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ON COMMUTATIVITY OF SEMIRINGS THROUGH IDENTITIES OF GENERALIZED DERIVATIONS

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ABSTRACT. The main purpose of this paper is to investigate the commuting conditions for prime MA-semirings through Jordan ideals and generalized derivations which are responsible to extend a few results of rings.

1. INTRODUCTION AND PRELIMINARIES

Semirings have notable applications in optimization theory, theory of automata, and in theoretical computer sciences (see [10, 11, 15]). A group of Russian mathematicians was able to establish novel probability theory based on additive inverse semirings, known as idempotent analysis (see [14, 16]) having interesting applications in quantum physics. The notion of Jordan ideals was introduced by Herstein [12] in rings which is further extended canonically by Sara [22] for semirings. Several papers have been produced on Jordan ideals, for reference one can see [5, 7, 17–19]. Javed et al. [13] introduced a special class of semirings known as MA-Semirings. The class of MA-semirings properly contains the class of rings and the class of distributive lattices. In fact every ring is an MA-semiring but the converse may not be true in general, one can find

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examples of MA-semirings in [5, 6, 13, 21]. In this paper we mostly use Lie type theory of MA-semirings (see [1–3, 6, 13, 21, 22]) that may be helpful to attract the algebraists to extend some remarkable results in this area.

Now, we present some important definitions and preliminaries. By a semiring S, we mean a semiring with absorbing zero '0' in which addition is commutative. A semiring S is said to be additive inverse semiring if for each $s \in S$ there is a unique $s' \in S$ such that $s + s' + s = s$ and $s' + s + s' = s'$, where s' denotes the pseudo inverse of s. An additive inverse semiring S is said to be an MA-Semiring if it satisfies $s + s' \in Z(S), \forall s \in S$, where $Z(S)$ is the center of S. Throughout the paper by semiring S we mean an MA-semiring unless stated otherwise. A semiring S is prime if $aSb = \{0\}$ implies that $a = 0$ or $b = 0$ and semiprime if $aSa = \{0\}$ implies that $a = 0$. S is 2-torsion free if for $s \in S$, $2s = 0$ implies $s = 0$. An additive mapping $d : S \longrightarrow S$ is a derivation if $d(st) = d(s)t + sd(t)$. The concept of generalized derivation was studied for MA-semirings in [2]. An additive mapping $F_d : S \longrightarrow S$ is a generalized derivation associated with a derivation d if $F_d(st) = F_d(s)t + sd(t)$ (see [9]). The commutator is defined as $[s, t] = st + t's$. By Jordan product we mean $s \circ t = st + ts$ for all $s, t \in S$. A mapping $f : S \longrightarrow S$ is commuting if $[f(s), s] = 0$, $\forall s \in S$. An additive subsemigroup G of S is called the Jordan ideal if $s \circ j \in G$ for all $s \in S, j \in G$. A mapping $f : S \longrightarrow S$ is centralizing if $[[f(s), s], r] = 0$, $\forall s, r \in S$. We include some MA-semiring identities useful for the sequel: If $d : S \to S$ is derivation and $s, t, z \in S$, then: $[s, st] = s[s, t]$, $[st, z] = s[t, z] + [s, z]t$, $[s, yz] = [s, t]z + t[s, z]$, $[s, t] + [t, s] = t(s + s') = s(t + t')$, $(st)' = s't = st'$, $[s, t]' = [s, t'] = [s', t]$, $s \circ (t + z) = s \circ t + s \circ z$, $d(x') = (d(x))'$, for more details, one can see [13, 21, 22]. From the literature, we recall a few results for MA-semirings which are very useful to establish the main results.

Lemma 1.1. *[1] Let G be a Jordan ideal of an MA-semiring S. Then for all $j \in G$ following containments hold:*

(a). $2[S, S]G \subseteq G$ *(b).* $2G[S, S] \subseteq G$ *(c).* $4j^2S \subseteq G$

(d). $4Sj^2 \subseteq G$, *(e).* $4jSj \subseteq G$.

Proof. We prove only (c) here, the other inclusions can be followed in the similar fashion.

For any $j \in G$ and $s \in S$, we have $[j, s] \in S$ and therefore by the definition of Jordan ideal $j \circ [j, s] \in J$. But then by the definition of MA-semiring and using commutator identities

$$\begin{aligned} j \circ [j, s] &= j(js + sj^{'}) + (js + sj^{'})j = j^2s + jsj^{'} + jsj + s(j^2)^{'} \\ &= j^2s + j(s^{'} + s)j + s^{'}(j^2) = j^2s + (s^{'} + s)j^2 + s^{'}(j^2) = j^2s + s^{'}(j^2). \end{aligned}$$

This shows that $2(j^2s + s^{'}j^2) \in G$. Also since $2j^2 \in G$, $2j^2 \circ s = 2sj^2 + 2j^2s \in G$. Therefore $4j^2s = 2j^2s + 2s^{'}j^2 + 2sj^2 + 2j^2s \in G$ and hence $4j^2S \subseteq G$. □

Lemma 1.2. *[1] Let S be a 2-torsion free prime MA-semiring and G a Jordan ideal of S. If $aGb = \{0\}$ then $a = 0$ or $b = 0$.*

Remark 1.1. *[1]*
a) If $aG = \{0\}$ or $Ga = \{0\}$, then $a = 0$.
b) If S is semiprime and $x \in Z(S)$, $x^2 = 0$, then $x = 0$.

Lemma 1.3. *Let G be a Jordan ideal and d be a derivation of a 2-torsion free prime MA-semiring S such that $d(G) = \{0\}$. Then $d = 0$.*

Proof. By Theorem 2.4 of [1], we have either $d = 0$ or $[G, S] = \{0\}$. If $[G, S] = \{0\}$, then for any $j \in G$ and $s \in S$, we have $2js = j \circ s \in G$. Therefore, by the 2-torsion freeness, we have $0 = d(js) = d(j)s + jd(s)$ and using hypothesis again, we get $Gd(s) = 0$. By Remark 1.1, we conclude that $d = 0$. □

Lemma 1.4. *[2] Let G be a Jordan ideal and d be a derivation of a 2-torsion free prime MA-semiring S such that for all $u \in G$, $d(u^2) = 0$. Then $d = 0$*

Lemma 1.5. *[6] Let G be a Jordan ideal of a 2-torsion free prime MA-semiring S. If $a \in S$ such that for all $v \in G$, $[a, v^2] = 0$, then $[a, S] = \{0\}$. In particular $[u^2, v^2] = 0$.*

Lemma 1.6. *[6] Let S be a 2-torsion free prime MA-semiring and G a Jordan ideal of S. If S is non-commutative such that for all $u, v \in G$, and $r \in S$, $a[r, uv]b = 0$, then $a = 0$ or $b = 0$.*

Oukhtite et al [17] proved some results on generalized derivations satisfying certain identities on Jordan ideals of rings. The main objective of this paper is to prove some results of [17] for the Jordan ideals of MA-semirings.

2. Main Results

Lemma 2.1. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and d a nonzero derivation of S. Setting $G_0 = G$. Then for any positive integer i, the set $G_i = \{x \in G_{i-1} : d(x) \in G_{i-1}\}$ nonzero Jordan ideal. Moreover if $G \cap Z(S) \neq \{0\}$, then $G_i \cap Z(S) \neq \{0\}$.*

Proof. Let $u, v \in G_1$. Then $d(u), d(v) \in G_0$. As G is closed under addition, $u + v \in G$ such that $d(u + v) = d(u) + d(v) \in G$. Therefore $u + v \in G_1$. Secondly for any $s \in S$ and $u \in G_1$, we have $d(u) \in G$ and therefore $s \circ d(u) \in G$. As $s \circ u \in G$, $d(s \circ u) = d(s) \circ u + s \circ d(u) \in G$. Therefore $s \circ d(u) \in G_1$, which shows that G_1 is Jordan ideal. Similarly, we conclude that each G_i is a Jordan ideal. From the definition, we easily see that $G_i \subseteq G_{i-1}, i = 1, 2, 3, \ldots$.

Next we show that each G_i is nonzero. Suppose that $G_0 = G \neq 0$. We consider As $d \neq 0$, there is at least one $0 \neq u \in G$ such that $d(u) \neq 0$. Therefore $G_1 \neq 0$. Consequently $G_2 \neq 0, G_3 \neq 0, \ldots$.

Finally let $0 \neq u \in G \cap Z(S)$ such that $d(u) \neq 0$. Then $u \in G_1 \cap Z(S)$, which shows that $G_1 \cap Z(S) \neq \{0\}$. □

Theorem 2.1. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(uv) + u^{'}v, r] = 0$ for all $u, v \in G$, then S is commutative.*

Proof. By the hypothesis, for all $u, v \in G, r \in S$, we have

$$[F_d(uv) + u^{'}v, r] = 0. \tag{2.1}$$

In (2.1) taking $v = u$, we get $[F_d(u^2) + (u^2)^{'}, r] = 0$ and therefore

$$F_d(u^2) + (u^2)^{'} \in Z(S). \tag{2.2}$$

Suppose that $G \cap Z(S) = \{0\}$. By lemma 1.1, replacing u by $2u^2$ and v by $2[s, jk]v, j, k \in G, s \in S$ in (2.1), we get

$$[4F_d(u^2[s, jk]v) + 4(u^2)^{'}[s, jk]v, r] = 0. \tag{2.3}$$

In view of (2.2), using the identities of MA-semirings, we can write $4F_d(u^2[s, jk]v) + 4(u^2)^{'}[s, jk]v$,

$$= 4[(F_d(u^2) + (u^2)^{'})s, jk]v + 4(u^2 d[s, jk])v + 4u^2[s, jk])d(v).$$

In view of Lemma 1.1, using 2-torsion free of S, we obtain

$$F_d(u^2[s,jk]v) + (u^2)'[s,jk]v \in G$$

and

$$F_d(u^2[s,jk]v) + (u^2)'[s,jk]v \in Z(S).$$

But then by the assumption that $G \bigcap Z(S) = \{0\}$, we have

$$F_d(u^2[r,jk]v) + (u^2)'[r,jk]v = 0. \tag{2.4}$$

In (2.4) replacing v by $4vw^2$ and using 2-torsion freeness, we obtain

$$F_d(u^2[r,jk]v)w^2 + u^2[r,jk]vd(w^2) + (u^2)'[r,jk]vw^2 = 0.$$

Using (2.4) again, we get $u^2[r,jk]vd(w^2) = 0$ and therefore by Lemma 1.6, we have either $u^2 = 0$ or $vd(w^2) = 0$. As $u^2 = 0$ implies $G = \{0\}$ which is not possible, therefore we have $Gd(w^2) = \{0\}$. By Remark 1.1, we have $d(w^2) = 0$ which by Lemma 1.4, implies that $d = 0$, a contradiction. Therefore, we must have $G \cap Z(S) \neq \{0\}$.

In (2.1) replacing v by $4vx^2, x \in G$ and using (2.1) again, we obtain $(F_d(uv) + u'v)[x^2, r] + [uvd(x^2), r] = 0$. In particular, for $r = x^2$, we have

$$(F_d(uv) + u'v)[x^2, x^2] + [uvd(x^2), x^2] = 0. \tag{2.5}$$

As S is MA-semiring, $[x,x] = [x,x]', \forall x \in S$, therefore $(F_d(uv) + u'v)[x^2, x^2]' + [uvd(x^2), x^2] = 0$ and hence

$$(F_d(uv) + u'v)[x^2, x^2] = [uvd(x^2), x^2]. \tag{2.6}$$

Using (2.6) into (2.5) and then by the 2-torsion freenessof S, we obtain $[uvd(x^2), x^2] = 0$, which further implies

$$uv[d(x^2), x^2] + u[v, x^2]d(x^2) + [u, x^2]vd(x^2) = 0. \tag{2.7}$$

By the definition of MA-semiring, we have $u + u' \in Z(S)$, we obtain

$$4d(x^2)u = 2(d(x) \circ x) \circ u + 2[(d(x) \circ x), u].$$

In view of Lemma 2.1, Lemma 1.1 and Remark 1.1, for all $u, x \in G_1$, we obtain

$$4d(x^2)u = 2(d(x) \circ x) \circ u + 2[(d(x) \circ x), u] \in G.$$

In (2.7) replacing v by $v \in G_1$ and u by $4d(x^2)u, x, u$, we obtain $4d(x^2)uv[d(x^2), x^2] + 4d(x^2)u[v, x^2]d(x^2) + [4d(x^2)u, x^2]vd(x^2) = 0$ and by MA-semiring identities and then rearranging the terms, we have

$$4d(x^2)(uv[d(x^2), x^2] + u[v, x^2]d(x^2) + [u, x^2]vd(x^2)) + [4d(x^2), x^2]uvd(x^2) = 0.$$

Using (2.7) again and hence using 2-torsion freeness of S, we obtain

$$[d(x^2), x^2]uvd(x^2) = 0, \forall x, u, v \in G_1. \tag{2.8}$$

In (2.8) replacing v by $2x^2$, we obtain

$$[d(x^2), x^2]uvx^2d(x^2) = 0, \forall x, u, v \in G_1. \tag{2.9}$$

Multiplying (2.8) by $(x^2)'$ from the right, we obtain

$$[d(x^2), x^2]uvd(x^2)(x^2)' = 0, \forall x, u, v \in G_1. \tag{2.10}$$

Adding (2.9) and (2.10), we obtain $[d(x^2), x^2]uG[d(x^2), x^2] = 0, \forall x, u \in G_1$. In view of Lemma 1.2, using Remark 1.1, we obtain $[d(x^2), x^2] = 0$ and by Lemma 1.5, we further obtain

$$[d(x^2), s] = 0 = 0, \forall x \in G_1, s \in S. \tag{2.11}$$

In (2.11) replacing x by $x + y, \forall x \in G_1$ and using it again, we get

$$[d(xy) + d(yx), s] = 0. \tag{2.12}$$

□

As $G_1 \cap Z(S) \neq \{0\}$, replacing y by $z \in G_1 \cap Z(S) - \{0\}$ and s by $x \in G_1$ in (2.12), we get $2[d(xz), x] = 2[d(x)z + xd(z), x] = 0$ and therefore

$$z[d(x), x] + x[d(z), x] = 0. \tag{2.13}$$

In (2.13) replacing z by $2z^2$, we obtain $2z^2[d(x), x] + 4zx[d(z), x] = 0$ which further implies that $2z(z[d(x), x] + x[d(z), x]) + 2zx[d(z), x] = 0$. Using (2.13) again, we obtain $2zx[d(z), x] = 0$ and therefore by the 2-torsion freeness $zG[d(z), x] = \{0\}$. By Lemma 1.2, we conclude that $[d(z), x] = 0$. Therefore (2.13) becomes $z[d(x), x] = 0$ and therefore $zS[d(x), x] = \{0\}$. By the primeness of S, we have $[d(x), x] = 0, \forall x \in G_1$. By Theorem 2.2 of [5], S is commutative.

On the similar lines of the proof of Theorem 2.1, we can establish the following:

Theorem 2.2. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(uv) + uv, r] = 0$ for all $u, v \in G$, then S is commutative.*

Theorem 2.3. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(uv) + vu^{'}, r] = 0$ for all $u, v \in G$, then S is commutative.*

Proof. By the hypothesis, we have for all $u, v \in G$

$$[F_d(uv) + vu^{'}, r] = 0. \tag{2.14}$$

Suppose that $G \cap Z(S) = \{0\}$. From (2.14) it is quite clear that $[F_d(u^2) + (u^2)^{'}, r] = 0$ and therefore $F_d(u^2) + (u^2)^{'} \in Z(S)$. In (2.14) replacing u by $2u^2$ and v by $2[s, jk]v$ where $j, k \in G, s \in S$, we obtain

$$4[F_d(u^2[s, jk]v) + 4[s, jk]v(u^2)^{'}, r] = 0$$

and therefore

$$[F_d(u^2[s, jk])v + u^2[s, jk]d(v) + [s, jk]v(u^2)^{'}, r] = 0.$$

As S is an MA-semiring, $v+v^{'} \in Z(S)$ and $v+v^{'}+v = v$, therefore $[(F_d(u^2[s, jk])+ [s, jk](u^2)^{'})v + u^2[s, jk]d(v) + [s, jk][u^2, v], r] = 0$, which further means that

$$(F_d(u^2[s, jk]) + [s, jk](u^2)^{'})v + u^2[s, jk]d(v) + [s, jk][u^2, v] \in Z(S).$$

In view of Lemma 1.1, following the similar arguments used in the proof of Theorem 2.1, we infer

$$(F_d(u^2[s, jk]) + [s, jk](u^2)^{'})v + u^2[s, jk]d(v) + [s, jk][u^2, v] \in G.$$

By our assumption $G \cap Z(S) = \{0\}$, we have

$$(F_d(u^2[s, jk]) + [s, jk](u^2)^{'})v + u^2[s, jk]d(v) + [s, jk][u^2, v] = 0. \tag{2.15}$$

In (2.15)replacing v by $4vu^2$ and using 2-torsion freeness, we obtain

$$(F_d(u^2[s, jk])+[s, jk](u^2)^{'})vu^2+u^2[s, jk]d(v)u^2+[s, jk][u^2, v]u^2+[u^2[s, jk]vd(u^2) = 0,$$

which further gives

$$((F_d(u^2[s, jk])+[s, jk](u^2)^{'})v+u^2[s, jk]d(v)+[s, jk][u^2, v])u^2+u^2[s, jk]vd(u^2) = 0.$$

Using (2.15) again, we get $u^2[s, jk]vd(u^2) = 0$. By Lemma 1.6, we have either $u^2 = 0$ or $Gd(u^2) = \{0\}$ which further implies by Remark 1.1, $G = \{0\}$ or

$d(u^2) = 0$. As $G \neq \{0\}$, therefore $d(u^2) = 0, \forall u \in G$. By Lemma 1.4, $d = 0$, a contradiction. Hence we conclude that $G \cap Z(S) \neq \{0\}$. In (2.14) replacing v by $4vx^2$ and r by u^2, we get $4[F_d(uvx^2) + vx^2u', x^2] = 0$, which implies $[F_d(uv)x^2 + uvd(x^2) + vx^2u', x^2] = 0$ and therefore $[F_d(uv), x^2]x^2 + [uvd(x^2) + vx^2u', x^2] = 0$. Using (2.14), we obtain $[vu, x^2]x^2 + [uvd(x^2) + vx^2u', x^2] = 0$ and therefore

$$[uvd(x^2), x^2] + [v[u, x^2], x^2] = 0. \tag{2.16}$$

In (2.16) replacing u by $4ux^2$, we obtain $[4ux^2vd(x^2), x^2] + [v[4ux^2, x^2], x^2] = 0$ and using MA-semiring commutators identities and 2-torsion freeness, we have

$$[ux^2vd(x^2), x^2] + [v[u, x^2], x^2]x^2 = 0. \tag{2.17}$$

Multiplying (2.16) by x^2 from the right we get

$$[uvd(x^2), x^2]x^2 + [v[u, x^2], x^2]x^2 = 0$$

and hence

$$[v[u, x^2], x^2]x^2 = [uvd(x^2), x^2](x^2)'. \tag{2.18}$$

Using (2.18) into (2.17), we get

$$[ux^2vd(x^2), x^2] + [uvd(x^2)(x^2)', x^2] = 0$$

and therefore

$$[u[x^2, vd(x^2)], x^2] = 0. \tag{2.19}$$

On the similar lines of the proof of Theorem 2.1, we infer $4vd(x^2)w \in G$, therefore replacing u by $4vd(x^2)w$ in (2.20), we obtain

$$4[vd(x^2)w[x^2, vd(x^2)], x^2] = 0$$

and using MA-semiring identities, we obtain

$$[vd(x^2), x^2]w[x^2, vd(x^2)] + vd(x^2)[w[x^2, vd(x^2)], x^2] = 0.$$

Using (2.19) again, we obtain $[vd(x^2), x^2]G[vd(x^2), x^2] = \{0\}$. By the Lemma 1.2, we obtain for all $v, x \in G$

$$[vd(x^2), x^2] = 0. \tag{2.20}$$

By our assumption $G \cap Z(S) \neq \{0\}$. Replacing v by $z \in G \cap Z(S) - \{0\}$, we have $zS[d(x^2), x^2] = \{0\}$, which further implies by the primeness $[d(x^2), x^2] = 0$ for all $x \in G$. By Lemma 1.5, $[d(x^2), s] = 0$, for all $x \in G, s \in S$, which is equation

(2.11) of Theorem 2.1, therefore the remaining part is same as the proof of Theorem 2.1. □

On the same lines of the proof of Theorem 2.3, we can prove the following:

Theorem 2.4. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(uv) + vu, r] = 0$ for all $u, v \in G$, then S is commutative.*

Theorem 2.5. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(u)F_d(v) + u^{'}v, r] = 0$ for all $u, v \in G$, then S is commutative.*

Proof. By the hypothesis, we have

$$[F_d(u)F_d(v) + u^{'}v, r] = 0, \ \forall u, v \in G, r \in S. \tag{2.21}$$

Assume that $G \cap Z(S) = \{0\}$. In (2.21) replacing v by $4v^2[s, xy]$, where $s \in S, x, y \in G$ and using 2-torsion freeness, we obtain $[(F_d(u)F_d(v^2) + u^{'}v^2)[s, xy] + F_d(u)v^2d[s, xy], r] = 0$, which further gives

$$(F_d(u)F_d(v^2) + u^{'}v^2)[s, xy] + F_d(u)v^2d[s, xy] \in Z(S).$$

By definition of G_1, $x \in G_1$ if $d(x) \in G$. In view of lemma 1.1, we see that for all $x, y, u, v \in G_1$,

$$\begin{aligned} &8((F_d(u)F_d(v^2) + u^{'}v^2)[s, xy] + F_d(u)v^2d[s, xy]) \\ &= 2(2[s(F_d(u)F_d(2v^2) + u^{'}2v^2), xy] + 2(F_d(u)(2v)^2d[s, xy]) \in G. \end{aligned}$$

In view of our assumption that $G \cap Z(S) = \{0\}$, by the 2-torsion freeness of S, we have for all $x, y, u, v \in G_1$

$$(F_d(u)F_d(v^2) + u^{'}v^2)[s, xy] + F_d(u)v^2d[s, xy] = 0. \tag{2.22}$$

In (2.22) replacing y by $4y^2s$ where $y \in G_1$, we get

$$(F_d(u)F_d(v^2) + u^{'}v^2)[s, xy^2s] + F_d(u)v^2d[s, xy^2s] = 0.$$

Using MA-semiring commutator identities, we further obtain

$$(F_d(u)F_d(v^2) + u^{'}v^2)[s, xy^2]s + F_d(u)v^2d[s, xy^2]s) = 0$$

and therefore

$$(F_d(u)F_d(v^2) + u^{'}v^2)[s, xy^2]s + F_d(u)v^2d[s, xy^2]s + F_d(u)v^2[s, xy^2]d(s) = 0.$$

Using (2.22) again, we obtain $F_d(u)v^2[s, xy^2]d(s) = 0$ and hence

(2.23) $$F_d(u)v^2x[s, y^2]d(s) + F_d(u)v^2[s, x]y^2d(s) = 0,\ \forall x, y, u, v \in G_1, s \in S.$$

As $8F_d(u)v^2[t, xw] = 2(F_d(u)4v^2)[t, xw] \in G_1,\ \forall u, v, w, x \in G_1, t \in S$, replacing x by $8F_d(u)v^2[t, xw]$ in (2.23) and using 2-torsion freeness, for all $x, y, u, v, w \in G_1, s \in S$, we get

$$F_d(u)v^2F_d(u)v^2[t, xw][s, y^2]d(s) + F_d(u)v^2[s, F_d(u)v^2[t, xw]]y^2d(s) = 0$$

and by using identities, we have

$$F_d(u)v^2F_d(u)v^2[t, xw][s, y^2]d(s) + F_d(u)v^2[s, F_d(u)v^2][t, xw]y^2d(s)$$
$$+F_d(u)v^2F_d(u)v^2[s, [t, xw]]y^2d(s) = 0.$$

Using 2-torsion freeness and (2.23) again, we obtain

(2.24) $$F_d(u)v^2[s, F_d(u)v^2][t, xw]y^2d(s) = 0.$$

By lemma 1.6, we obtain either $F_d(u)v^2[s, F_d(u)v^2] = 0$ or $y^2d(s) = 0$. For the second possibility, we obtain either $G_1 = \{0\}$ or $d = 0$ and both contradict the hypothesis. On the other hand suppose that

(2.25) $$F_d(u)v^2[s, F_d(u)v^2] = 0$$

In (2.25) replacing s by st and using (2.25) again, we obtain $F_d(u)v^2s[t, F_d(u)v^2] = 0$ and hence by the primeness of S, we have either $F_d(u)v^2 = 0$ or $[t, F_d(u)v^2] = 0$. The second one implies that $F_d(u)v^2 \in Z(S)$. Also since $4F_d(u)v^2 \in G_1$ and by assumption $G_1 \cap Z(S) = \{0\}$, therefore $4F_d(u)v^2 = 0$ which implies by the 2-torsion freeness that $F_d(u)v^2 = 0$. We conclude that in both the cases

(2.26) $$F_d(u)v^2 = 0,\ \forall u, v \in G_1.$$

In (2.26) replacing u by $2u^2r$, we get $2F_d(uw^2)v^2 = 0$ which implies $F_d(u)w^2v^2 + ud(w^2)v^2 = 0$. Using (2.26) again and then using Remark1.1, we get $d(w^2)v^2 = 0$, which further implies either $d = 0$ or $G_1 = \{0\}$ which both contradict the hypothesis. Hence our supposition is wrong and $G \cap Z(S) \neq \{0\}$. For any $z \in G \cap Z(S) - \{0\}$, replacing v by $v \circ z = 2vz$ in (2.21), we get for all $u, v \in G$,

$$2[F_d(u)F_d(vz) + u^{'}vz, r] = 0,$$

which implies that

$$[F_d(u)F_d(v)z + F_d(u)vd(z) + u^{'}vz, r] = 0,$$

and therefore

$$z[F_d(u)F_d(v)+u'v,r]+[F_d(u)vd(z),r]=0,$$

and using (2.21) again, we get

$$[F_d(u)vd(z),r]=0. \tag{2.27}$$

In (2.27) replacing z by $2z^2$ and using (2.27) again, we get $4F_d(u)vd(z)[z,r]=0$, and therefore by the 2-torsion freeness, we further get $F_d(u)Gd(z)[z,r]=\{0\}$. By Lemma 1.2 we obtain either $F_d(u)=0$ or $d(z)[z,r]=0$. If $F_d(u)=0$, then $d(u)=0$ and by Lemma 1.3 $d=0$, a contradiction. On the other hand if $d(z)[z,r]=0$, then after appropriate replacements we obtain $d(z)=0$ or $[z,r]=0$ which further implies $d(z)=0$ or $[d(z),r]=0$. Suppose that $[d(z),r]=0$. Then $d(z)\in Z(S)$, therefore (2.27) becomes $[F_d(u)v,r]d(z)=0$ which further implies $[F_d(u)v,r]Sd(z)=\{0\}$. By the primeness of S, we have either $d(z)=0$ or $[F_d(u)v,r]=0$. Suppose that

$$[F_d(u)v,r]=0. \tag{2.28}$$

In (2.28) replacing v by $4v^2s$, where $s\in S$, we obtain $2[F_d(u)2v^2s,r]=0$. Using 2-torsion freeness and (2.28) again, we obtain

$$F_d(u)v^2[s,r]=0. \tag{2.29}$$

In (2.29) replacing s by ts and using (2.29) again, we obtain $F_d(u)v^2S[s,r]=\{0\}$, which by the primeness of S implies either S is commutative or $F_d(u)v^2=0$. Suppose that

$$F_d(u)v^2=0. \tag{2.30}$$

In (2.30) replacing u by $4uw^2$ and using (2.30) again, we obtain $Gd(w^2)v^2=0$. By Remark 1.1, we have

$$d(w^2)v^2=0. \tag{2.31}$$

In (2.31) replacing v by $u+v$ and using (2.31) again, we get

$$d(w^2)vu+d(w^2)uv=0. \tag{2.32}$$

In (2.32) replacing u by $2u^2$ and using (2.32) again, we obtain $d(w^2)Gu^2=\{0\}$. By Lemma 1.2, we obtain either $d(w^2)=0$ or $u^2=0$ which respectively imply either $d=0$ or $G=\{0\}$ and both contradict the hypothesis.

Now we suppose the case when $d(z) = 0$, for all $z \in G \cap Z(S)$. In (2.21) replacing v by $4vx^2s$, we obtain

$$4[F_d(u)F_d(vx^2) + u^{'}vx^2s + F_d(u)v^2d(x^2), r] = 0,\ \forall u, v, x \in G, r \in S.$$

Taking $r = x^2$ and using (2.21) again

$$[F_d(u)vd(x^2), x^2] = 0. \tag{2.33}$$

In(2.33) replacing u by $4ux^2$, we obtain

$$[F_d(u)(4x^2v)d(x^2) + 4ud(x^2)vd(x^2), x^2] = 0$$

and using (2.33) we obtain

$$[ud(x^2)vd(x^2), x^2] = 0. \tag{2.34}$$

As above $4d(x^2)u \in G, \forall u, x \in G_1$, Replacing u by $4d(x^2)u$ in (2.34), we obtain for all $u, v, x \in G_1$,

$$[d(x^2), x^2]ud(x^2)vd(x^2) = 0,$$

and therefore

$$[d(x^2), x^2]ud(x^2)G_1d(x^2) = \{0\}.$$

By Lemma 1.2, we obtain either $d(x^2) = 0$ or $[d(x^2), x^2]ud(x^2) = 0$. If $d(x^2) = 0$, then $d = 0$, a contradiction. On the other hand $[d(x^2), x^2]G_1d(x^2) = \{0\}$ for all $x \in G_1$. As above $d(x^2) = 0$ leads to $d = 0$, therefore for all $x \in G_1$, we have $[d(x^2), x^2] = 0$. Applying Lemma 1.5, we have

$$[d(x^2), r] = 0, \forall r \in S, x \in G_1. \tag{2.35}$$

In (2.35) replacing x by $s \circ z = 2sz$ and using the fact that $d(z) = 0$, we obtain $[sd(s) + d(s)s, r] = 0$ and replacing s by s^2 it further gives $d(s^2)[s^2, r] = 0$ and hence $d(s^2)S[s^2, r] = \{0\}$. By the primeness we obtain either $d(s^2) = 0$ or $[s^2, r] = 0$. If $d(s^2) = 0$, then by Lemma 1.4 $d = 0$, a contradiction. Secondly if $[s^2, r] = 0$, then we can easily see that S is commutative. □

On the similar lines of the proof of Theorem 2.5, we can establish the following:

Theorem 2.6. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(u)F_d(v) + uv, r] = 0$ for all $u, v \in G$, then S is commutative.*

Theorem 2.7. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies*

$$[F_d(u)F_d(v) + vu^{'}, r] = 0, \; for \; all \; u, v \in G, \tag{2.36}$$

then S is commutative.

Proof. Firstly suppose that $G \cap Z(S) = \{0\}$. In (2.36), replacing v by $4[s, pq][t, xy]$, where $s, t \in S, p, q, x, y \in G$ and then using the definition of MA-semiring, we obtain

$$[F_d(u)F_d([s, pq])[t, xy] + [s, pq][t, xy](u^{'} + u + u^{'}) + F_d(u)[s, pq]d([t, xy]), r] = 0.$$

As $u + u^{'} \in Z(S)$, therefore after simplification, we obtain

$$(F_d(u)F_d([s, pq]) + [s, pq]u^{'})[t, xy] + [s, pq][u, [t, xy]] + F_d(u)[s, pq]d([t, xy]) \in Z(S).$$

On the other hand, as above we can easily see that for all $u, v, x, y \in G_1, \; s, t \in S$,

$$(F_d(u)F_d([s, pq]) + [s, pq]u^{'})[t, xy] + [s, pq][u, [t, xy]] + F_d(u)[s, pq]d([t, xy]) \in G.$$

By the assumption $G \cap Z(S) = \{0\}$, we have

$$(F_d(u)F_d([s, pq]) + [s, pq]u^{'})[t, xy] + [s, pq][u, [t, xy]] + F_d(u)[s, pq]d([t, xy]) = 0. \tag{2.37}$$

In (2.37) replacing t by txy and using MA-semiring identities, we get

$$\begin{aligned} &(F_d(u)F_d([s, pq]) + [s, pq]u^{'})[t, xy]xy + [s, pq][u, [t, xy]xy] \\ &+F_d(u)[s, pq]d([t, xy])xy + F_d(u)[s, pq][t, xy]d(xy) = 0. \end{aligned}$$

Using (2.37) again, we obtain

$$[s, pq][t, xy][u, xy] + F_d(u)[s, pq][t, xy]d(xy) = 0. \tag{2.38}$$

In (2.38) replacing p by $4sp^2$, we obtain

$$s[s, p^2q][t, xy][u, xy] + F_d(u)s[s, p^2q][t, xy]d(xy) = 0. \tag{2.39}$$

Multiplying (2.39) by s from the left, we have

$$s[s, pq][t, xy][u, xy] + sF_d(u)[s, pq][t, xy]d(xy) = 0,$$

which further implies

$$s[s, pq][t, xy][u, xy] = s^{'}F_d(u)[s, pq][t, xy]d(xy). \tag{2.40}$$

Using (2.40) into (2.39), we obtain

(2.41) $$[F_d(u), s][s, p^2q][t, xy]d(xy) = 0, \forall p, q, u, x, y \in G_1, s, t \in S.$$

In (2.41) replacing t by $[F_d(u), s][s, p^2q]t$ and using (2.41) again, we get

$$[F_d(u), s][s, p^2q][[F_d(u), s][s, p^2q], xy]Sd(xy) = 0.$$

As S is prime, we have either $d(xy) = 0$ or $[F_d(u), s][s, p^2q][[F_d(u), s][s, p^2q], xy] = 0$. If $d(xy) = 0$, then by Lemma 1.4, $d = 0$, a contradiction. On the other hand, if $[F_d(u), s][s, p^2q][[F_d(u), s][s, p^2q], xy] = 0$, then by the MA-semiring identities, we can write

(2.42)
$$[F_d(u), s][s, p^2q]x[[F_d(u), s][s, p^2q], y] + [F_d(u), s][s, p^2q][[F_d(u), s][s, p^2q], x]y = 0.$$

In (2.42) replacing y by $2y[t, r], t, r \in S$,

$$[F_d(u), s][s, p^2q]x[[F_d(u), s][s, p^2q], y[t, r]]$$
$$+[F_d(u), s][s, p^2q][[F_d(u), s][s, p^2q], x]y[t, r] = 0,$$

and after simplification, it further gives

$$[F_d(u), s][s, p^2q]x[[F_d(u), s][s, p^2q], y][t, r]$$
$$+[F_d(u), s][s, p^2q]xy[[F_d(u), s][s, p^2q], [t, r]] \quad +[F_d(u), s][s, p^2q][[F_d(u), s][s, p^2q], x]y[t, r] = 0.$$

Using (2.42) again, we obtain

$$[F_d(u), s][s, p^2q]xG[[F_d(u), s][s, p^2q], [t, r]] = \{0\}.$$

By Lemma 1.2 and hence by Remark 1.1, we obtain either $[F_d(u), s][s, p^2q] = 0$ or $[[F_d(u), s][s, p^2q], [t, r]] = 0$. Firstly assume that

(2.43) $$[[F_d(u), s][s, p^2q], [t, r]] = 0.$$

In (2.43) replacing r by rt, we get

$$[[F_d(u), s][s, p^2q], [t, r]]t + [t, r][[F_d(u), s][s, p^2q], t] = 0,$$

and using (2.43) again, we obtain

(2.44) $$[t, r][[F_d(u), s][s, p^2q], t] = 0.$$

In (2.44) replacing r by $rm, m \in S$ and using (2.44) again, we obtain

$$[t, r]S[[F_d(u), s][s, p^2q], t] = \{0\}$$

and taking $r = [F_d(u), s][s, p^2q]$, we can write

$$[[F_d(u), s][s, p^2q], t]S[[F_d(u), s][s, p^2q], t] = \{0\}$$

By the primeness, we get $[[F_d(u), s][s, p^2q], t] = 0$, which implies

$$[F_d(u), s][s, p^2q] \in Z(S)$$

and therefore (2.41) becomes

$$[F_d(u), s][s, p^2q]S[t, xy]d(xy) = \{0\},$$

which further gives either S is commutative or $[F_d(u), s][s, p^2q] = 0$. Assume that

$$[F_d(u), s][s, p^2q] = 0. \tag{2.45}$$

In (2.45) replacing q by $2[t, xy]q$, we get

$$2[F_d(u), s]p^2[t, xy][s, q] + [F_d(u), s][s, p^2(2[t, xy])]q = 0,$$

using (2.45) again and the 2-torsion freeness, we obtain

$$2[F_d(u), s]p^2[t, xy][s, q] = 0.$$

Using Lemma 1.6, we have either $[F_d(u), s]p^2 = 0$ or $[S, G_1] = \{0\}$. If $[S, G_1] = \{0\}$, then by Theorem 2.3 of [1], S is commutative. From the secondly possibility, since $G_1 \neq \{0\}$, we obtain $[F_d(u), s] = 0$. Therefore the hypothesis becomes $[F_d(v)F_d(u) + v'u, s] = 0, \forall u, v \in G_1, s \in S$. Therefore for the assumption $G_1 \cap Z(S) = \{0\}$, following the same arguments of the proof of Theorem 2.4, we obtain $d(u^2) = 0, \forall u \in G_1$ and therefore by the Lemma 1.4, we get $d = 0$, a contradiction. Consequently $G \cap Z(S) \neq \{0\}$.

In (2.36) replacing v by $4vx^2$, we obtain

$$4[F_d(u)F_d(v)x^2 + F_d(u)vd(x^2) + vx^2u', r] = 0, \ for\ all\ u, v \in G.$$

As for each $s \in S$, $s + s' + s = s$ and $s + s' \in Z(S)$, therefore

$$[F_d(u)F_d(v)x^2 + vu'x^2 + F_d(u)vd(x^2) + vux^2 + vx^2u', r] = 0,$$

which further implies

$$[(F_d(u)F_d(v) + vu')x^2 + F_d(u)vd(x^2) + v[u, x^2], r] = 0. \tag{2.46}$$

In (2.46) replacing r by x^2 and using (2.36) again, we find

$$[F_d(u)vd(x^2) + v[u, x^2], x^2] = 0. \tag{2.47}$$

In (2.47) replacing u by $4ux^2$, we get

$$4[F_d(ux^2)vd(x^2) + v[ux^2, x^2], x^2] = 0$$

which further gives

$$[F_d(u)x^2vd(x^2) + ud(x^2)vd(x^2) + v[u, x^2]x^2, x^2] = 0,$$

and therefore

(2.48) $$[(F_d(u)x^2 + ud(x^2))vd(x^2), x^2] + [v[u, x^2], x^2]x^2 = 0.$$

In (2.47) replacing v by $4x^2v$, we obtain

$$[F_d(u)x^2vd(x^2), x^2] + x^2[v[u, x^2], x^2] = 0,$$

which further implies

(2.49) $$[F_d(u)x^2vd(x^2), x^2] = x^2[v^{'}[u, x^2], x^2].$$

Using (2.49) into (2.48), we obtain

$$[ud(x^2)vd(x^2), x^2] + [v[u, x^2], x^2]x^2 + x^2[v^{'}[u, x^2], x^2] = 0,$$

and therefore

(2.50) $$[ud(x^2)vd(x^2), x^2] + [[v[u, x^2], x^2], x^2] = 0.$$

In (2.50) replacing u by $4ux^2$ and using 2-torsion freeness, we obtain

$$[ux^2d(x^2)vd(x^2), x^2] + [[v[ux^2, x^2], x^2], x^2] = 0,$$

and therefore

(2.51) $$[ux^2d(x^2)vd(x^2), x^2] + [[v[u, x^2], x^2], x^2]x^2 = 0.$$

Multiplying (2.50) by x^2 from the right, we obtain

$$[ud(x^2)vd(x^2), x^2]x^2 + [[v[u, x^2], x^2], x^2]x^2 = 0,$$

which further implies

(2.52) $$[u^{'}d(x^2)vd(x^2)x^2, x^2] = [[v[u, x^2], x^2], x^2]x^2.$$

Using (2.52) into (2.51), we obtain

$$[ux^2d(x^2)vd(x^2), x^2] + [u^{'}d(x^2)vd(x^2)x^2, x^2] = 0,$$

and therefore

(2.53) $$[u[x^2, d(x^2)vd(x^2)], x^2] = 0.$$

By the definition of G and G_1, we observe that

$$4d(x^2)vd(x^2)u = 4(d(x) \circ x)yd(x^2)u = 2[d(x) \circ x, vd(x^2)u] + 2(d(x) \circ x) \circ (vd(x^2)u).$$

Therefore $4d(x^2)vd(x^2)u \in G$, for all $u, v, x \in G_1$. In (2.53) replacing u by $4d(x^2)vd(x^2)u$, for all $u, x, v \in G_1$ and then using 2-torsion freeness, we obtain

$$[d(x^2)vd(x^2), x^2]G_1[d(x^2)vd(x^2), x^2] = \{0\}.$$

By Lemma 1.2, we obtain

$$[d(x^2)vd(x^2), x^2] = 0. \tag{2.54}$$

As above $8wd(x^2)v \in G$, for all $w \in G$, $x, v \in G_1$, therefore replacing v by $8wd(x^2)v$ in (2.54) and then by the 2-torsion freeness of S, we obtain

$$[d(x^2)wd(x^2)vd(x^2), x^2] = 0.$$

Using (2.54) again, we can find $[d(x^2)w, x^2]d(x^2)G_1d(x^2) = \{0\}$. By Lemma 1.2, we obtain either $[d(x^2)w, x^2]d(x^2) = 0$ or $d(x^2) = 0$. If $d(x^2) = 0$, then by Lemma 1.4, $d = 0$, a contradiction. Therefore we must have

$$[d(x^2)w, x^2]d(x^2) = 0. \tag{2.55}$$

From (2.54), using MA-semiring identities, we can write

$$[d(x^2)v, x^2]d(x^2) + d(x^2)v[d(x^2), x^2] = 0,$$

and using (2.55) in particular for $w = v \in G_1$, we find $d(x^2)v[d(x^2), x^2] = 0, \forall v, x \in G_1$ and therefore by Lemma 1.2, we have either $d(x^2) = 0$ or $[d(x^2), x^2] = 0$. If $d(x^2) = 0$, then $d = 0$, a contradiction. On the other hand if $[d(x^2), x^2] = 0$, then following the same arguments as above, we conclude the required result. $\square$

On the similar lines of the proof of Theorem 2.7 we can establish the following:

Theorem 2.8. *Let G be a nonzero Jordan ideal of a 2-torsion free prime MA-semiring S and F_d be a generalized derivation associated with a nonzero derivation d. If F_d satisfies $[F_d(u)F_d(v) + vu, r] = 0$ for all $u, v \in G$, then S is commutative.*

3. Open Problems

The results of this paper are proved for the generalized derivations satisfying central identities on Jordan ideals of prime MA-semirings. It would be interesting to investigate the results of this paper for semiprime MA-semirings and for the Lie ideals instead of Jordan ideals.

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