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PAIRWISE α-PARACOMPACT SUBSETS.

Mouna Benbettiche[1] and Hassan Z. Hdeib

ABSTRACT. In this paper we define pairwise-α-paracompact subsets as a generalisation of α-paracompact subsets in topological spaces and study their properties and its relations with other bitopological spaces. Several theorems are obtained concerning pairwise-α-paracompact sets. We introduced the definition of p-paracompact bitoplogical spaces and study their properties and its relations with other bitopological spaces.

1. INTRODUCTION

The study of bitopological spaces was initiated by Kelly (1963, [5]). A bitopological space is a triple (X, τ_1, τ_2) where X is a non-empty set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular (p-regular), pairwise normal (p-normal) and obtained generalizations of several standard results such as Urysohn's lemma and Tietze extension theorem. Fletcher in [3] gave the definitions of $\tau_1\tau_2$-open and p-open covers in bitopological spaces. A cover $\underset{\sim}{U}$ of the bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open if $\underset{\sim}{U} \subset \tau_1 \cup \tau_2$, if in addition, $\underset{\sim}{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2, it is called p-open. He also defined pairwise paracompact bitopological

[1] *corresponding author*

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space (X, τ_1, τ_2), if it is p-Hausdörrf space and every τ_1-open cover has a τ_2-open, τ_2-locally finite refinement and every τ_2-open cover has a τ_1-open, τ_1-locally finite refinement, also (Datta, 1979), (Raghavan and Reilly, 1986), (Ganster and Reilly, 1992), (Hdeib and Fora, 1982).

Aull (1966) introduced the notion of α-paracompact subsets as a generalization of compact subsets. A subset M of a topological space (X, τ) is α-paracompact (σ-paracompact) if every open cover by members of τ has an open locally finite (σ-locally finite) refinement by members of τ.

2. Results

Definition 2.1.

(a) *A bitopological space (X, τ_1, τ_2) is called τ_1-paracompact with respect to τ_2 if every τ_1-open cover of X has a τ_2-locally finite τ_2-open cover.*

(b) *A bitopological space (X, τ_1, τ_2) is called B-paracompact if it is τ_1-paracompact with respect to τ_2 and τ_2-paracompact with respect to τ_1.*

Definition 2.2.

(a) *A subset K of a bitopological space (X, τ_1, τ_2) is said to be τ_1-α-paracompact with respect to τ_2 if every τ_1-open cover of K has a τ_2-locally finite, τ_2-open refinement.*

(b) *A subset K of a bitopological space is said to be B-α-paracompact if it is τ_1-α-paracompact with respect to τ_2 and τ_2-α-paracompact with respect to τ_1.*

Theorem 2.1. *Let (X, τ_1, τ_2) be a τ_1-paracompact with respect to τ_2, bitopological space . Let K be τ_1-closed subset of X then K is τ_1-α-paracompact with respect to τ_2.*

Proof. Let $\underset{\sim}{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be a τ_1-open of K. Then $\underset{\sim}{U}^* = \{U_\alpha \mid \alpha \in \Lambda\} \cup \{X - K\}$ is a τ_1-open cover of X. Hence $\underset{\sim}{U}^*$ has τ_2-locally finite, τ_2-open refinement say H'.

Let $H^* = \{H \in H' \mid H \cap K \neq \phi\}$. Then H^* is a τ_2-open, τ_2-locally finite refinement of $\underset{\sim}{U}$. Hence the result. $\square$

Theorem 2.2. *Let (X, τ_1, τ_2) be τ_2-paracompact with respect to τ_1, bitopological space. Let K be a τ_1-closed subset of X then K is τ_2-α-paracompact with respect to τ_1.*

The proof follows by a similar method used in the Theorem (2.1).

Corollary 2.1. *Let (X, τ_1, τ_2) be a B-paracompact space. Let A be τ_1-closed, τ_2-closed subset of (X, τ_1, τ_2) , then A is a B-α-paracompact.*

It follows as a result of Theorem (2.1) and (2.2).

Theorem 2.3. *If every proper τ_1-closed subset K of a bitopological space (X, τ_1, τ_2) is τ_1-α-paracompact with respect to τ_2, then (X, τ_1, τ_2) is τ_1-paracompact with respect to τ_2.*

Definition 2.3. *(Swart, 1971) A cover $\underset{\sim}{U}$ for a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open cover if $\underset{\sim}{U} \subseteq \tau_1 \cup \tau_2$.*

Definition 2.4. *(Fletcher, 1969) A cover $\underset{\sim}{U}$ for a bitopological space (X, τ_1, τ_2) is called p-open cover if it contains at least one non empty member of τ_1 and at least one non empty member of τ_2.*

Clearly, a $\tau_1\tau_2$-open cover for a bitopological space is finer than a p-open cover for it.

Definition 2.5. *[2] A pairwise open cover V of a bitopological space (X, τ_1, τ_2) is said to be parallel refinement of a pairwise open cover U of X if for each $i = 1, 2$, every τ_i-open set of V is in some τ_i-open set of U.*

Definition 2.6. *[4]*

(a) *A family $\underset{\sim}{A}$ of subsets of a bitopological space (X, τ_1, τ_2) is said to be p_1-locally finite if $\underset{\sim}{A} \cap \tau_i$ is locally finite in $(X, \tau_i),\ i = 1, 2$.*

(b) *A family $\underset{\sim}{A}$ of subsets of a bitopological space (X, τ_1, τ_2) is said to be p_2-locally finite if $\underset{\sim}{A} \cap \tau_i$ is locally finite in (X, τ_j) for each $i \neq j,\ i = 1, 2$.*

Definition 2.7. *[4] A space (X, τ_1, τ_2) is said to be pairwise paracompact of type one (denoted by p_1-paracompact) if every pairwise open cover (p-open cover) of X has a p_1-locally finite $\tau_1\tau_2$-open parallel refinement.*

Definition 2.8. *[4] A space (X, τ_1, τ_2) is said to be pairwise paracompact of type two (denoted by p_2-paracompact) if every pairwise open cover (p-open cover) of X has a p_2-locally finite $\tau_1\tau_2$-open parallel refinement.*

Theorem 2.4. *Let (X, τ_1, τ_2) be a bitopological space. Then the following are equivalent:*

(a) *Every τ_1-closed proper subset of X is τ_2-α-paracompact with respect to τ_1 and every τ_2-closed proper subset of X is τ_1-α-paracompact with respect to τ_2.*

(b) *(X, τ_1, τ_2) is p_1-paracompact.*

Proof.

$(a) \Rightarrow (b)$

Let $\underset{\sim}{U} = \{U_\beta \mid \beta \in \Gamma\} \bigcup \{V_\alpha \mid \alpha \in \Lambda\}$ be a p-open cover of X where U_β is τ_1-open, $\forall\ \beta \in \Gamma$ and V_α is τ_2-open, $\forall \alpha \in \Lambda$.

We have two cases:

(i) If $\bigcup_{\alpha\in\Lambda} V_\alpha = X$ then choose $\beta_0 \in \Gamma$ s.t. $U_{\beta_0} \neq \phi$. Since $\{V_\alpha \mid \alpha \in \Lambda\}$ is τ_2-open cover of $X - U_{\beta_0}$ and $X - U_{\beta_0}$ is a τ_1-closed subset of X which is τ_2-α-paracompact with respect to τ_1, $\{V_\alpha \mid \alpha \in \Lambda\}$ has a τ_1-locally finite τ_1-open refinement say $\{V'_\alpha \mid \alpha \in \Lambda\}$. Now $\{V'_\alpha \mid \alpha \in \Lambda\} \bigcup \{U_{\beta_0}\}$ is $\tau_1\tau_2$-open p_1-locally finite refinement of $\underset{\sim}{U}$.

(ii) If $\bigcup_{\alpha\in\Lambda} V_\alpha \neq X$ then $K = X - \bigcup_{\alpha\in\Lambda} V_\alpha$ is a τ_2-closed proper subset of X which is by assumption is τ_1-α-paracompact with respect to τ_2. Since $K \subset \bigcup_{\beta\in\Gamma} U_\beta$ we get $\{U_\beta \mid \beta \in \Gamma\}$ has a τ_2-open τ_2-locally finite refinement, say $\{U'_\beta \mid \beta \in \Gamma\}$.

If $\bigcup_{\beta'\in\Gamma} U_{\beta'} = X$ there is nothing to prove but if $\bigcup_{\beta'\in\Gamma} U_{\beta'} \neq X$ then $X - \bigcup_{\beta'\in\Gamma} U_{\beta'}$ is a τ_1-closed subset of X contained in $\bigcup_{\alpha\in\Lambda} V_\alpha$, so by the assumption there is a τ_1-open τ_1-locally finite refinement say $\{V'_\alpha \mid \alpha \in \Lambda\}$ of $\{V_\alpha \mid \alpha \in \Lambda\}$. Therefor $\{U'_\beta \mid \beta \in \Gamma\} \bigcup \{V'_\alpha \mid \alpha \in \Lambda\}$ is a $\tau_1\tau_2$-open p_1-locally finite refinement of $\underset{\sim}{U}$. Hence the result.

$(b) \Rightarrow (a)$

Let $K \neq \phi$ be a τ_1-closed proper subset of X and $\underset{\sim}{U} = \{V_\alpha \mid \alpha \in \Lambda\}$ is a τ_2-open cover of K. $\underset{\sim}{V} = \{X - k\} \bigcup \{V_\alpha : \alpha \in \Lambda\}$ is a p-open cover of X, since (X, τ_1, τ_2) is p_1-paracompact, $\underset{\sim}{V}$ has p_1-locally finite $\tau_1\tau_2$-open refinement say $\{V_\beta \mid \beta \in \Gamma\} \bigcup \{U'_\alpha \mid \alpha \in \Lambda\}$ where $V_\beta \in \tau_1, \forall\ \beta$ and $U'_\alpha \in \tau_2, \forall\alpha$. Hence $\{U'_\alpha \mid \alpha \in \Lambda\}$ is τ_2-open τ_2-locally finite refinement of $\underset{\sim}{U}$. Hence K is τ_2-α-paracompact with respect to τ_1.

Similary we can show that every τ_2-closed proper subset of X is τ_1-α-paracompact with respect to τ_2. □

Theorem 2.5. *Let (X, τ_1, τ_2) be a p-T_2-bitopological space. Let K be a τ_1-α-paracompact w.r.t. τ_2 and $x \notin K$. Then there is a τ_2-open set V contaning K and $x \notin \overline{V}^{\tau_2}$, In particular K is τ_2-closed.*

Proof. Let $x \notin K$, then $\forall\ y \in K$ there is a τ_1-open set V_y containing y, $x \notin \overline{V_y}^{\tau_2}$. Now $\{V_y \mid y \in K\}$ is a τ_1-open cover of K, so it has a τ_2-open, τ_2-locally finite refinement say $\underset{\sim}{H}$. Let $V = \bigcup\{H \mid H \in \underset{\sim}{H}\}$.

Let $\overline{V}^{\tau_2} = \bigcup\{\overline{H}^{\tau_2} \mid H \in \underset{\sim}{H}\}$. Since $\overline{H}^{\tau_2} \subset \overline{V_y}^{\tau_2}$ we get $x \notin \overline{V}^{\tau_2}$. □

Theorem 2.6. *Let (X, τ_1, τ_2) be a p-T_2-bitopological space. Let K be a τ_2-α-paracompact w.r.t. τ_1 and $x \notin K$. Then there is a τ_1-open set V contaning K and $x \notin \overline{V}^{\tau_1}$, In particular K is τ_1-closed.*

The proof follows by a similar method used in the Theorem (2.5).

Corollary 2.2. *Let (X, τ_1, τ_2) be a p-T_2-bitopological space. Let K be a B-α-paracompact subset of X. Then K is τ_1-closed and τ_1-open subset of X.*

Theorem 2.7. *Let (X, τ_1, τ_2) be a τ_1-regular w.r.t. τ_2 bitopological space. Let $x \in U$ where U is τ_1-open then $\exists\ \tau_1$-open set V s.t. $x \in V \subset \overline{V}^{\tau_2} \subset U$.*

Proof. Let $x \notin X - U$ and $X - U$ is τ_1-closed. Then there is a τ_1-open set V contaning x and a τ_2-open set W contaning $X - U$ and $X - W \subset U$. Since $V \subset X - W$ and $\overline{V}^{\tau_2} \subset X - W \subset U$ therfore $x \in V \subset \overline{V}^{\tau_2} \subset U$. □

Theorem 2.8. *Let (X, τ_1, τ_2) be a τ_2-regular w.r.t. τ_1 bitopological space. Let $x \in U$ where U is τ_2-open then there is a τ_2-open set V s.t. $x \in V \subset \overline{V}^{\tau_1} \subset U$.*

The proof follows by a similar method used in the Theorem (2.7).

Theorem 2.9. *Let (X, τ_1, τ_2) be a τ_2-regular w.r.t. τ_1, bitopological space. Let M be a τ_1-α-paracompact w.r.t. τ_2. Then $\overline{M}^{\tau_1}$ is τ_1-α-paracompact w.r.t. τ_2.*

Proof. Let $\underset{\sim}{U}$ be a τ_1-open cover of $\overline{M}^{\tau_1}$. Then there is a τ_2-open, τ_2-locally finite refinement of $\underset{\sim}{U}$ that covers M say $\{V_\alpha\}$. For each $x \in V_\alpha$, $\exists\, W_{x\alpha}$ s.t. $x \in W_{x\alpha} \subset \overline{W}^{\tau_1}_{x\alpha} \subset V_\alpha$. Then $\{W_{x\alpha}\}$ is a τ_2-open cover of M and has a τ_1-open, τ_1-locally finite refinement $\{H_b\}$. $\overline{M}^{\tau_1} \subset \bigcup \overline{H_b}^{\tau_1} \subset \bigcup V_\alpha$, we get $\{V_\alpha\}$ is τ_2-open, τ_2-locally finite cover of $\overline{M}^{\tau_1}$. Hence $\overline{M}^{\tau_1}$ is τ_1-α-paracompact w.r.t. τ_2. □

Theorem 2.10. *Let (X, τ_1, τ_2) be a τ_1-regular w.r.t. τ_2, bitopological space. Let M be a τ_2-α-paracompact w.r.t. τ_1. Then $\bar{M}^{\tau_2}$ is τ_2-α-paracompact w.r.t. τ_1.*

The proof follows by a similar method used in the Theorem (2.9).

Definition 2.9. *A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be a p-continuous (p-closed respectively) if $f : (X, \tau_1) \longrightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \longrightarrow (Y, \sigma_2)$ are continuous (closed respectively) functions.*

Theorem 2.11. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a p-continious, p-closed function s.t. $\forall\, y \in Y, f^{-1}(y)$ is a p_1-α-paracompact subset of X, then X is p_1-paracompact if Y is so.*

Proof. Let $\underset{\sim}{U}$ be a p-open cover of X. Then $\underset{\sim}{U}$ has p_1-locally finite p-open parallel refinement that covers $f^{-1}(y)$, $\forall\, y \in Y$, i.e. $f^{-1}(y) \subset (\bigcup_{\alpha\in\Lambda_1} V_\alpha) \bigcup (\bigcup_{\alpha\in\Lambda_2} V'_\alpha)$ where $\{V_\alpha \mid \alpha \in \Lambda_1\}$ is τ_1-open τ_1-locally finite and $\{V'_\alpha \mid \alpha \in \Lambda_2\}$ is τ_2-open τ_2-locally finite.

Let $O_y = Y - f(X - \bigcup_{\alpha\in\Lambda_1} V_\alpha)$, $O'_y = Y - f(X - \bigcup_{\alpha\in\Lambda_2} V'_\alpha)$, then $f^{-1}(O_y) \subset \bigcup_{\alpha\in\Lambda_1} V_\alpha$ and $f^{-1}(O'_y) \subset \bigcup_{\alpha\in\Lambda_2} V'_\alpha$.

Then $\underset{\sim}{O} = \{O_y \mid y \in Y\} \bigcup \{O'_y \mid y \in Y\}$ is a p-open cover of Y. Since Y is a p_1-paracompact, $\underset{\sim}{O}$ has a p-open p_1-locally finite parallel refinement say $\underset{\sim}{H} = \underset{\sim}{H}_1 \cup \underset{\sim}{H}_2$ where $\underset{\sim}{H}_1$ is τ_1-open τ_1-locally finite and $\underset{\sim}{H}_2$ is τ_2-open τ_2-locally finite refinement.

Let $\underset{\sim}{S} = \{f^{-1}(H_1) \cap V_\alpha \mid H_1 \in \underset{\sim}{H}_1 \ , \ \alpha \in \Lambda_1\} \bigcup \{f^{-1}(H_2) \cap V'_\alpha \mid H_1 \in \underset{\sim}{H}_2 \ , \ \alpha \in \Lambda_2\}$. Then $\underset{\sim}{S}$ is a p_1-locally finite p-open parallel refinement of $\underset{\sim}{U}$. Hence X is a P_1-paracompact space. □

Theorem 2.12. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a p-continious, p-closed function s.t. $\forall\, y \in Y, f^{-1}(y)$ is a p_2-α-paracompact subset of X, then X is p_2-paracompact space if Y is so.*

Proof. Let $\underset{\sim}{U}$ be a p-open cover of X. Then $\underset{\sim}{U}$ has p_2-locally finite p-open parallel refinement that covers $f^{-1}(y)$, $\forall\, y \in Y$, i.e. $f^{-1}(y) \subset (\bigcup_{\alpha\in\Lambda_1} V_\alpha) \bigcup (\bigcup_{\alpha\in\Lambda_2} V'_\alpha)$ where $\{V_\alpha \mid \alpha \in \Lambda_1\}$ is τ_1-open τ_2-locally finite and $\{V'_\alpha \mid \alpha \in \Lambda_2\}$ is τ_2-open τ_1-locally finite.

Let $O_y = Y - f(X - \bigcup_{\alpha\in\Lambda_1} V_\alpha)$, $O'_y = Y - f(X - \bigcup_{\alpha\in\Lambda_2} V'_\alpha)$, then $f^{-1}(O_y) \subset \bigcup_{\alpha\in\Lambda_1} V_\alpha$ and $f^{-1}(O'_y) \subset \bigcup_{\alpha\in\Lambda_2} V'_\alpha$.

Then $\underset{\sim}{O} = \{O_y \mid y \in Y\} \bigcup \{O'_y \mid y \in Y\}$. Then $\underset{\sim}{O}$ is a p-open cover of Y.

Since Y is a p_2-paracompact, $\underset{\sim}{O}$ has a p_2-locally finite p-open parallel refinement say $\underset{\sim}{H} = \underset{\sim}{H}_1 \cup \underset{\sim}{H}_2$ where $\underset{\sim}{H}_1$ is τ_1-open τ_2-locally finite and $\underset{\sim}{H}_2$ is τ_2-open τ_1-locally finite.

Let $\underset{\sim}{S} = \{f^{-1}(H_1) \cap V_\alpha \mid H_1 \in \underset{\sim}{H}_1 \ and\ \alpha \in \Lambda_1\} \bigcup \{f^{-1}(H_2) \cap V'_\alpha \mid H_2 \in \underset{\sim}{H}_2 \ , and\ \alpha \in \Lambda_2\}$. Then $\underset{\sim}{S}$ is a p_2-locally finite p-open parallel refinement of $\underset{\sim}{U}$. Hence X is a P_2-paracompact space. □

Definition 2.10. *A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$is said to be a B-closed iff it maps τ_1-closed sets onto τ_2-closed sets and τ_2-closed sets onto τ_1-closed sets.*

Theorem 2.13. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a p-continious, B-closed function s.t. $\forall\, y \in Y, f^{-1}(y)$ is a B-α-paracompact. Then (X, τ_1, τ_2) is B-paracompact if Y is so.*

Proof. Let $\underset{\sim}{U}$ be a τ_1-open cover of X. Since $f^{-1}(y)$ is B-α-paracompact, for each $y \in Y$, $\underset{\sim}{U}$ has τ_2-open τ_2-locally finite refinement in X which covers $f^{-1}(y)$, say $\underset{\sim}{A}_y = \{A_\alpha \mid \alpha \in \Lambda_y\}$.

Let $O_y = Y - f(X - \bigcup_{\alpha\in\Lambda_y} A_\alpha))$, then $f^{-1}(O_y) \subset \bigcup_{\alpha\in\Lambda_y} A_\alpha$. Since f is B-closed O_y is a τ_1-open set in Y and $y \in O_y\ \forall\, y \in Y$.

Now $\underset{\sim}{Q} = \{O_y \mid y \in Y\}$ is τ_1-open cover of Y. Since Y is B-paracompact $\underset{\sim}{Q}$ has τ_2-open τ_2-locally finite refinement say $\underset{\sim}{V}$.

Let $\underset{\sim}{S} = \{f^{-1}(V) \cap A \mid A \in \underset{\sim}{A}_y, V \in \underset{\sim}{V}\}$ then $\underset{\sim}{S}$ is a τ_2-open τ_2-locally finite refinement of $\underset{\sim}{U}$. By a similar method we can show that every τ_2-open cover of X has a τ_1-open τ_1-locally finite refinement. Hence (X, τ_1, τ_2) is a B-paracompact space. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JORDAN
QEEN RANIA ST, AMMAN,
JORDAN.
Email address: mon9170448@ju.edu.jo

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JORDAN
QEEN RANIA ST, AMMAN,
JORDAN.
Email address: zahdeib@ju.edu.jo