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PROPERTIES OF REGULAR FUNCTIONS OF A QUATERNION VARIABLE MODIFIED WITH TRI-COMPLEX QUATERNION

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ABSTRACT. In a quaternion structure composed of four real dimensions, we derive a form wherein three complex numbers are combined. Thereafter, we examined whether this form includes the algebraic properties of complex numbers and whether transformations were necessary for its application to the system. In addition, we defined a regular function in quaternions, expressed as a combination of complex numbers. Furthermore, we derived the Cauchy-Riemann equation to investigate the properties of the regular function in the quaternions coupled with the complex number.

1. INTRODUCTION

The quaternion, first introduced by Hamilton [5] in 1894, is a number system that extends a complex number and is a useful structure for representing a single real number and a three-dimensional space vector. Therefore, it is extensively used in both physics and engineering fields. In contrast to complex numbers, the product of quaternions exhibits non-commutative properties; hence, the algebraic properties of quaternions are different from those of complex numbers. In addition, owing to the characteristics caused by the non-commutativity of

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the product for quaternions, the derivative of a quaternionic function depends on whether the differential operator is applied from the left or right. Therefore, there is a restriction on the derivative in the direction of the differential operation.

Because the basis of quaternions is based on the basis of complex numbers, we can deal with complex numbers in the structure of quaternions. In fact, mathematicians have interpreted quaternions as a combination of complex numbers. For example, in the studies by Nôno et al. [14–16], Naser [12], and Kim et al. [7–10], the theory of the quaternion functions as a complex bivariate function with $z_1 = x_0 + x_1e_1$ and $z_2 = x_2 + x_3e_1$, where $x_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$ and $e_1^2 = -1$ as variables was proposed by expressing the quaternion q as $q = z_1 + z_2e_2$.

Goto and Nôno [4] proposed the construction of a commutative algebra $\mathbb{C}(\mathbb{C})$ identified with $\mathbb{C}$ as the subalgebra of the four-dimensional real matrix algebra $M(4, \mathbb{R})$. They provided a regularity of functions of two complex variables with values in $\mathbb{C}(\mathbb{C})$ and also provided certain properties of regular functions. Further, Kajiwara et al. [6] obtained certain results for the regeneration in complex, quaternion, and Clifford analysis, and for the inhomogeneous Cauchy-Riemann system of quaternions and Clifford analysis in ellipsoids. In particular, they proved the existence of a complex-valued harmonic function f_2 in D such that the quaternion-valued function f_1+f_2j was hyperholomorphic in D for any complex valued harmonic function f_1, which satisfies the condition of integrability in a pseudoconvex domain D of $\mathbb{C}^4$. Kim et al. [7–10] defined the function of the modified quaternion using its algebraic property. The (hyper-)holomorphicity and (hyper-)regularity of the modified quaternion function were defined, and the properties of the corresponding quaternion variables were studied. In addition, they proposed the Cauchy and Dirac operators to investigate their feasibility in quaternion functions, and ways to respond to functions that satisfy (hyper-)regularity in contrast to existing functional theory. Naser [12] and Nôno [13] obtained certain properties of quaternionic hyperholomorphic functions and researched the hyperholomophic and hyperconjugate harmonic functions of octonion variables, and dual quaternion functions. In addition they investigated the applications of quaternion functions. Further, in 2011, Koriyama et al. [11] and Naser [12] studied the hyperholomorphic and holomorphic functions in quaternion analysis, and they obtained certain results of regularities in octonion functions and holomorphic mappings. Nguyen [13], Sangwine, and Bihan [17] characterized the quaternion generalization of the presented Cauchy-Riemann equation. In 1935, Fueter [2,3] and Deavours [1] proposed a definition for regular quaternionic functions, which resulted in analogues of the Cauchy theorem, Cauchy integral formula, and Laurent expansion. A general representation of the components of the holomorphic function and the relationship between them has been suggested. Further, the symmetric properties of the components of the holomorphic quaternion function and derivatives of all orders were demonstrated. When considering complete derivative, Sudbery [18] is the result of integrating the left and right derivatives within the framework of the developed theory. Certain holomorphic generalizations of holomorphic complex functions have been discussed in detail to demonstrate the specificity of constructing a holomorphic quaternion function.

This paper is motivated by an attempt to convert a quaternary structure into a complex binary number and proposes a method of studying quaternion functions using the functional theory and analytic properties of complex numbers. In this study, we expanded on complex numbers that can express the quaternion structure. In addition, from the underlying characteristics of the quaternion, a quaternion was constructed by combining it with a complex number rather than a real number. Furthermore, based on these results, this paper proposes a new structured quaternion system using complex numbers.

2. Preliminaries

We focus on the basis of the quaternion, which results in the non-commutativity property of the product of quaternions. The quaternion consists of bases $e_1,\ e_2,\ e_3$, with the following rules:

$$e_1^2 = e_2^2 = e_3^2 = -1,$$
$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.$$

A quaternion q can be expressed as $q = x_0 + e_1x_1 + e_2x_2 + e_3x_3$. Further, utilizing the non-commutative product of the basis of a quaternion, the quaternion q can be expressed as

$$
\begin{aligned}
q &= x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_1e_2x_3 + e_2e_1x_3 \\
&= (x_0 + e_3x_3) + (x_1 + e_2x_3)e_1 + (x_2 + e_1x_3)e_2 \\
&= z_0 + z_1e_1 + z_2e_2, \qquad (2.1)
\end{aligned}
$$

where $z_0 = x_0 + e_3x_3$, $z_1 = x_1 + e_2x_3$, $z_2 = x_2 + e_1x_3$ and $x_j \in \mathbb{R}$ $(j = 0, 1, 2, 3)$. When the quaternion q is expressed in the form of (2.1), the form of $z_0 + z_1e_1 + z_2e_2$ is called the tri-complex form, with q referred to as the tri-complex quaternion. Subsequently, the following expressions were organized and classified for the complex number z_r $(r = 1, 2, 3)$ formed by combining each basis:

$$
\begin{gathered}
\overline{z_0} = x_0 - e_3x_3 \ , \ \overline{z_1} = x_1 - e_2x_3 \ , \ \overline{z_2} = x_2 - e_1x_3, \\
\widetilde{z_1} := z_1e_2 = -x_3 + e_2x_1 \ , \ \widetilde{z_2} := z_2e_1 = -x_3 + e_1x_2, \\
\overline{\widetilde{z_1}} = -x_3 - e_2x_1 \ , \ \overline{\widetilde{z_2}} = -x_3 - e_1x_2, \\
\widehat{z_1} := \widetilde{\overline{z_1}} = \overline{z_1}e_2 = x_3 + e_2x_1 \ , \ \widehat{z_2} := \widetilde{\overline{z_2}} = \overline{z_2}e_1 = x_3 + e_1x_2, \\
z_1e_1 = e_1\overline{z_1}, \ z_2e_1 = e_1z_2, \ z_1e_2 = e_2z_1, \ z_2e_2 = e_2\overline{z_2}, \\
\widetilde{z_1}e_1 = e_1\overline{\widetilde{z_1}}, \ \widetilde{z_2}e_1 = e_1\widetilde{z_2}, \ \widetilde{z_1}e_2 = e_2\widetilde{z_1}, \ \widetilde{z_2}e_2 = e_2\overline{\widetilde{z_2}}, \\
\overline{\widetilde{z_1}} = -\widetilde{\overline{z_1}} = -\widehat{z_1}, \ \overline{\widetilde{z_2}} = -\widetilde{\overline{z_2}} = -\widehat{z_2}.
\end{gathered}
$$

Let q and p be the tri-complex quaternions such that

$$
q = z_0 + z_1e_1 + z_2e_2, \qquad p = w_0 + w_1e_1 + w_2e_2,
$$

where $w_0 = y_0 + e_3y_3$, $w_1 = y_1 + e_2y_3$, $w_2 = y_2 + e_1y_3$ and $y_j \in \mathbb{R}$ $(j = 0, 1, 2, 3)$. Thus, the addition and subtraction of the tri-complex quaternions are expressed as

$$
\begin{aligned}
q \pm p &= (z_0 + z_1e_1 + z_2e_2) \pm (w_0 + w_1e_1 + w_2e_2) \\
&= (z_0 \pm w_0) + (z_1 \pm w_1)e_1 + (z_2 \pm w_2)e_2 = w \pm z.
\end{aligned}
$$

Because quaternion multiplication is not commutative, using the tri-complex form, the multiplication of tri-complex quaternions is also not commutative. In fact, it is evident that

$$\begin{aligned} qp =&(z_0 + z_1e_1 + z_2e_2)(w_0 + w_1e_1 + w_2e_2) \\ =&(z_0w_0 - z_1\overline{w_1} - z_2\overline{w_2}) + (z_0w_1 + z_1\overline{w_0} + z_2\widetilde{\overline{w_1}})e_1 \\ &+ (z_0w_2 + z_1\widetilde{\overline{w_2}} + z_2\overline{w_0})e_2 \end{aligned}$$

and

$$\begin{aligned} pq =&(w_0 + w_1e_1 + w_2e_2)(z_0 + z_1e_1 + z_2e_2) \\ =&(w_0z_0 - w_1\overline{z_1} - w_2\overline{z_2}) + (w_0z_1 + w_1\overline{z_0} + w_2\widetilde{\overline{z_1}})e_1 \\ &+ (w_0z_2 + w_1\widetilde{\overline{z_2}} + w_2\overline{z_0})e_2. \end{aligned}$$

The conjugate of a tri-complex quaternion q is expressed by

$$q^* := \overline{z_0} - z_1e_1 - z_2e_2.$$

Further, the modulus of q, denoted by $M(q)$, is written as $M(q) = qq^* = q^*q$; thereafter, using the tri-complex form,

$$\begin{aligned} M(q) =&(z_0 + z_1e_1 + z_2e_2)(\overline{z_0} - z_1e_1 - z_2e_2) \\ =&z_0\overline{z_0} + z_1\overline{z_1} + z_2\overline{z_2} - 2x_3^2 \in \mathbb{R}. \end{aligned}$$

The inverse of q is expressed as

$$q^{-1} = \frac{z^*}{N_z}.$$

We defined the form of the tri-complex quaternions and presented the basic operations, modulus, and inverse utilizing the form of these numbers. Thereafter, we consider defining a function for a set of tri-complex quaternions and propose a derivative operator to derive the derivative of this function. The differential operators over the tri-complex quaternions are as follows:

$$D := \frac{\partial}{\partial z_0} - e_1\frac{\partial}{\partial \overline{z_1}} - e_2\frac{\partial}{\partial \overline{z_2}} \quad \text{and} \quad D^* = \frac{\partial}{\partial \overline{z_0}} + e_1\frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}},$$

where

$$\frac{\partial}{\partial z_0}=\frac{\partial}{\partial x_0}-e_3\frac{\partial}{\partial x_3}\ ,\ \frac{\partial}{\partial z_1}=\frac{\partial}{\partial x_1}-e_2\frac{\partial}{\partial x_3}\ ,\ \frac{\partial}{\partial z_2}=\frac{\partial}{\partial x_2}-e_1\frac{\partial}{\partial x_3}$$
$$\frac{\partial}{\partial \overline{z_0}}=\frac{\partial}{\partial x_0}+e_3\frac{\partial}{\partial x_3}\ ,\ \frac{\partial}{\partial \overline{z_1}}=\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_3}\ ,\ \frac{\partial}{\partial \overline{z_2}}=\frac{\partial}{\partial x_2}+e_1\frac{\partial}{\partial x_3}$$
$$\frac{\partial}{\partial \widetilde{z_1}}=-\frac{\partial}{\partial x_3}-e_2\frac{\partial}{\partial x_1}\ ,\ \frac{\partial}{\partial \widetilde{z_2}}=-\frac{\partial}{\partial x_3}-e_1\frac{\partial}{\partial x_2}$$
$$-\frac{\partial}{\partial \widetilde{z_1}}=\frac{\partial}{\partial z_1}e_2\ ,\ -\frac{\partial}{\partial \widetilde{z_2}}=\frac{\partial}{\partial z_2}e_1.$$

Let Ω be a domain in $\mathbb{C}^3$. We considered a function f defined in Ω and with values in $\mathbb{C}^3$, for $f := f_0 + f_1e_1 + f_2e_2$, and denoted by

$$q=(z_0,z_1,z_2)\mapsto f(z)=f_0(z_0,z_1,z_2)+f_1(z_0,z_1,z_2)e_1+f_2(z_0,z_1,z_2)e_2,$$

where $f_0 = u_0 + e_3u_3\ ,\ f_1 = u_1 + e_2u_3\ ,\ f_2 = u_2 + e_1u_3$ and $u_j\ (j = 0, 1, 2, 3)$ are real valued functions. A function $f := f_0 + f_1e_1 + f_2e_2$ is called a tri-complex quaternionic function. Thus, we obtained results of calculating the differential operators with a tri-complex quaternionic function as follows:

$$\begin{aligned} Df &= \left(\frac{\partial}{\partial z_0}-e_1\frac{\partial}{\partial \overline{z_1}}-e_2\frac{\partial}{\partial \overline{z_2}}\right)(f_0+f_1e_1+f_2e_2)\\ &= \left(\frac{\partial f_0}{\partial z_0}+\frac{\partial \overline{f_1}}{\partial z_1}+\frac{\partial \overline{f_2}}{\partial z_2}\right)+\left(\frac{\partial f_1}{\partial z_0}-\frac{\partial \overline{f_0}}{\partial z_1}-\frac{\partial \widetilde{f_1}}{\partial z_2}\right)e_1\\ &\quad+\left(\frac{\partial f_2}{\partial z_0}-\frac{\partial \widetilde{f_2}}{\partial z_1}-\frac{\partial \overline{f_0}}{\partial z_2}\right)e_2 \end{aligned}$$

and

$$\begin{aligned} D^*f &= \left(\frac{\partial}{\partial \overline{z_0}}+e_1\frac{\partial}{\partial \overline{z_1}}+e_2\frac{\partial}{\partial \overline{z_2}}\right)(f_0+f_1e_1+f_2e_2)\\ &= \left(\frac{\partial f_0}{\partial \overline{z_0}}-\frac{\partial \overline{f_1}}{\partial z_1}-\frac{\partial \overline{f_2}}{\partial z_2}\right)+\left(\frac{\partial f_1}{\partial \overline{z_0}}+\frac{\partial \overline{f_0}}{\partial z_1}+\frac{\partial \widetilde{f_1}}{\partial z_2}\right)e_1\\ &\quad+\left(\frac{\partial f_2}{\partial \overline{z_0}}+\frac{\partial \widetilde{f_2}}{\partial z_1}+\frac{\partial \overline{f_0}}{\partial z_2}\right)e_2. \end{aligned}$$

3. Properties of the regularity for a tri-complex quaternionic function

The regularity corresponding to the tri-complex quaternionic functions and their properties was defined.

Definition 3.1. *A function $f = f_0 + f_1 e_1 + f_2 e_2$ is said to be a tri-complex quaternionic regular in Ω and with values in $\mathbb{C}^3$ if the following conditions are satisfied:*

(1) f_j $(j = 0, 1, 2)$ *are holomorphic functions in* Ω,
(2) $D^* f = 0$ *in* Ω.

In addition, f is abbreviated as a TCQ-regular function. From the second condition of the definition of a TCQ-regular function, we obtain

$$\begin{aligned} D^* f &= \left(\frac{\partial}{\partial \overline{z_0}} + e_1 \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}} \right) (f_0 + f_1 e_1 + f_2 e_2) \\ &= \left(\frac{\partial f_0}{\partial \overline{z_0}} - \frac{\partial \overline{f_1}}{\partial z_1} - \frac{\partial \overline{f_2}}{\partial z_2} \right) + \left(\frac{\partial f_1}{\partial \overline{z_0}} + \frac{\partial \overline{f_0}}{\partial z_1} + \frac{\partial \widetilde{f_1}}{\partial z_2} \right) e_1 \\ &\quad + \left(\frac{\partial f_2}{\partial \overline{z_0}} + \frac{\partial \widetilde{f_2}}{\partial z_1} + \frac{\partial \overline{f_0}}{\partial z_2} \right) e_2 = 0. \end{aligned}$$

Thus, the following equations are induced

$$\begin{cases} \frac{\partial f_0}{\partial \overline{z_0}} - \frac{\partial \overline{f_1}}{\partial z_1} - \frac{\partial \overline{f_2}}{\partial z_2} = 0, \\ \frac{\partial f_1}{\partial \overline{z_0}} + \frac{\partial \overline{f_0}}{\partial z_1} + \frac{\partial \widetilde{f_1}}{\partial z_2} = 0, \\ \frac{\partial f_2}{\partial \overline{z_0}} + \frac{\partial \widetilde{f_2}}{\partial z_1} + \frac{\partial \overline{f_0}}{\partial z_2} = 0. \end{cases} \tag{3.1}$$

In this paper, Equation (3.1) is referred to as the corresponding Cauchy-Riemann equations for tri-complex quaternionic functions.

Proposition 3.1. *Let Ω be a domain in $\mathbb{C}^3$, with f being a TCQ-regular in Ω with values in $\mathbb{C}^3$. Then, we obtain*

$$Df = f' = \frac{\partial f}{\partial z_0} + \frac{\partial f}{\partial \overline{z_0}} = 2 \frac{\partial f}{\partial x_0}.$$

Proof. Because f is a TCQ-regular in Ω, we obtain the following calculation process:

$$\begin{aligned}
Df &= \left(\frac{\partial}{\partial z_0} - e_1\frac{\partial}{\partial \overline{z_1}} - e_2\frac{\partial}{\partial \overline{z_2}}\right)(f_0 + f_1e_1 + f_2e_2)\\
&= \left(\frac{\partial f_0}{\partial z_0} + \frac{\partial \overline{f_1}}{\partial z_1} + \frac{\partial \overline{f_2}}{\partial z_2}\right) + \left(\frac{\partial f_1}{\partial z_0} - \frac{\partial f_0}{\partial z_1} - \frac{\partial \widetilde{f_1}}{\partial z_2}\right)e_1\\
&\quad + \left(\frac{\partial f_2}{\partial z_0} - \frac{\partial \widetilde{f_2}}{\partial z_1} - \frac{\partial f_0}{\partial z_2}\right)e_2\\
&= \left(\frac{\partial}{\partial z_0} + \frac{\partial}{\partial \overline{z_0}}\right)f_0 + \left(\frac{\partial}{\partial z_0} + \frac{\partial}{\partial \overline{z_0}}\right)f_1e_1, + \left(\frac{\partial}{\partial z_0} + \frac{\partial}{\partial \overline{z_0}}\right)f_2e_2\\
&= \left(\frac{\partial}{\partial z_0} + \frac{\partial}{\partial \overline{z_0}}\right)f = 2\frac{\partial f}{\partial x_0}.
\end{aligned}$$

Thus, Df can be expressed as $2\frac{\partial f}{\partial x_0}$. □

Proposition 3.2. *Let f be a homogeneous polynomial of degree n with respect to the variables x_j $(j = 0, 1, 2, 3)$. If f is a TCQ-regular function in $\mathbb{C}^3$, then we have*

$$f(q) = \frac{1}{n!}\frac{\partial^n f(q)}{\partial x_0^n}q^n. \tag{3.2}$$

Proof. Because $f(q)$ is homogeneous polynomial, the following form is obtained:

$$f(q) = \frac{1}{n}\frac{\partial f(q)}{\partial x_0}q.$$

Specifically, the term $\frac{\partial f(q)}{\partial x_0}$ is a homogeneous polynomial of degree $n-1$; thus, we obtain

$$\frac{\partial f(q)}{\partial x_0} = \frac{1}{n-1}\frac{\partial^2 f(q)}{\partial x_0^2}q.$$

If the above process is repeated, Equation (3.2)can be finally obtained. □

In this section, let Ω be a domain in $\mathbb{C}^3$. Consequently, we obtain the following properties from a TCQ-regular function: Let ω be a differential form such that

$$\omega = -\,dz_0 \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} + dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}\, e_1 \tag{3.3}$$

$$+\, dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}\, e_2. \tag{3.4}$$

Now, we provide a corresponding Cauchy theorem for a TCQ-function. Consequently, using Stokes' theorem, we obtain:

Theorem 3.1. *Let Ω be a domain in $\mathbb{C}^3$ and D be any domain in Ω with a smooth boundary bdD such that $\overline{D} \subset \Omega$. If f is a TCQ-regular function in Ω, then we have*

$$\int_{bdD} \omega f = 0.$$

Proof. Further, let ω be the differential 1D-form such that (3.3). Then, we have

$$\begin{aligned}
\omega f \;=\;& -dz_0 \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} f_0 - dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}\ \overline{f_1} \\
& -dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}\ \overline{f_2} + dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}\ \overline{f_0} e_1 \\
& -dz_0 \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} f_1 e_1 + dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}\ \overline{f_0} e_2 \\
& -dz_0 \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} f_2 e_2 - dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2}\ f_1 e_{12} \\
& +dz_0 \wedge d\overline{z_0} \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}\ f_2 e_{12}.
\end{aligned}$$

In addition, $d(\omega f)$ is calculated as follows: for $dV = dz_0 \wedge d\overline{z_0} \wedge dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2}$ we have

$$\begin{aligned}
d(\omega f) \;=\;& \Big(\frac{\partial f_0}{\partial \overline{z_0}} + \frac{\partial f_1}{\partial \overline{z_0}} e_1 + \frac{\partial f_2}{\partial \overline{z_0}} e_2 - \frac{\partial \overline{f_1}}{\partial z_1} + \frac{\partial \overline{f_0}}{\partial z_1} e_1 + \frac{\partial f_2}{\partial z_1} e_1 e_2, \\
& -\frac{\partial \overline{f_2}}{\partial z_2} + \frac{\partial \overline{f_0}}{\partial z_2} e_2 + \frac{\partial f_1}{\partial z_2} e_2 e_1\Big) dV \\
=\;& \Big(\frac{\partial f_0}{\partial \overline{z_0}} - \frac{\partial \overline{f_1}}{\partial z_1} - \frac{\partial \overline{f_2}}{\partial z_2} + \Big(\frac{\partial f_1}{\partial \overline{z_0}} + \frac{\partial \overline{f_0}}{\partial z_1} + \frac{\partial \widetilde{f_1}}{\partial z_2}\Big) e_1 \\
& + \Big(\frac{\partial f_2}{\partial \overline{z_0}} + \frac{\partial \widetilde{f_2}}{\partial z_1} + \frac{\partial \overline{f_0}}{\partial z_2}\Big) e_2\Big),
\end{aligned}$$

Because f is a TCQ-regular in Ω, it satisfies the corresponding Cauchy-Riemann equations for tri-complex quaternionic functions. Hence, we obtain $d(\omega f) = 0$. Furthermore, by using the Stokes' theorem over quaternions, the equation

$$0 = \int_D d(\omega f) = \int_{bdD} \omega f$$

is satisfied. □

4. Conclusion

This paper focuses on the structure of a quaternion composed of real and three-dimensional space vectors considering the basis of complex numbers. In particular, we derived the composition of a quaternion combined with three complex numbers using the property $e_1 e_2 = -e_2 e_1$. Because the quaternion is isomorphic to $\mathbb{C}$, which is composed of two complex numbers, the conjugate and modulus were defined, and the inverse of the product was defined when the quaternion number was composed of three complex numbers. In addition, regularity was suggested by defining a function in a set of tri-complex quaternions. There have been various attempts and results in the non-commutative quaternion function, and regularity is regarded as an important condition for deriving analytic and various other properties of a quaternary function. Moreover, a function for which regularity can be defined can represent the result derived from holomorphicity instead of holomorphicity.

Therefore, in this paper, a new type of quaternion called a tri-complex quaternion was defined, to propose a differential operator suitable for application to the function defined in this system, and the regularity was defined using this operator. Further, the Cauchy-Riemann equation corresponding to the tri-complex quaternionic functions was defined from the condition of the defined regularity, and the properties were verified in the quaternion function using the Cauchy-Riemann equation. Furthermore, we plan the following research based on this study: propose various research perspectives on odd-numbered dimensions of complex numbers, and express the scalability of complex dimensions by examining the homomorphism of complex numbers with odd-numbered dimensions. In the future, we plan to expand the generalized Cauchy-Riemann system using the regularity and Cauchy-Riemann equation defined in this paper, and thereafter investigate the form of series expansion by applying it to a polynomial function that defines a new type of quaternion.

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REFERENCES

1. C. A. DEAVOURS: *The Quaternion Calculus*, Amer. Math. Monthly **80** (1973), 995–1008.
2. R. FUETER: *Die Funktionentheorie der Differentialgleichungen* $\Delta u = 0$ *und* $\Delta\Delta u = 0$ *mit vier reellen Variablen*, Comm. Math. Helv. **7**(1) (1934), 307–330.
3. R. FUETER: *Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen*, Comm. Math. Helv. **8**(1) (1935), 371–378.

4. S. GOTO, K. NÔNO: *Regular functions with values in a Commutative subalgebra* $\mathbb{C}(\mathbb{C})$ *of Matrix algebra* $M(4,\mathbb{R})$, Bull. Fukuoka Univ. Ed. **61**(III) (2012), 9–15.
5. W. R. HAMILTON: *Elements of Quaternions*, Chelsea, New York, 1866.
6. J. KAJIWARA, X. D. LI, K. H. SHON: *Regeneration in complex, quaternion and Clifford analysis*, Finite or Infinite Dimensional Complex Anal. Appl. Springer, Boston, MA. (2004), 287–298.
7. J. E. KIM, S. J. LIM, K. H. SHON: *Regularity of functions on the reduced quaternion field in Clifford analysis*, Abstr. Appl. Anal. **2014** (2014), Artical id 654798.
8. J. E. KIM, K. H. SHON: *Polar Coordinate Expression of Hyperholomorphic Functions on Split Quaternions in Clifford Analysis*, Adv. Appl. Clifford Alg. **25**(4) (2015), 915–924.
9. J. E. KIM, S. J. LIM, K. H. SHON: *Taylor series of functions with values in dual quaternion*, Pure & Appl. Math. **20**(4) (2013), 251–258.
10. J. E. KIM: *Hyperholomorphic functions with values in a modified form of quaternions*, Problemy Analiza-Issues of Analysis **9**(1) (2020), 83–95.
11. H. KORIYAMA, H. MAE, K. NÔNO: *Hyperholomorphic Functions and Holomorphic functions in Quaternionic Analysis*, Bull. Fukuoka Univ. Ed. **60**(III) (2011), 1–9.
12. M. NASER: *Hyperholomorphic functions*, Siberian Math. J. **12** (1971), 959–968.
13. T. V. NGUYEN: *Differential operators associated to the Cauchy-Fueter operator in quaternion algebra*, Adv. Appl. Clifford Alg. **21**(3) (2011), 591–605.
14. K. NÔNO: *Hyperholomorphic functions of a quaternion variable*, Bull. Fukuoka Univ. Ed. **32** (1983), 21–37.
15. K. NÔNO: *Characterization of domains of holomorphy by the existence of hyper-conjugate harmonic functions*, Rev. Roumaine Math. Pures Appl. **31**(2) (1986), 159–161.
16. K. NÔNO: *Domains of Hyperholomorphic in* $\mathbb{C}^2 \times \mathbb{C}^2$, Bull. Fukuoka Univ. Ed. **36** (1987), 1–9.
17. S. J. SANGWINE, N. LE BIHAN: *Quaternion polar representation with a complex modulus and complex argument inspired by the Cayley-Dickson form*, Adv. Appl. Clifford Al. **20**(1) (2010), 111–120.
18. A. SUDBERY: *Quaternionic analysis*, Math. Proc. Camb. Phil. Soc. **85**(2) (1979), 199–225.

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