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ALEXANDROFF SPACES AND GRAPHIC TOPOLOGY

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ABSTRACT. This work studies and gives some conditions for an Alexandroff space to be graphic topological space by using some basic properties of graphic topology such as locally finitely property. That is, we offer some answer for the open problem which is recalled in [3] (Problem 2 page 658).

1. INTRODUCTION

Topologies for discrete set attracted many attentions since the paper of Golomb [2]. In [4, 6], it was introduced an Alexandroff topology on some graphs, this means a topology satisfying any intersection of open sets is an open set. The authors take account that a graph $G = (V, E)$ is connected if and only if it is connected as topological space. After that, Jafarian Amiri et al. introduce a topology τ_G in every locally finite graph $G = (V, E)$, i.e. each vertex in V is adjacent to a finite number of vertices. The topology τ_G is an Alexandroff topology but mainly the graph G is connected, the topological space (G, τ_G) can be disconnected as the complete graph K_n and the cycle graph C_n of size n, for $n \geq 2$.

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The topology τ_G is defined as follows:

Let $G = (V, E)$ is a simple undirected locally finite graph and with no isolated vertices, that is for each $x \in V$, there exist $y \in V$ such that the edge $\{x, y\} \in E$, this means x and y are adjacent ($x \sim y$).

Let

$$\mathcal{N}(x) = \{y \in V; \{x, y\} \in E\}$$

be the neighborhood of x. Its cardinality is denoted by $d_G(x)$ or $d(x)$.
The topology τ_G on the set V is the topology which has the collection $\mathcal{N}_G$ as a subbasis, where $\mathcal{N}_G = \{\mathcal{N}(x) : x \in V\}$.
We say the pair (G, τ_G) is a graphic topological space or (G, τ_G) is topological graph.

After that, this definition is generalized as follows: A space (X, T) is said graphic topological space if there exists a graph $G = (X, E)$ such that $T = \tau_G$. The fact that (V, τ_G) is an Alexandroff space give rise to this open problem, [3]: are there some conditions for an Alexandroff space to be graphic?

In section 3, we give one answer to the above problem by using subbasis [5].

The organization of this paper is as follows: In section 2 we recall some basic definitions, notations and preliminaries results. Then, section 3 contains some examples of graphic topological spaces and one answer to the problem recalled in [3].

2. Basic concepts

In what follows, (X, T) or X denotes a topological space and $G = (V, E)$ a simple locally finite graph without isolated vertex. We have the graphic topology on V and the most interesting property of this topology is that it is an Alexandroff topology.

Definition 2.1. *A topological space is called Alexandroff space if any intersection of open sets is also an open set.*

Theorem 2.1. *[3] Let G be a locally finite graph without isolated vertex and vertex set V. Then, the topological space (V, τ_G) is an Alexandroff space.*

Definition 2.2. *[3] Let (X, T) be a topological space. X is called graphic, if there exists a subset E of $X \times X$ such that $\tau_G = T$, where $G = (X, E)$.*

Since a graphic space has to be Alexandroff space, we begin by recalling some facts about such topological spaces that will be used later.

A topological space X is an Alexandroff space if and only if each point of X has a minimal open neighborhood $S(x)$. In this case, $S(x)$ is the intersection of all open subset containing x. It is shown in [7] that the study of an Alexandroff space is based on the study of the minimal open neighborhoods and $\{S(x); x \in X\}$ is a minimal basis for T. Also, if two minimal basis are equal for a space X, then the corresponding topologies coincide.

We recall also the following interesting results.

Theorem 2.2. *[7] Suppose that $\mathcal{B}$ is a collection of subsets of X and for any $x \in X$, there exists a minimal set $m(x) \in \mathcal{B}$ which contains x. Then, $\mathcal{B}$ is a basis for an Alexandroff topology on X. Also, the minimal open neighborhood of $x \in X$ is $m(x)$.*

Now, set V a vertex set of a graph G equipped with the graphic topology τ_G. For each $x \in V$, we denote $\mathcal{O}(x)$ the minimal open subset containing x and

$$\alpha_G = \{\mathcal{O}(y) \mid y \in V\}$$

a basis of the space (V, τ_G). In fact, it is the minimal one.

Proposition 2.1. *[3] Assume that $G = (V, E)$ is a locally finite graph without isolated vertex. Then, for all $x \in V$,*

$$\mathcal{O}(x) = \cap_{y \sim x} \mathcal{N}(y).$$

And so $\mathcal{O}(x)$ is finite, for all $x \in V$.

For completeness we recall the following corollary and give its proof.

Corollary 2.1. *If we consider $G = (V, E)$ a locally finite graph without isolated vertex. Then, for all $x, y \in V$, $y \in \mathcal{O}(x)$ if and only if $\mathcal{N}(x) \subseteq \mathcal{N}(y)$.*

Proof. From Proposition 2.1, $y \in \mathcal{O}(x)$ equivalent to $y \in \cap_{z \sim x} \mathcal{N}(z)$, that is $y \in \mathcal{N}(z)$ for all $z \in \mathcal{N}(x)$.

So, $y \in \mathcal{O}(x)$ if and only if for all $z \in \mathcal{N}(x)$, $z \in \mathcal{N}(y)$ and the corollary follows. □

Proposition 2.2. *[3] For any $G = (V, E)$ a locally finite graph without isolated vertex and $z \in V$. Then*

$$\mathcal{O}(z) \subseteq \{z\} \cup \{y \in V; d(z, y) = 2\}$$

and so

(i) $\mathcal{O}(z) \cap \mathcal{N}(z) = \emptyset$.
(ii) $\mathcal{N}(z) \subseteq \mathcal{O}(z)^c$.
(ii) *If $z \sim y$, then $\mathcal{O}(z) \cap \mathcal{O}(y) = \emptyset$.*

Example 1. *Consider the following graph $G_1 = (V, E_1)$.*

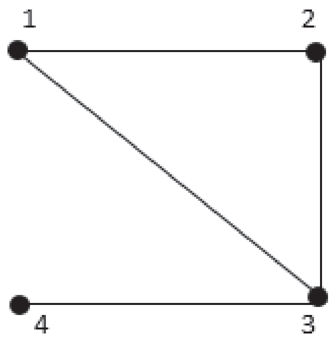

FIGURE 1.

In this graph, $\mathcal{N}_{G_1} = \{\{2, 3\}, \{1, 3\}, \{1, 2, 4\}, \{3\}\}$, the minimal basis is $\alpha_{G_1} = \{\{1\}, \{2\}, \{3\}, \{1, 2, 4\}\}$ and

$$\tau_{G_1} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2, 4\}, \{2, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, V\}.$$

Example 2. *Consider the following graph $G_2 = (V, E_2)$.*

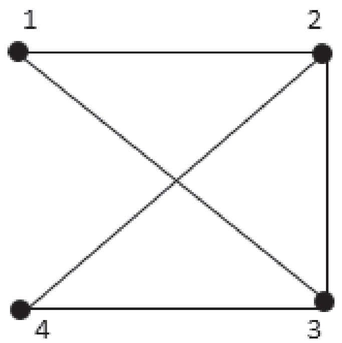

FIGURE 2.

In this graph, $\mathcal{N}_{G_2} = \{\{2,3,4\},\{1,3,4\},\{1,2,4\},\{2,3\}\}$ *and the minimal basis is* $\alpha_{G_2} = \{\{1,4\},\{2\},\{3\},\{4\}\}$*. So,*

$$\tau_{G_2} = \{\emptyset,\{2\},\{3\},\{4\},\{1,4\},\{2,3\},\{3,4\},\{2,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},V\}.$$

Example 3. *Consider the following graph* $G_3 = (V, E_3)$*.*

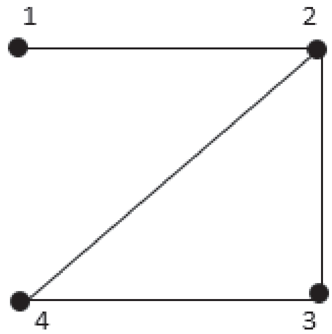

FIGURE 3.

$\mathcal{N}_{G_1} = \{\{2\},\{1,3,4\},\{2,4\},\{2,3\}\}$ *and the minimal basis is*

$$\alpha_{G_3} = \{\{1,3,4\},\{2\},\{3\},\{4\}\},$$

$$\tau_{G_3} = \{\emptyset,\{2\},\{3\},\{4\},\{1,3,4\},\{2,4\},\{2,3\},\{3,4\},\{2,3,4\},V\}.$$

We remark that there is a bijective correspondence between τ_{G_1} and τ_{G_3} and this is due to the fact that there is an isomorphism between the two graphs.

Definition 2.3. *Two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are said isomorphic if there exists a bijection* $\psi : V_1 \to V_2$ *with* $\{x, y\} \in E_1$ *if, and only if* $\{\psi(x), \psi(y)\} \in E_2$*.*

Definition 2.4. *Let* (X_1, T_1) *and* (X_2, T_2) *be two topological spaces.* X_1 *and* X_2 *are called homeomorphic if there exists a bijection* $\psi : X_1 \to X_2$ *with* ψ *and its inverse* ψ^{-1} *are continuous.*

When two graphs are isomorphic, the graphic topological spaces are homeomorphic. But two homeomorphic graphic space can be not isomorphic. As example, The graphic graph $G = (V, E)$ where $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{3,4\}\}$ is homeomorphic to the graphic complete graph of order 4, K_4 but they are not isomorphic.

3. Graphic Spaces

In this section, we investigate some necessary or sufficient condition for an Alexandroff space to be graphic.

For (V,T) an Alexandroff space, $\beta = \{S(x); x \in V\}$ denotes its minimal basis. From the Proposition 2.1, we have to suppose that $S(x)$ is finite for all $x \in V$.

When V is finite, we have the following sufficient condition result.

Theorem 3.1. *[3] Suppose that (V,T) is a finite Alexandroff space and for $x, y \in V$,*

$$S(x) = S(y) \text{ or } S(x) \cap S(y) = \emptyset.$$

Then, (V,T) is a graphic space and the neighborhood of an element x is given by

$$\mathcal{N}(x) = \{y \in V; S(x) \cap S(y) = \emptyset\}.$$

Remark 3.1. *In the Theorem 3.1, $T = \tau_G$ and so*

$$\alpha_G = \mathcal{B} = \{S(x); x \in V\},$$

since the minimal basis is unique for an Alexandroff space.

Remark 3.2. *The Theorem 3.1 still true if we consider (V,T) an Alexandroff space, where $S(x)$ is finite for all $x \in V$.*

The Theorem 3.1 guides us to recall the following definition introduced in [7].

Definition 3.1. *The minimal open neighborhood $S(x)$ in an Alexandroff space is called irreducible if $S(y) = S(x)$ whenever $S(y) \subset S(x)$.*

Theorem 3.2. *Let (V,T) be nontrivial Alexandroff space. If for all $x \in V$, $S(x)$ is finite and irreducible, then (V,T) is a graphic.*

Proof. Let x and y two elements of V. Suppose that $S(x) \cap S(y) \neq \emptyset$ and let $z \in S(x) \cap S(y)$. Then, $S(z) \subseteq S(x)$ and $S(z) \subseteq S(y)$ and so $S(z) = S(x) = S(y)$. From Theorem 3.1 and the Remark 3.2, we conclude that (V,T) is a graphic space. □

Example 4. *Let (V,T) be an Alexandroff space such that $V = \{1,2,3,4,5,6\}$ and*

$$\tau = \{\emptyset, V, \{2\}, \{3\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \{2,4,5,6\}, \{2,3,4,5,6\}\}.$$

Not that (V,T) is graphic topological space which it coincides with a graphic topological space (V, T_G) with the graph $G = (V,E)$.

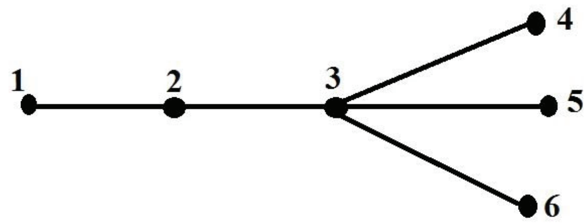

FIGURE 4.

Also not that $S(1) = \{1, 3\}$*,* $S(2) = \{2\}$*,* $S(3) = \{3\}$ *and* $S(4) = S(5) = S(6) = \{2, 4, 5, 6\}$.

We observe that (V, T) *is graphic but* $S(6)$ *is not irreducible, since* $S(2) \subset S(6)$ *but* $S(2) \neq S(6)$.

Remark 3.3.

- *In any trivial topological space* (V, T) *with finite set* V *we have that for all* $x \in V, S(x)$ *is finite and irreducible, since* $S(x) = V$ *for all* $x \in V$.
- *Any discrete space* (V, T) *with finite set* V *is graphic, since it will be coincide with the complete graph* $K = (V, E)$

To study the typical topological spaces, we begin by the following result.

Theorem 3.3. *Suppose that* (V, T) *is a finite space of order* n*,* $n \geq 2$*. If there exists* $x \in V$ *such that* $|S(x)| = n$*, then* (V, T) *is not a graphic space.*

Proof. Suppose that (V, T) is a graphic space. Then, $\mathcal{N}(x) \subseteq S(x)^c = (V)^c = \phi$, that is, $\mathcal{N}(x) = \phi$. This implies that x is an isolated vertex and this contradicts with graphically of V. □

Corollary 3.1. *Any finite trivial topological space* (V, T) *is not a graphic. However, if* (V, T) *is a discrete space, then it is graphic and the graph* G *can be the complete graph of order* n *and vertex set* V.

Another interesting type of topological spaces, those which are called Hausdorff spaces.

Definition 3.2. *A topological space* (V, T) *is called Hausdorff space if for any two elements* $x, y \in V$*, there exist two disjoint open sets* O_1 *and* O_2 *such that* $x \in O_1$ *and* $y \in O_2$.

Theorem 3.4.

(i) *If* (V, T) *is a finite Hausdorff Alexandroff space, then* (V, T) *is graphic space.*

(ii) *A finite graphic space* (V, τ_G) *is Hausdorff space if, and only if,* τ_G *is the discrete topology on* V.

Proof.

(i) Let (V, T) be a finite Hausdorff Alexandroff space. For all $x, y \in V$, $x = y$ or $x \neq y$.

If $x \neq y$, then there exists two open sets O_x and O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \phi$. Suppose that $S(x) \neq S(y)$. Since $S(x)$ and $S(y)$ are the minimal open sets containing x and y, respectfully, then $S(x) \subseteq O_x$ and $S(y) \subseteq O_y$. Hence, $S(x) \cap S(y) \subseteq O_x \cap O_y = \emptyset$ and so $S(x) \cap S(y) = \emptyset$. Therefore, by Theorem 3.1 (V, T) is a graphic space.

(ii) Let (V, τ_G) be a Hausdorff space. We will prove that for all $x \in V$, $\{x\}$ is open set, that is $S(x) = \{x\}$. Suppose that $y \in S(x)$ and $x \neq y$. Since (V, τ_G) is Hausdorff space, then there exists two open set O_x and O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$. Then, $S(x) \cap S(y) \subseteq O_x \cap O_y = \emptyset$, that is, $S(x) \cap S(y) = \emptyset$. But this is contradiction, because $y \in S(y)$ and $y \in S(x)$. Hence, $S(x) = \{x\}$ that is, τ_G is discrete topology.

The converse is trivial. □

Theorem 3.5. *Assume that* (V, T) *is a finite Alexandroff topological space and* $\mathcal{B} = \{S(x); x \in V\}$ *its minimal basis.* (V, T) *is graphic space if, and only if, there exists a subbasis* $\mathcal{N} = \{N_x; x \in V\}$ *of* β *satisfying*

(i) *For all* $x \in V$, $x \notin N_x$ *and* $N_x \neq \emptyset$.

(ii) $x \in N_y \Leftrightarrow y \in N_x$.

Proof. Suppose that (V, T) is a graphic space. Then, there exists a simple graph $G = (V, E)$ with vertices set V and without isolated point such that $\tau_G = T$. Take $\mathcal{N} = \{N_x;\ x \in V\}$, where $N_x = \mathcal{N}(x)$ the neighborhood of x in G. $\mathcal{N}$ is a subbasis of τ_G by definition of τ_G. We get

(i) For all $x \in V$, $x \notin \mathcal{N}(x)$ since the graph G is simple and then no loops. G without isolated node, so $N_x \neq \emptyset$, $\forall x \in V$.

(ii) For all $x, y \in V$, $x \sim y$ equivalent to $y \sim x$, that is $x \in N_y \Leftrightarrow y \in N_x$.

Conversely, suppose $N = \{N_x : x \in V\}$ is a subbasis of β such that

(i) For all $x \in V$, $x \notin N_x$ and $N_x \neq \emptyset$.
(ii) $x \in N_y \Leftrightarrow y \in N_x$.

We define $G = (V, E)$ by $\{x, y\} \in E$ if and only if $x \in N_y$. Then,

$$\mathcal{N}(x) = \{y \in V; \{x, y\} \in E\} = \{y \in V : y \in N_x\},$$

that is $\mathcal{N}(x) = N_x$.

From the hypothesis (i), the graph G is simple and without isolated nodes. As it is proved in [3], $\mathcal{N} = \{\mathcal{N}(x);\ x \in V\}$ is a subbasis of the minimal basis α_G inducing τ_G. But $\mathcal{N}$ is a subbasis of β and so $\alpha_G = \beta$ and so, $T = \tau_G$ and (V, T) is a graphic space. □

Theorem 3.6. *Assume (V, T) is an Alexandroff space such that $S(x)$ is finite for all $x \in V$. If for all $x \neq y \in V, y \notin \overline{S(x)}$ or $x \notin \overline{S(y)}$, then (V, T) is graphic.*

Proof. Let $x, y \in V$ be two distinct elements of V. We have $y \notin \overline{S(x)}$ or $x \notin \overline{S(y)}$. Since y and x play symmetric roles, we can suppose that $y \notin \overline{S(x)}$. Then, $y \in \overline{S(x)}^c$.

Therefore, $\overline{S(x)}^c$ is open set containing y and so $S(y) \subseteq \overline{S(x)}^c$. Hence,

$$S(x) \cap S(y) \subseteq S(x) \cap \overline{S(x)}^c \subseteq S(x) \cap S(x)^c = \emptyset.$$

That is, $S(x) \cap S(y) = \emptyset$. From Theorem 3.1, (V, T) is a graphic space. □

4. Conclusion

In this paper, we recall some properties of graphic topological spaces and we give some conditions on a topological space in order to get a graphic topology. Also, necessary and sufficient conditions on the minimal basis of an Alexandroff space are given in order to be graphic. That is, we give some answer for the open problem which is recalled in [3] (Problem 2 page 658).

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