(G)$ contains an element of cardinality $n-1(n>4)$ if that partition contains a singleton. This is not possible since any $n-1$ element set of $V(G)$ is not independent. Therefore, $|V(G)| \leq 2$. Since $V(G)$ is even, $|V_1| = |V_2| = 1$, $G = K_2$, a contradiction.

Case (ii): Let $|V_2| = |V_1|+1$. Then V_2 contains an isolate and every element u_i of V_1 as well as v_j of V_2, $1 \leq i,j \leq k$ has $k-i+1$ neighbors. Since $\chi_{NDC}(G) = 2$, any χ_{NDC} - partition of $V(G)$ containing a singleton will contain an elements of cardinality $n-1(n \geq 2)$. If $n \geq 3$ then the isolated vertex cannot appear as a singleton.

Therefore, from case(i) and case(ii) G is not χ_{NDC} - excellent if $G \neq K_2$. □

Remark 2.1.

(i) *$\chi_{NDC} - excellent$ graphs have no isolates.*

(ii) *Let T be a tree such that each support vertex has exactly one pendant vertex. Then, T is not $\chi_{NDC} - excellent$.*

Proof. Let T be a tree satisfying the coditions of hypothesis. Let v be a pendant vertex of u in T. Let w be adjacent with u. Since w is not a pendant vertex, w is adjacent with some vertex say $y(\neq v)$ in T. Let $\pi = \{\{v,w\},\{u\},\{y\},\ldots\}$. The codes of u, v and w with respect to π are,

$C_\pi(u) = (2,0,0,\ldots)$,
$C_\pi(v) = (0,1,0,\ldots)$,
$C_\pi(w) = (0,1,1,\ldots)$.

Let z be any vertex. Since w is not a pendant vertex of u, w is adjacent with some vertex say y. Therefore, w has a 1 in some place of π. The code of z with respect to $\{v, w\}$ is either 1 or 0 according as z is adjacent with w or not. If z is adjacent with w then the code of z with respect to $\{v, w\}$ is 1 and hence different from the code of u, v and w with respect to $\{v, w\}$. If z is not adjacent with w, then the code of z with respec to $\{v, w\}$ is 0. If z is adjacent with u then z has code 1 with respect to $\{u\}$ and hence different from the codes of u, v and w with respect to first two places. If z is not adjacent with u, then the code of z in the first two places are 0 and hence different from u, v and w. Thus, if there exists a NDC-partition in which $\{v\}$ appears then there exists another NDC-partition with lesser cardinality in which v is combined with some other vertex. Thus in any χ_{NDC}-partition of T, v does not belong to the partition as a singleton. Therefore, T is not χ_{NDC}-excellent. □

Remark 2.2. *Let $G = tK_2$, $t \geq 2$. Then G is $\chi_{NDC} - excellent$ and $\delta(G) = 1$. There also exist connected graphs like P_5, P_9 and P_{13} which are $\chi_{NDC} - excellent$ and in these graphs $\delta(G) = 1$.*

Theorem 2.2. *Suppose G_1 and G_2 are two graphs such that $\chi_{NDC}(G_1 \cup G_2) = Max\{\chi_{NDC}(G_1), \chi_{NDC}(G_2)\}$. Then, $G_1 \cup G_2$ is not $\chi_{NDC} - excellent$.*

Proof. Suppose $G_1 \cup G_2$ is $\chi_{NDC} - excellent$. Let $v \in V(G_1)$. Then, there exists a $\chi_{NDC} - partition$ of $G_1 \cup G_2$ such that $\{v\}$ is an element of that partition. Let $\pi = \{\{v\}, V_2, \ldots, V_k\}$ be a $\chi_{NDC} - partition$ of $G_1 \cup G_2$ where $k = \chi_{NDC}(G_1 \cup G_2) = \chi_{NDC}(G_2)(say)$. Let $\pi_2 = \{V_2 - V(G_1), \ldots, V_k - V(G_1)\}$. Then π_2 is a $\chi_{NDC} - partition$ of G_2, a contradiction, since $\chi_{NDC}(G_2) = k$. Therefore, $G_1 \cup G_2$ is not $\chi_{NDC} - excellent$. □

Conjecture 1. *P_n is $\chi_{NDC} - excellent$ for $n \geq 5$.*

Remark 2.3. *Let G be a graph with $\chi_{NDC}(G) > \frac{n}{2}$. Then any χ_{NDC}-partition of G contains a singleton element.*

Proof. Given $\chi_{NDC}(G) > \frac{n}{2}$. If each element of a χ_{NDC}-partition contains atleast 2 elements, then the total number of vertices is greater than n, a contradiction. □

Theorem 2.3. *Let G be a vertex transitive graph with $\chi_{NDC}(G) > \frac{n}{2}$. Then G is $\chi_{NDC} - excellent$.*

Proof. From the previous remark, there exists a χ_{NDC}-partition containing a singleton say$\{u\}$. since G is vertex transitive, for any v in $V(G)$ there exists a

χ_{NDC}-partition such that $\{v\}$ is an element of that partition. Hence the theorem. $\square$

Theorem 2.4. *If $\chi_{NDC}(G) = n$ then G as well as $\mu(G)$ are $\chi_{NDC} - excellent$.*

Proof. Since $\chi_{NDC}(G) = n$, every vertex is a singleton in the unique χ_{NDC}-partition of G. Since $\chi_{NDC}(\mu(G)) = n + 1$, any vertex u' as well as v in $\mu(G)$ can be made to lie as a singlton in a χ_{NDC}-partition of $\mu(G)$. $\square$

Remark 2.4. *Since $\chi_{NDC}(K_n)$, $\chi_{NDC}(nK_2)$ and $\chi_{NDC}(K_{r,sK_2})$ are equal to the order of the respective graphs, $\chi_{NDC}(\mu(G))$ is χ_{NDC}-excellent when G is any one of these graphs.*

Theorem 2.5. *Let G be a graph of order $n \geq 2$ admitting NDC. Then, G^+ is not $\chi_{NDC} - excellent$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(G^+) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$. Suppose v'_i appears as a singleton in a χ_{NDC}-partition of G. For any v_j, v_k, $\{v_j, v_k\}$ cannot appear in any χ_{NDC}-partition of G^+, since in that case v'_j and v'_k will have the same code. Hence no χ_{NDC}-partition can contain v'_i as a singleton. Thus, G^+ is not $\chi_{NDC} - excellent$. $\square$

Theorem 2.6. *$K_{m,m} - 1F$ is $\chi_{NDC} - excellent$.*

Proof. Let $G = K_{m,m} - 1F$. Let $V(G) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m\}$ and $E(G) = \{u_iv_j, 1 \leq i, j \leq m\} - \{u_iv_i, 1 \leq i \leq m\}$. We know that, $\chi_{NDC}(G) = m + 2$, theorem 3.3.1. Let $\pi = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{m-2}, v_{m-2}\}, \{u_{m-1}\}, \{v_{m-1}\}, \{u_m\}, \{v_m\}\}$. Then, π is a χ_{NDC}-partition of G. Hence u_{m-1},u_m,v_{m-1} and v_m are χ_{NDC}-good vertices in G. Since G is symmetric, it follows that every vertex of G is χ_{NDC}-good. Therefore, G is χ_{NDC}-excellent. $\square$

Theorem 2.7. *Let $m \geq 4$. H_m is not χ_{NDC}-excellent.*

Proof: Let H_m be the graph obtained by $Hajós$ construction from two copies of K_m. Let the vertex set of one of the copies be $\{u_1, u_2, \ldots, u_m\}$ and that of the other be $\{v_1, v_2, \ldots, v_m\}$. Let u_1 and v_1 be identified. Remove u_1u_m and v_1v_m. Join u_m and v_m. Give color C_1 to u_1 and u_m, color C_i to u_i, $2 \leq i \leq m-1$. Give the colors C_2 to C_m to v_2 to v_m. Let $\pi = \{\{u_1\}, \{u_2, v_2\}, \{u_3, v_3\}, \ldots, \{u_{m-1}, v_m\}, \{u_m, v_{m-1}\}\}$ be a partition of $V(H_m)$. Let u_1 be adjacent with u_3 and v_3(without loss of generality). Consider $\{u_3, v_3\}$. $C_\pi(u_3) = (1, 1, 0, 1, \ldots, 1, 1) = C_\pi(v_3)$.

Hence π is not a NDC-partition. Therefore, identified vertex can not appear as a singleton in a NDC-partition of H_m. Therefore, H_m is not χ_{NDC}-excellent. $\square$

Remark 2.5. *H_3 is C_5 and H_2 is K_2 which are χ_{NDC}-excellent.*

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