

## A NUMERICAL STUDY OF A HOMOGENEOUS BEAM WITH A TIP MASS

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**ABSTRACT.** In this paper, we prove the existence and uniqueness of the weak solution of a flexible beam that is clamped at one end and free at the other; a mass is also attached to the free end of the beam. Also, we construct a finite element method, based on piecewise cubic Hermitian shape functions. Next, we derive error estimates for the semi-discrete Galerkin approximations. The results are derived from [2]. Finally, we implement the results of numerical schemes developed.

### 1. INTRODUCTION

In this work, we present some sufficient conditions for the existence and uniqueness of the weak solution and we develop a numerical method for an Euler-Bernoulli beam equation. The hybrid system submitted to our study consists of a flexible beam that is clamped at one end and free at the other end; a mass is also attached to the free end. The equations of motion for this system

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are given by:

$$(1.1) \quad w_{tt}(x, t) + w_{xxxx}(x, t) = 0, \quad x \in (0, 1), t > 0,$$

$$(1.2) \quad w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, \quad t \geq 0,$$

$$(1.3) \quad -w_{xxx}(1, t) + mw_{tt}(1, t) = u(t), \quad t \geq 0,$$

where  $m > 0$  is the tip mass,  $w$  is the amplitude of the vibration and  $u(t)$  is the boundary control force applied at the free end of the beam; a subscript letter denotes the partial derivation with respect to that variable. For simplicity, and without loss of generality, the length of the beam, the mass per unit length, and the flexural rigidity of the beam are chosen to be unity. The following linear feedback control law is proposed in [5]:

$$u(t) = -\alpha w_t(1, t) + \beta w_{xxxt}(1, t),$$

$t \geq 0$  ( $\alpha$  and  $\beta$  are positive constants). Then, the closed-loop system becomes:

$$(1.4) \quad w_{tt}(x, t) + w_{xxxx}(x, t) = 0, \quad x \in (0, 1), t > 0,$$

$$(1.5) \quad w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, \quad t \geq 0,$$

$$(1.6) \quad w_{xxx}(1, t) = mw_{tt}(1, t) + \alpha w_t(1, t) - \beta w_{xxxt}(1, t) \quad t \geq 0.$$

The total mechanical energy  $\mathbb{E} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  to the above system is given by

$$(1.7) \quad \mathbb{E}(t) = \frac{1}{2} \int_0^1 w_{xx}^2 dx + \frac{1}{2} \int_0^1 v^2 dx + \frac{K}{2} \gamma^2.$$

It has been shown in [5] that

$$(1.8) \quad \frac{d}{dt} \mathbb{E}(t) = -\frac{K}{\beta} w_{xxx}^2(1) - \frac{Km\alpha}{\beta^2} v^2(1) \leq 0.$$

The expression (1.8) shows that the energy  $\mathbb{E}(t)$  decreases over time and therefore defines a Lyapunov function.

Following [9, 14], for  $\beta = 0$ , it has been shown that the system with this feedback law  $u(t) = -\alpha w_t(1, t)$  leads to an asymptotic stability but not to an exponential stability. In [5] on the other hand, for  $\beta$  a positive constant, the authors showed that there is uniform stability of the system. To show that the system is exponentially stable for any positive reals  $\alpha$  and  $\beta$ , the energy multiplier method is used. In addition, they analyzed the spectrum of the system for a special case where  $m = \alpha\beta$  and proved that the spectrum determines the exponential decay rate for the case considered for almost all  $\alpha > 0$ . It has been shown in [7] that

a sequence of generalized eigenfunctions of problem (1.4)–(1.6) forms a Riesz basis on the suitable Hilbert space.

Our main contribution is, on the one hand, to show the existence, the uniqueness and the higher regularity of the weak solution of the system after having formulated it as an evolution problem. To do this, we base ourselves on the work done in [3, 10]. The proof of existence being based on the Faedo-Galerkin method as well as the proofs of existence developed in Evans [6], Lions [8]. The major challenge of this article is the taking into account of the boundary condition (1.6) which makes the resolution of the problem more difficult than the classical cases approached in [3, 10]. We are therefore led to use certain strategies that will allow us to solve this problem. On the other hand, it is a question of developing a convergent numerical method, which faithfully reproduces certain properties of this problem such as stability and energy decay.

This article is subdivided into five sections. In the section 2 of this paper, we formulate the system (1.4)–(1.6) as a Cauchy problem and study the stability of the closed-loop system through Lyapunov method. In section 3, from the weak formulation, by referring to the properties of existence and uniqueness expressed in Lions [8] and Temam [15], we show the existence and uniqueness of the weak solution. In section 4, we define the method to be used, namely the method of the finite elements of Hermite and we study the error estimates for the semi-discrete scheme as well as convergence. In the last section, we implement the results of numerical schemes developed.

## 2. FORMULATION OF THE SYSTEM IN THE CONTEXT OF THE $C_0$ -SEMIGROUP OF CONTRACTIONS THEORY

**2.1. Semigroup formulation.** To analyse the system given by (1.4)–(1.6), we first introduce the following spaces. We introduce the functional space

$$H_E^2(0, 1) = \{w \in H^2(0, 1), w(0) = w_x(0) = 0\},$$

where  $H^m(0, 1)$  is defined by

$$H^m(0, 1) = \{w : [0, 1] \longrightarrow \mathbb{R} \mid w, w^1 = \frac{\partial w}{\partial t}, \dots, w^m = \frac{\partial^m w}{\partial t^m} \in L^2(0, 1)\},$$

we denote by  $\|\cdot\|_m$  the associated norm of the space  $H^m(0, 1)$ , and

$$L^2(0, 1) = \{w : [0, 1] \longrightarrow \mathbb{R} \mid \int_0^1 w^2 dx < \infty\}$$

with  $\|\cdot\|$  the associated norm of the space. Moreover, we introduce the energy space:

$$(2.1) \quad \begin{aligned} \mathcal{H} &:= \{z = (w, v, \gamma)^T \mid w \in H_E^2(0, 1), v \in L^2(0, 1), \gamma \in \mathbb{R}\} \\ &= H_E^2(0, 1) \times L^2(0, 1) \times \mathbb{R}, \end{aligned}$$

the superscript  $T$  stands for the transpose.

In the space  $\mathcal{H}$ , we define the inner-product:

$$(2.2) \quad \langle z_1, z_2 \rangle_{\mathcal{H}} = \int_0^1 (w_1)_{xx}(x)(w_2)_{xx}(x)dx + \int_0^1 v_1(x)v_2(x)dx + K\gamma_1\gamma_2,$$

where  $z_1 = (w_1, v_1, \gamma_1)^T \in \mathcal{H}$ ;  $z_2 = (w_2, v_2, \gamma_2)^T \in \mathcal{H}$  and  $K = \frac{\beta^2}{m + \alpha\beta} > 0$ .

We denote by  $\|\cdot\|_{\mathcal{H}}$  the norm associated to the inner-product in the space  $\mathcal{H}$ .

$D(0, 1) :=$  the space of smooth functions with compact support,

$D'(0, 1) :=$  the space of continuous linear functions.

Next, we define an unbounded linear operator  $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \longrightarrow \mathcal{H}$  with the domain

$$(2.3) \quad \begin{aligned} D(\mathcal{L}) &= \{(w, v, \gamma)^T \mid w \in H^4(0, 1) \cap H_E^2(0, 1), v \in H_E^2(0, 1), \\ &\quad w_{xx}(1) = 0, \gamma = -w_{xxx}(1) + \frac{m}{\beta}v(1)\}, \forall t > 0 \end{aligned}$$

defined by

$$\mathcal{L} \begin{bmatrix} w \\ v \\ \gamma \end{bmatrix} = \begin{bmatrix} v \\ -w_{xxxx} \\ -\frac{1}{\beta}\gamma - \frac{1}{\beta}(\alpha - \frac{m}{\beta})v(1) \end{bmatrix}.$$

The equations (1.4) – (1.6) can be written in the following abstract form

$$(2.4) \quad \begin{cases} z_t = \mathcal{L}z(t) \\ z(0) = z_0 \in \mathcal{H}, \end{cases}$$

where

$$z(t) = (w(\cdot, t), w_t(\cdot, t), \frac{m}{\beta}w_t(1, t) - w_{xxx}(1, t))^T; z_0 = (w_0, w_1, \frac{m}{\beta}w_1(1) - w_0'''(1))^T,$$

for all  $t > 0$ . Therefore, this result follows immediately from the theory of operator semigroups (see [13]).

**Theorem 2.1.** *The operator  $\mathcal{L}$  defined by (2.3) – (2.4) generates a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ , denoted by  $\{S(t)\}_{t \geq 0}$ .*

*Proof.* (see [5]). □

The next result follows immediately from Theorem 2.1.

**Theorem 2.2.** *The equation (2.4) has a unique solution*

$$z(t) = S(t)z_0 \in C([0, \infty); \mathcal{H}), \forall z_0 \in \mathcal{H}.$$

**2.2. Stability analysis using Lyapunov.** The studied systems can only be known approximately (the parameters of the system may not be known precisely or terms may be missing in the equations). Thus, the approximations used therefore call into question the validity or the relevance of numerical solutions. To overcome this difficulty, several concepts of stability have been introduced in the study, such as Lyapunov's stability, Lagrange's stability and others. Here, we use the Lyapunov method which allows us to study the stability of the system without knowing the explicit solution of the studied system.

We give the definition of a Lyapunov function in a Hilbert space to fix ideas:

**Definition 2.1.** *The functional  $p : \mathcal{H} \rightarrow \mathbb{R}$  is called the Lyapunov functional of the evolution problem (2.4) if the following propositions are verified:*

- (i)  $p(z) > 0, \forall z \in \mathcal{H} - \{0\}$ ,
- (ii)  $p(0) = 0$ ,
- (iii)  $\dot{p}(z_0) \leq 0 \forall z_0 \in \mathcal{H}$ .

Consider the following Lyapunov candidate, the functional  $p : \mathcal{H} \rightarrow \mathbb{R}$  defined by:

$$(2.5) \quad p(z) = \frac{1}{2} \int_0^1 w_{xx}^2 dx + \frac{1}{2} \int_0^1 w_t^2 dx + \frac{K}{2} \left( \frac{m}{\beta} w_t(1) - w_{xxx}(1) \right)^2.$$

Analogously as in (1.8), for all classical solutions  $z$  it follows that:

$$(2.6) \quad \frac{d}{dt} p(z) = \frac{d}{dt} \|z\|_{\mathcal{H}}^2 \leq 0,$$

hence, time evolution of the Lyapunov functional  $p$  along the classical solutions is non-increasing. We can say that the system (2.4) is stable in the sense of

Lyapunov. Thus, from Theorem 2.1, the decay of energy along the classical solutions can be extended to mild solutions:

**Theorem 2.3.** *Assume that  $z(t)$  is the mild solution of (2.4) for some  $z_0 \in \mathcal{H}$ . Then  $z(t) \rightarrow 0$  in  $\mathcal{H}$  when  $t \rightarrow \infty$ .*

Now, the system of equations (1.4) – (1.6) is written in the weak form, and the existence and uniqueness of the weak solution are demonstrated.

### 3. EXISTENCE, UNIQUENESS AND HIGHER REGULARITY OF THE WEAK SOLUTION

**3.1. Weak formulation.** Multiplying (1.4) by  $\varphi(x) \in H_E^2(0, 1)$  and integrating over  $(0, 1)$ , we have:

$$(3.1) \quad \int_0^1 w_{tt}\varphi dx + \int_0^1 w_{xxxx}\varphi dx = 0, \forall \varphi \in H_E^2(0, 1), t > 0.$$

Integrating twice by parts and taking into account the boundary conditions, it follows:

$$(3.2) \quad \int_0^1 w_{tt}\varphi dx + \int_0^1 w_{xx}\varphi_{xx} dx + w_{xx}(1, t)\varphi(1) = 0, \forall \varphi \in H_E^2(0, 1), t > 0.$$

$$(3.3) \quad \int_0^1 w_{tt}\varphi dx + \int_0^1 w_{xx}\varphi_{xx} dx + mw_{tt}(1, t)\varphi(1) + \alpha w_t(1, t)\varphi(1) - \beta w_{xxxt}(1, t)\varphi(1) = 0, \forall \varphi \in H_E^2(0, 1), t > 0.$$

In the definition of the weak formulation a very important element is the appropriate space setting. We rely on the work done in [1] for more appropriate space choices. We define the Hilbert space  $X$  by

$$X = \mathbb{R}^2 \times H_E^2(0, 1) = \{\hat{w} = (w(1), w_x(1), w); w \in H_E^2(0, 1)\}$$

with the inner product:

$$(3.4) \quad \langle \hat{w}_1, \hat{w}_2 \rangle_X = \langle (w_1)_{xx}, (w_2)_{xx} \rangle_{L^2(0,1)}.$$

We also define the Hilbert space

$$Z = \mathbb{R}^2 \times L^2(0, 1) = \{\hat{y} = (y(1), y_x(1), y); y \in L^2(0, 1)\}$$

with the inner product:

$$(3.5) \quad \langle \hat{y}_1, \hat{y}_2 \rangle_Z = my_1(1)y_2(1) + \langle y_1, y_2 \rangle_{L^2(0,1)}.$$

Assume that the injection  $X \subset Z$  is continuous. There is a canonical map  $T : Z' \longrightarrow X'$ . Identifying  $Z'$  with  $Z$  and using  $T$  as a canonical embedding from  $Z'$  into  $X'$ , we can write

$$X \subset Z \equiv Z' \subset X'$$

where all the injections are continuous and dense (provided  $X$  is reflexive). One says that  $Z$  is the pivot space, where  $X'$  and  $Z'$  are respectively the space of continuous linear functionals on  $X$  and  $Z$ . Consider the bilinear forms:

$$c_1 : X \times X \longrightarrow \mathbb{R} \quad (\hat{w}_1, \hat{w}_2) \mapsto c_1(\hat{w}_1, \hat{w}_2) = \langle \hat{w}_1, \hat{w}_2 \rangle_X$$

and

$$c_2 : Z \times Z \longrightarrow \mathbb{R} \quad (\hat{y}_1, \hat{y}_2) \mapsto c_2(\hat{y}_1, \hat{y}_2) = \alpha y_1(1)y_2(1) - \beta(y_{1x})_{xx}(1)y_2(1).$$

Here, the term  $w_{xxxt}(1)$  needs to be considered. Then, the bilinear form  $c_2(.,.)$  with the first order boundary term in  $t$  requires a slight generalization of the standard theory (as presented for example in section 7.2 of [6]).

**Definition 3.1.** *Let  $T > 0$  be fixed. We say that  $\hat{w} = (w(1), w_x(1), w)$  is a weak solution of problem (1.4) – (1.6) on  $(0, 1)$  if  $\hat{w} \in L^2(0, T; X) \cap H^1(0, T; Z) \cap H^2(0, T; X')$  and satisfies*

$$(3.6) \quad \langle \hat{w}_{tt}, \hat{\phi} \rangle_{X, X'} + c_2(\hat{w}_t, \hat{\phi}) + c_1(\hat{w}, \hat{\phi}) = 0$$

for almost everywhere  $t \in (0, T)$  and for all  $\hat{\phi} \in X$  with the following initial conditions:

$$(3.7) \quad \hat{w}(0) = \hat{w}_0 = (w_0(1), (w_0)_x(1), w_0) \in X$$

$$(3.8) \quad \hat{w}_t(0) = \hat{v}_0 = (v_0(1), (v_0)_x(1), v_0) \in Z.$$

In the Definition 3.1, the bilinear form  $\langle ., . \rangle_{X, X'}$  is the duality pairing between  $X$  and  $X'$ . Moreover, the duality pairing on  $X' \times X$  can be identified with the unique extension of the inner product in  $Y$ . In (3.7),  $w_0(1)$  and  $(w_0)_x(1)$  are the boundary traces of  $w_0 \in H_E^2(0, 1)$ , but in (3.8),  $v_0(1)$  and  $(v_0)_x(1)$  are additionally given values. Note that in the case where  $\hat{w} \in H^2(0, T; X)$ , the formulation (3.6) is equivalent to equality (3.3).

In the next section, we present two measure theoretic lemmas which will be used in the following discussions. We just remind you what intermediate spaces are. For more details see [8].

The space  $[X, Z]_\theta$  with  $0 \leq \theta \leq 1$  is the intermediate space defined as in [8], for  $X$  and  $Z$  two Hilbert spaces,  $X \subset Z$ ,  $X$  dense in  $Z$  with continuous injection by means of domains of positive self-adjoint operators. Moreover for  $\theta = 0$ ,  $[X, Z]_0 = X$  and for  $\theta = 1$ ,  $[X, Z]_1 = Z$ . Since  $X \subset [X, Z]_\theta \subset Z$ , each space being dense in the following one, we have also by duality for  $\theta \in ]0, 1[$ :

$$Z' \subset [X, Z]_\theta' \subset X'$$

each space being dense in the following one.

**3.2. Existence and uniqueness of the solution.** In order to give a meaning to the initial conditions (3.7) and (3.8) we shall use the following lemma (special case of Theorem 3.1 in [8]).

**Lemma 3.1.** *Let  $X$  and  $Z$  be two Hilbert spaces, such that  $X$  is dense and continuously embedded in  $Z$ . Assume that  $w \in L^2(0, T; X)$  and  $v = w_t \in L^2(0, T; Z)$  then  $w \in C([0, T]; [X, Z]_{\frac{1}{2}})$  after, possibly, a modification on a set of measure zero.*

We need the following duality theorem and Theorem 3.1 for the proof of Theorem 3.2.

**Lemma 3.2.** *Let  $X$  and  $Z$  be two Hilbert spaces, such that  $X$  is dense and continuously embedded in  $Z$ . For all  $\theta \in ]0, 1[$ ,*

$$[X, Z]_\theta' = [Z', X']_{1-\theta}.$$

*with equivalent norms.*

**Theorem 3.1.** *Let  $H_E^2(0, 1)$  be a subspace of  $H^2(0, 1)$ . Then there exists a infinite sequence of functions  $\{\varphi_i\}_{i=1}^\infty$  such that:  $\{\varphi_i\}_{i=1}^\infty$  is an orthogonal basis of  $H_E^2(0, 1)$  and  $\{\varphi_i\}_{i=1}^\infty$  is an orthonormal basis of  $L^2(0, 1)$ .*

*Proof.* see [3, 10] □

**3.2.1. Existence of the weak solution.**

**Theorem 3.2.** *The weak formulation (3.6) – (3.8) has a unique solution  $\hat{w}$  such that:*

$$(3.9) \quad \hat{w} \in L^\infty(0, T; X), \hat{w}_t \in L^\infty(0, T; Z),$$

$$(3.10) \quad \hat{w} \in C([0, T]; [X, Z]_{\frac{1}{2}}),$$



$$(3.11) \quad \hat{w}_t \in C([0, T]; [X, Z]_{\frac{1}{2}}').$$

The following proof is an adaption of the proof of Theorem 8.1 in [8] which is based on the Faedo-Galerkin's method. This method consists of three steps.

*Proof.* Let  $\{\hat{\varphi}_i\}_{i=1}^{\infty}$  be a sequence of functions that is an orthonormal basis for  $Z$ , and an orthogonal basis for  $X$  according to Theorem 3.1. We introduce the following finite dimensional spaces spanned by  $\{\hat{\varphi}_i\}_{i=1}^m$  given by:

$$(3.12) \quad \forall m \in \mathbb{N}, \widehat{W}_m := \text{span}\{\hat{\varphi}_1, \dots, \hat{\varphi}_m\} = \left\{ \sum_{j=1}^m \alpha_j \hat{\varphi}_j; \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R} \right\}.$$

**Step 1 (Construction of approximate solutions):** We seek  $\hat{w} = \hat{w}_m(t) \in \widehat{W}_m$  the approximate solution of the problem in the form

$$\hat{w}_m(t) = \sum_{i=1}^m g_{im}(t) \hat{\varphi}_i,$$

where  $g_{im}(t) \in \mathbb{R}$  ( $0 \leq t \leq T$ ,  $i = 1, \dots, m$ ) solves the weak formulation (3.6) on  $\widehat{W}_m$ . For a fixed  $m \in \mathbb{N}$ , it follows

$$(3.13) \quad < (\hat{w}_m)_{tt}, \hat{\varphi} >_Z + c_1(\hat{w}_m, \hat{\varphi}) + c_2((\hat{w}_m)_t, \hat{\varphi}) = 0 \quad \forall \hat{\varphi} \in \widehat{W}_m.$$

And (3.13) is completed with the initial conditions:

$$(3.14) \quad \hat{w}_m(0) = \hat{w}_{m0}, \quad \hat{w}_{m0} = \sum_{i=1}^m \alpha_{im} \hat{\varphi}_i \longrightarrow \hat{w}_0 \text{ in } X \text{ when } m \longrightarrow \infty,$$

$$(3.15) \quad \hat{v}_m(0) = \hat{v}_{m0}, \quad \hat{v}_{m0} = \sum_{i=1}^m \beta_{im} \hat{\varphi}_i \longrightarrow \hat{v}_0 \text{ in } Z \text{ when } m \longrightarrow \infty,$$

with  $\alpha_{im} = g_{im}(0)$  and  $\beta_{im} = (g_{im})_t(0)$ . The ordinary differential equation of the second order thus obtained admits a unique solution  $\hat{w}_m \in C^2([0, T]; X)$  of (3.13) – (3.15) for  $0 \leq t \leq T$ .

**Step 2 (A-priori estimates on approximate solutions):** Let  $\widehat{E} : \mathbb{R} \times X \longrightarrow \mathbb{R}$  an energy functional, analogous to the Lyapunov functional in (2.5):

$$(3.16) \quad \widehat{E}(t, \hat{w}) = \frac{1}{2} \int_0^1 \hat{w}_{xx}^2 dx + \frac{1}{2} \int_0^1 \hat{w}_t^2 dx + \frac{1}{2} K \left( \frac{m}{\beta} \hat{w}_t(1) - \hat{w}_{xxx}(1) \right)^2,$$

$$(3.17) \quad \widehat{E}(t, \hat{w}) = \frac{1}{2} \left\| (w, w_t, \frac{m}{\beta} w_t(1) - w_{xxx}(1)) \right\|_{\mathcal{H}}.$$

For a solution  $\hat{w}_m \in C^2([0, \tau]; \widehat{W}_m)$  of (3.13) on some interval  $[0, \tau]$  and taking  $\hat{\varphi} = (\hat{w}_m)_t$  in (3.13), a straightforward calculation yields

$$\frac{d}{dt} \widehat{E}(t, \hat{w}_m) \leq 0,$$

for all  $t \in [0, \tau]$ . Dissipation of the functional  $\widehat{E}$  corresponds to the decay in (2.6) for the classical solution. Hence:

$$\widehat{E}(t, \hat{w}_m) \leq \widehat{E}(0, \hat{w}_{m0}), \quad t \geq 0,$$

which implies that:

$$(3.18) \quad \{\hat{w}_m\}_{m \in \mathbb{N}} \text{ is bounded in } C([0, T]; X),$$

$$(3.19) \quad \{(\hat{w}_m)_t\}_{m \in \mathbb{N}} \text{ is bounded in } C([0, T]; Z).$$

Considering the boundedness results in (3.18)-(3.19), for all  $\hat{\varphi} \in X$ , we have:

$$(3.20) \quad |c_1(\hat{w}_m(t), \hat{\varphi}) + c_2((\hat{w}_m)_t(t), \hat{\varphi})| \leq M \|\hat{\varphi}\|_X, \quad \forall t \in [0, T],$$

where  $M$  is a positive constant which does not depend on  $m$ . Let  $m \in \mathbb{N}$  be fixed. Also, we consider  $\hat{\varphi} \in X$  and  $\hat{\varphi} = \hat{\varphi}_1 + \hat{\varphi}_2$  such  $\hat{\varphi}_1 \in \widehat{W}_m$  and  $\hat{\varphi}_2$  orthogonal to  $\widehat{W}_m$  in  $Z$ . Then we obtain  $\langle (\hat{w}_m)_{tt}, \hat{\varphi} \rangle_Z = \langle (\hat{w}_m)_{tt}, \hat{\varphi}_1 \rangle_Z$ . From (3.13) and (3.20), we have:

$$(3.21) \quad \langle (\hat{w}_m)_{tt}, \hat{\varphi} \rangle_Z = -c_1(\hat{w}_m, \hat{\varphi}_1) - c_2((\hat{w}_m)_t, \hat{\varphi}_1) \leq M \|\hat{\varphi}_1\|_X \leq M \|\hat{\varphi}\|_X.$$

This implies that

$$(3.22) \quad \{(\hat{w}_m)_{tt}\}_{m \in \mathbb{N}} \text{ is bounded in } C([0, T]; X').$$

**Step 3 (Passage to the limit):** According to the Eberlein-Šmulian Theorem in [4], there exist subsequences  $\{\hat{w}_{m_l}\}_{l \in \mathbb{N}}$ ,  $\{(\hat{w}_{m_l})_t\}_{l \in \mathbb{N}}$  and  $\{(\hat{w}_{m_l})_{tt}\}_{l \in \mathbb{N}}$  with  $\hat{w} \in L^2(0, T; X)$ ,  $\hat{w}_t \in L^2(0, T; Z)$  and  $\hat{w}_{tt} \in L^2(0, T; X')$  such that:

$$(3.23) \quad \{\hat{w}_{m_l}\} \rightharpoonup \hat{w} \text{ in } L^2(0, T; X),$$

$$(3.24) \quad \{(\hat{w}_{m_l})_t\} \rightharpoonup \hat{w}_t \text{ in } L^2(0, T; Z),$$

$$(3.25) \quad \{(\hat{w}_{m_l})_{tt}\} \rightharpoonup \hat{w}_{tt} \text{ in } L^2(0, T; X').$$

Let  $m_0 \in \mathbb{N}$ . For all functions  $\hat{\varphi} \in L^2(0, T; \widehat{W}_{m_0})$  of the form

$$(3.26) \quad \hat{\varphi}(t, x) = \sum_{j=1}^{m_0} \kappa_j(t) \phi_j(x),$$

where  $\kappa_j \in L^2(0, T; \mathbb{R})$  and for all  $m_l \geq m_0$ , the formulation (3.13) becomes

$$(3.27) \quad \int_0^T \langle \hat{w}_{m_l} \rangle_{tt}, \hat{\varphi} \rangle_Z + c_1(\hat{w}_{m_l}, \hat{\varphi}) + c_2((\hat{w}_{m_l})_t, \hat{\varphi}) dt = 0.$$

Therefore, passing to the limit in (3.27) for  $m = m_l$ , when  $l \rightarrow \infty$  and using the convergence results (3.23)-(3.25), we obtain

$$(3.28) \quad \int_0^T \langle \hat{w}_{tt}, \hat{\varphi} \rangle_{X, X'} + c_1(\hat{w}, \hat{\varphi}) + c_2(\hat{w}_t, \hat{\varphi}) dt = 0,$$

consequently  $\langle \hat{w}_{tt}, \hat{\varphi} \rangle_{X, X'} + c_1(\hat{w}, \hat{\varphi}) + c_2(\hat{w}_t, \hat{\varphi}) = 0$  on  $[0, T]$  for all  $\hat{\varphi} \in L^2(0, T; X)$ .

The functions  $\hat{\varphi}$  of (3.26) being dense in  $L^2(0, T; X)$  and therefore (3.28) is well-defined for any  $\hat{\varphi} \in L^2(0, T; X)$ . We obtain the expression of the weak formulation (3.6) for almost everywhere on  $[0, T]$ . Hence  $\hat{w}$  is the solution of the weak formulation.

Concerning additional regularities, by definition of weak solution and (3.18)-(3.19),  $\hat{w}$  satisfies (3.9). As for (3.10), it immediately arises from the Lemma 3.1, after, possibly a modification on a set of measure zero, and finally, the regularity (3.11) is deduced from the Lemma 3.1 and from Lemma 3.2.  $\square$

Now, we prove that the solution  $\hat{w}$  satisfies initial conditions (3.7)-(3.8). Let  $\hat{\varphi} \in C^2([0, T]; X)$  such that  $\hat{\varphi}(T) = 0$  and  $\hat{\varphi}_t(T) = 0$ . Integrating (3.6) on  $[0, T]$ , we have:

$$(3.29) \quad \int_0^T [\langle \hat{w}_{tt}, \hat{\varphi} \rangle_{X, X'} + c_1(\hat{w}, \hat{\varphi}) + c_2(\hat{w}_t, \hat{\varphi})] d\tau = 0.$$

Integrating twice by parts on  $[0, T]$  under the duality pairing, we obtain:

$$(3.30) \quad \begin{aligned} & \int_0^T [\langle \hat{w}, \hat{\varphi}_{tt} \rangle_Z + c_1(\hat{w}, \hat{\varphi}) + c_2(\hat{w}_t, \hat{\varphi})] d\tau \\ &= \langle \hat{w}_t(0), \hat{\varphi}(0) \rangle_{X, X'} - \langle \hat{w}(0), \hat{\varphi}_t(0) \rangle_Z. \end{aligned}$$

For a fixed  $m$ , similarly from (3.13), it follows:

$$(3.31) \quad \begin{aligned} & \int_0^T [\langle \hat{w}_m, \hat{\varphi}_{tt} \rangle_Z + c_1(\hat{w}_m, \hat{\varphi}) + c_2((\hat{w}_m)_t, \hat{\varphi})] d\tau \\ &= \langle \hat{w}_{m0}, \hat{\varphi}(0) \rangle_Z - \langle \hat{w}_{m0}, \hat{\varphi}_t(0) \rangle_Z. \end{aligned}$$

Using (3.14)-(3.15) and (3.23)-(3.25), passing to the limit in (3.31) along the convergent subsequence, we obtain:

$$(3.32) \quad \int_0^T \left[ \langle \hat{w}, \hat{\varphi}_{tt} \rangle_Z + c_1(\hat{w}, \hat{\varphi}) + c_2((\hat{w})_t, \hat{\varphi}) \right] d\tau \\ = \langle \hat{v}_0, \hat{\varphi}(0) \rangle_Z - \langle \hat{w}_0, \hat{\varphi}_t(0) \rangle_Z.$$

Comparing (3.30) with (3.32), we deduce that  $\hat{w}(0) = \hat{w}_0$  and  $\hat{w}_t(0) = \hat{v}_0$  so initial conditions (3.7) and (3.8) are verified.

### 3.2.2. Uniqueness of the Weak Solution.

**Theorem 3.3.** *The solution  $\hat{w}$  of weak formulation (3.6) with the initial conditions (3.7) – (3.8) is unique.*

*Proof.* We use an adaption of proof of Theorem 8.1 in [8] to prove that a only weak solution of (3.6) is  $\hat{w} \equiv 0$ . For this, let fix  $0 \leq s \leq T$  and let introduce this auxiliary function:  $\hat{\psi} : ]0, T[ \rightarrow \mathbb{R}$ ,

$$\hat{\psi}(t) := \begin{cases} \int_t^s \hat{w}(\tau) d\tau, & 0 < t < s, \\ 0, & t \geq s. \end{cases}$$

Taking  $\hat{\psi}(t) = \hat{\varphi}(t)$  in (3.6) and by integration by parts of (3.6) on  $[0, T[$ , we have

$$(3.33) \quad \int_0^s \left[ \langle \hat{w}_t(\tau), \hat{w}(\tau) \rangle_Z - c_1(\hat{\psi}_t(\tau), \hat{\psi}(\tau)) + c_2(\hat{w}(\tau), \hat{w}(\tau)) \right] d\tau = 0.$$

We deduce from (3.33), the following equality

$$(3.34) \quad \int_0^s \frac{d}{dt} \left[ \frac{1}{2} \|\hat{w}(\tau)\|_Z^2 - \frac{1}{2} c_1(\hat{\psi}(\tau), \hat{\psi}(\tau)) \right] d\tau = - \int_0^s c_2(\hat{w}(\tau), \hat{w}(\tau)) d\tau.$$

This is equivalent to

$$\left[ \frac{1}{2} \|\hat{w}(\tau)\|_X^2 - \frac{1}{2} c_1(\hat{\psi}(\tau), \hat{\psi}(\tau)) \right]_0^s = - \int_0^s c_2(\hat{w}(\tau), \hat{w}(\tau)) d\tau.$$

Therefore,

$$\frac{1}{2} \|\hat{w}(s)\|_X^2 + \frac{1}{2} c_1(\hat{\psi}(0), \hat{\psi}(0)) \leq 0.$$

The bilinear form  $c_1(\cdot, \cdot)$  is coercive,  $\hat{w}(s) \equiv 0$  and  $\hat{\psi}(0) = 0$ . Since  $s \in ]0, T[$  was arbitrary, then  $\hat{w} \equiv 0$ .  $\square$

**3.3. Higher regularity results.** We recall the Lemma 8.1 (here Lemma 3.3) of [8] which will be used in the Theorem 3.4. Before let's give the following definition.

**Definition 3.2.** Let  $Z$  be a Banach space. Then

$$C_w([0, T]; Z) = \{w \in L^\infty(0, T; Z) : t \mapsto \langle f, w(t) \rangle \text{ is continuous on } [0, T], \forall f \in Z'\}$$

denotes the space of weakly continuous functions with values in  $Z$ .

**Lemma 3.3.** Let  $X$  and  $Z$  be two Banach spaces,  $X \subset Z$  with continuous injection,  $X$  being reflexive. Then

$$L^\infty(0, T; X) \cap C_w(0, T; Z) = C_w(0, T; X).$$

**Theorem 3.4.** The weak solution  $\hat{w}$  of (3.6) – (3.8) satisfies

$$(3.35) \quad \hat{w} \in C([0, T]; X),$$

$$(3.36) \quad \hat{w}_t \in C([0, T]; Z),$$

after, possibly, a modification on a set of measure zero.

*Proof.* This proof is an adaption of standard strategies to the situation at hand (cf. Section 8.4 of [8] pp. 297-301 and Section 2.4 of [15]). Using Lemma 3.3, it follows from (3.9)-(3.10) that  $\hat{w} \in C_w([0, T]; X)$ . Likewise, (3.9) and (3.11) imply  $\hat{w}_t \in C_w([0, T]; Z)$ .

We set  $\xi \in C^\infty(\mathbb{R})$  a fixed scalar cutoff function such as  $\xi(x) = 1$  if  $x \in J \subset \subset [0, T]$  and  $\xi(x) = 0$  else. The function  $\xi\hat{w}$  is then compactly supported. Let  $\eta^\varepsilon$  be a standard mollifier in time. The following notation is introduced:

$$\hat{w}^\varepsilon = \eta^\varepsilon * \xi\hat{w} \in C_c^\infty(\mathbb{R}, X).$$

$\hat{w}^\varepsilon$  converges to  $\hat{w}$  in  $X$  and  $(\hat{w}^\varepsilon)_t$  converges to  $\hat{w}_t$  a.e in  $Z$  for all element on  $J$ . Hence  $\hat{E}(t, \hat{w}^\varepsilon)$  converges to  $\hat{E}(t, \hat{w})$  a.e on  $J$ . Since  $\hat{w}^\varepsilon$  is smooth, a straightforward calculation on  $J$  gives:

$$(3.37) \quad \frac{d}{dt} \hat{E}(t, \hat{w}^\varepsilon) = -\frac{K}{\beta} [(\hat{w}_x^\varepsilon)_{xx}(t)]^2 - \frac{Km\alpha}{\beta^2} [(\hat{w}^\varepsilon)_t(t)]^2.$$

Passing to the limit in (3.37) when  $\varepsilon \rightarrow 0$ :

$$(3.38) \quad \frac{d}{dt} \hat{E}(t, \hat{w}) = -\frac{K}{\beta} [(\hat{w}_x)_{xx}(t)]^2 - \frac{Km\alpha}{\beta^2} [(\hat{w})_t(t)]^2.$$

holds in the sense of distributions on  $J$ . Since  $J$  was arbitrary, (3.38) holds on all compact subintervals of  $[0, T]$ . Let  $t \in [0, \infty[$  be fixed and  $\lim_{n \rightarrow \infty} t_n = t$ .

Taking the sequence  $(\sigma_n)_{n \in \mathbb{N}}$  defined by

$$(3.39) \quad \sigma_n = \frac{1}{2} \|(\hat{w}(t) - \hat{w}(t_n))\|_X^2 + \frac{1}{2} \|\hat{w}_t(t) - \hat{w}_t(t_n)\|_Z^2 + \\ \frac{K}{2} \left[ \left( \frac{m}{\beta}(w_t)(t) - (w_x)_{xx}(t) \right) - \left( \frac{m}{\beta}(w_t)(t_n) - (w_x)_{xx}(t_n) \right) \right]^2 - \\ \frac{m}{2}(w_t)^2(t) - \frac{m}{2}(w_t)^2(t_n).$$

We have for all  $n \in \mathbb{N}$ :

$$(3.40) \quad \sigma_n = \hat{E}(t, \hat{w}) + \hat{E}(t_n, \hat{w}) - \langle \hat{w}(t), \hat{w}(t_n) \rangle_X - \\ \langle \hat{w}_t(t), \hat{w}_t(t_n) \rangle_Z - K \left( \frac{m}{\beta}(w_t)(t) - (w_x)_{xx}(t) \right) \left( \frac{m}{\beta}(w_t)(t_n) - (w_x)_{xx}(t_n) \right) \\ - \frac{m}{2}(w_t)^2(t) - \frac{m}{2}(w_t)^2(t_n).$$

Since  $\hat{w}, \hat{w}_t$  are weakly continuous and  $\hat{E}$  is continuous in  $t$ , passing to the limit in (3.40), it follows:

$$\sigma_n \longrightarrow 0, \text{ when } n \longrightarrow \infty.$$

Therefore, this implies that

$$(3.41) \quad \|\hat{w}(t) - \hat{w}(t_n)\|_X^2 \longrightarrow 0 \text{ when } n \longrightarrow \infty,$$

$$(3.42) \quad \|\hat{w}_t(t) - \hat{w}_t(t_n)\|_Z^2 \longrightarrow 0 \text{ when } n \longrightarrow \infty.$$

Thus, we get  $\hat{w} \in C([0, T]; X)$  and  $\hat{w}_t \in C([0, T]; Z)$ .

□

#### 4. SEMI-DISCRETE SCHEME

**4.1. Semi-discrete Scheme: Space Discretization.** Suppose  $V^h$  is a finite dimensional subspace of  $H_E^2(0, 1)$ . (At this stage the symbol  $h$  is used only to indicate that we are considering approximation in a finite dimensional space.) We can now formulate the semi-discrete problem for our general linear Euler-Bernoulli problem.

The Galerkin finite element approximation is referred to as **Problem  $F^h$** .

**Problem  $F^h$** 

Find  $w_h \in C^2((0, \infty), V^h)$  such as for all  $t > 0$

$$(4.1) \quad c((w_h)_{tt}(t), v) + c_2((w_h)_t(t), v) + c_1(w_h(t), v) = 0 \text{ for all } v \in V^h$$

$$\text{with } w_h(0) = w_0^h \text{ and } (w_h)_t(0) = w_1^h.$$

The initial conditions  $w_0^h$  and  $w_1^h$  are approximations in  $V^h$  and must be chosen appropriately. We will show the existence of the solution to **Problem  $F^h$**  in next section.

**Definition 4.1.** (bilinear forms)

$$c_1(w, v) = \int_0^1 w_{xx} v_{xx} dx, \quad \forall w, v \in H_E^2(0, 1);$$

$$c_2(w, v) = \alpha w(1)v(1) - \beta w_{xxx}(1)v(1), \quad \forall w, v \in H_E^2(0, 1);$$

$$c(w, v) = \int_0^1 w v dx + m w(1)v(1) \quad \forall w, v \in L^2(0, 1).$$

**Remark 4.1.** Note that the bilinear forms  $c_1$ ,  $c_2$  and  $c$  are symmetrical.

**Proposition 4.1.** There exists a constant  $C_K$  such that the bilinear form  $c_2$  is non-negative, symmetric and bounded on  $H_E^2(0, 1)$ , i.e. there exists a positive constant  $C_K$  such that

$$|c_2(w, v)| \leq C_K \|w\|_{H_E^2(0,1)} \|v\|_{H_E^2(0,1)}, \text{ for all } w, v \in H_E^2(0, 1),$$

with  $C_K = \alpha + \beta$

**4.1.1. An ordinary differential equation system.** In this subsection, let's consider  $(\varphi_i)_{i=1, \dots, N}$  a basis of  $V^h$ . The **Problem  $F^h$**  is equivalent to an ordinary differential equation problem as demonstrated below. There exists functions  $W_i(t)$  such that  $w_h(x, t) = \sum_{i=1}^N W_i(t) \varphi_i(x)$ , where  $W(t)$  is the vector representation of the function  $w_h$  defined as follows:

$$W = [W_1, W_2, \dots, W_N]^T.$$

The Galerkin approximation of (3.6) is written as follows:

$$(4.2) \quad \int_0^1 (w_h)_{tt} \varphi_j dx + \int_0^1 (w_h)_{xx} (\varphi_j)_{xx} dx + m (w_h)_{tt}(1) (\varphi_j)(1) + \alpha (w_h)_t(1) (\varphi_j)(1) - \beta (w_h)_{xxx}(1) (\varphi_j)(1) = 0,$$

for all  $j = 1, \dots, N$  and  $t > 0$ , with initial conditions  $w_h(., 0) = w_{h,0} \in V^h$ ;  $(w_h)_t(., 0) = w_{h,1} \in V^h$ .

Then, equation (4.2) is equivalent to the following equation:

$$(4.3) \quad SW_{tt} + BW_t + AW = 0.$$

**Notation:** The matrices  $A$ ,  $B$  and  $S$  are defined by:

$$A_{ij} = c_1(\varphi_i, \varphi_j) = \int_0^1 (\varphi_i)_{xx}(\varphi_j)_{xx} dx \quad \forall i, j = 1, \dots, N,$$

$$B_{ij} = c_2(\varphi_i, \varphi_j) = \alpha(\varphi_i)(1)(\varphi_j)(1) - \beta(\varphi_i)_{xxx}(1)(\varphi_j)(1), \quad \forall i, j = 1, \dots, N,$$

$$S_{ij} = c(\varphi_i, \varphi_j) = \int_0^1 \varphi_i \varphi_j dx + m(\varphi_i)(1)(\varphi_j)(1) \quad \forall i, j = 1, \dots, N.$$

$S$  is the mass matrix and  $A$  the rigidity matrix.  $A$  is symmetric, defined and positive, therefore  $A$  is invertible. The matrix  $S$  is also symmetric, defined and positive, therefore  $S$  is invertible. Using the theory of linear differential equations the problem (4.3) has a unique solution. This implies, the existence and the uniqueness of the solution of **Problem  $F^h$** . Note that  $S$  and  $A$  are tridiagonal matrices by blocks while  $B$  is diagonal.

**4.1.2. Piecewise cubic Hermite polynomials.** In this subsection, we use the well-know piecewise cubics (see [16]). They are used successfully as basis functions for the Galerkin approximation in the beams problem. Hermite cubics are sufficiently accurate for beams problems. The discretization space is therefore (see [16])

$$V^h = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_{2n-1}, \varphi_{2n}\},$$

which is dimension  $N = 2n$ . With the separation of variables, the approximate solution  $w_h \in V^h$  which we seek can be written as follows:

$$w_h(x, t) = \sum_{j=1}^n [\bar{w}_j(t) \varphi_{2j-1}(x) + (\bar{w}_j)_x(t) \varphi_{2j}(x)].$$

An advantage of this choice of discrete space and its basis is that it yields the simple relations:  $w_h(1, t) = W_{N-1}(t)$  and  $(w_h)_x(1, t) = W_N(t)$ .



## 4.2. Interpolation.

### 4.2.1. Interpolation operator.

**Definition 4.2.** We define

$$\Pi w = \sum_{i=1}^N w(x_i) \varphi_i, \quad w \in H^2(0, 1).$$

Note that if  $v = \Pi w$ , then  $v(x_i) = w(x_i)$ . Also, it is easy to see that  $\Pi w \in H_E^2(0, 1)$  if  $w \in H_E^2(0, 1)$ .

**4.2.2. Interpolation error.** In this subsection we quote standard interpolation estimates, as found in for instance, (see [11], [12]). We will use  $\widehat{K}_b$  to denote a generic constant which depends on the constants in Sobolev's lemma and let  $J$  be a bounded or unbounded interval, either an open interval containing zero or of the form  $[0, T)$  or  $[0, \infty)$ . The seminorm of order  $k$  for the product space  $H^k = H^k(0, 1) \times \mathbb{R} \times \mathbb{R}$  is defined by  $|\hat{w}|_k^2 = |w|_k^2$ .

**Lemma 4.1.** *There exists a constant  $\widehat{K}_b$  such that, for all  $w \in H^k(0, 1)$  with  $k \geq 2$ ,*

$$\|w - \Pi w\|_m \leq \widehat{K}_b h^{k^* - m} |w|_{k^*}, \quad m = 0, 1, \dots, k^*$$

where  $k^* = \min\{k; 4\}$ . As  $H_E^2(0, 1) \subset H^2(0, 1)$  and the energy norm is equivalent to the  $H^2$  norm, the following interpolation estimate holds.

**Corollary 4.1.** *There exists a constant  $\widehat{K}_b$  such that, for all  $w \in H^k(0, 1) \cap H_E^2(0, 1)$  with  $k \geq 2$ ,  $\|w - \Pi w\|_{H_E^2(0, 1)} \leq \widehat{K}_b h^{k^* - 2} |w|_{k^*}$ .*

For our problem  $k = 4$  it follows that  $k^* = 4$ . This means that the following result applies to the interpolation operator that we use.

**Corollary 4.2.** *There exists a constant  $\widehat{K}_b$  such that, for all  $w \in H^4(0, 1) \cap H_E^2(0, 1)$ ,  $\|w - \Pi w\|_{H_E^2(0, 1)} \leq \widehat{K}_b h^2 |w|_4$ .*

**4.3. Approximation.** We have a Hilbert space  $H_E^2(0, 1)$ , a finite dimensional subspace  $V^h$ , an interpolation operator  $\Pi$  and an estimate for the interpolation error  $w - \Pi w$ . We now introduce a projection of  $H_E^2(0, 1)$  onto the subspace  $V^h$ .

#### 4.3.1. Projection.

**Definition 4.3.**  $P$  is a projection of  $H_E^2(0, 1)$  onto  $V^h$  with respect to the inner product  $c_1$ .

To obtain an estimate for the error  $e_h(t) = w(t) - w_h(t)$ , the projection is used. The definition implies that for any  $w \in H_E^2(0, 1)$ ,

$$c_1(w - Pw, v) = 0, \quad \forall v \in V^h.$$

We use  $P$  to denote the projection  $Pw$  of a function  $w$ , that is,  $(Pw)(t) = Pw(t)$  for each  $t \in J$ . The projection is used to split the error  $e_h(t) = w(t) - w_h(t)$  as follows:  $e(t) = Pw(t) - w_h(t)$  and  $e_p(t) = w(t) - Pw(t)$ ; i.e.  $e_h(t) = e(t) + e_p(t)$ .

Due to the important role that it will play in the theory, we display the properties of this projection:

$$\|w - Pw\|_{H_E^2(0,1)} \leq \|w - v\|_{H_E^2(0,1)} \quad \text{for all } v \in V^h,$$

$$\|Pw - v\|_{H_E^2(0,1)} \leq \|w - v\|_{H_E^2(0,1)} \quad \text{for all } v \in V^h \quad \text{and} \quad \|Pw\|_{H_E^2(0,1)} \leq \|w\|_{H_E^2(0,1)}.$$

**Lemma 4.2.** *There exists a constant  $\widehat{K}_b$  such that, for all  $w \in H^k(0, 1) \cap H_E^2(0, 1)$  with  $k \geq 2$ ,  $\|Pw - w\|_{H_E^2(0,1)} \leq \widehat{K}_b |w|_{k^*} h^{k^*-2}$  and  $\|\Pi w - Pw\|_{H_E^2(0,1)} \leq \widehat{K}_b |w|_{k^*} h^{k^*-2}$ .*

*Proof.*  $\|Pw - w\|_{H_E^2(0,1)} \leq \|\Pi w - w\|_{H_E^2(0,1)}$  and  $\|\Pi w - Pw\|_{H_E^2(0,1)} \leq \|w - \Pi w\|_{H_E^2(0,1)}$ . Now, we use the result for the interpolation error in Corollary 4.1.  $\square$

**Corollary 4.3.** *There exists a constant  $\widehat{K}_b$  such that, for all  $w \in H^4(0, 1) \cap H_E^2(0, 1)$   $\|Pw - w\|_{H_E^2(0,1)} \leq \widehat{K}_b |w|_4 h^2$  and  $\|\Pi w - Pw\|_{H_E^2(0,1)} \leq \widehat{K}_b |w|_4 h^2$ .*

We are able to determine the estimate for the projection error, in the next subsection we will determine the estimate for the error  $e$  to finally find an estimate for the error  $e_h$ .

**4.3.2. Fundamental estimate.** The error  $e$  estimate of our problem is the same as in [2], then we have:

**Lemma 4.3.** *Let  $w$  the solution of (3.6). Assume that  $w \in C^1([0, T]; H_E^2(0, 1)) \cap C^2((0, T); H_E^2(0, 1))$ , then for any  $t \in (0, T)$ ,*

$$\|e(t)\|_{H_E^2(0,1)} + \|e_t(t)\|_{L^2(0,1)} \leq \sqrt{24e^{3t}} B_T$$

where

$$\begin{aligned}
 (4.4) \quad B_T &= \int_0^T \|(e_p)_{tt}(s)\|_{L^2(0,1)} ds + 3C_K \max \|(e_p)_t(t)\|_{H_E^2(0,1)} \\
 &+ 3C_K \int_0^T \|(e_p)_{tt}(s)\|_{H_E^2(0,1)} ds + \|e_t(0)\|_{L^2(0,1)} \\
 &+ \sqrt{1+C_K} \|e(0)\|_{H_E^2(0,1)} + \sqrt{C_K} \|(e_p)_t(0)\|_{H_E^2(0,1)}.
 \end{aligned}$$

**4.4. Convergence.** The rate of convergence is also directly influenced by the choice of the initial values  $w_0^h$  and  $w_1^h$  for the solution  $w_h$  of **Problem  $F^h$** .

In the following theorem we consider the case where  $w_0^h = \Pi w_0$  and  $w_1^h = \Pi w_1$ .

**Corollary 4.4.** *Let  $w$  the solution of (3.6). Assume that  $C^1([0, T]; H_E^2(0, 1)) \cap C^2((0, T); H_E^2(0, 1))$ . Then, for any  $t \in (0, T)$ ,*

$$\begin{aligned}
 \|e_h(t)\|_{H_E^2(0,1)} + \|(e_h)_t(t)\|_{L^2(0,1)} &\leq \|w(t) - Pw(t)\|_{H_E^2(0,1)} \\
 &+ \|w_t(t) - Pw_t(t)\|_{L^2(0,1)} + \sqrt{24e^{3t}} B_T,
 \end{aligned}$$

where

$$\begin{aligned}
 B_T &= \int_0^T \|(w_{tt} - Pw_{tt})(s)\|_{L^2(0,1)} ds + 3C_K \max \|(w_t - Pw_t)(t)\|_{H_E^2(0,1)} \\
 &+ 3C_K \int_0^T \|(w_{tt} - Pw_{tt})(s)\|_{H_E^2(0,1)} ds + \|Pw_1^h - w_1\|_{L^2(0,1)} \\
 &+ \sqrt{1+C_K} \|Pw_0^h - w_0\|_{H_E^2(0,1)} + \sqrt{C_K} \|w_1 - Pw_1\|_{H_E^2(0,1)}.
 \end{aligned}$$

**Theorem 4.1.** *Let  $w_0^h = \Pi w_0$ ,  $w_1^h = \Pi w_1$  and  $w$  the solution of (3.6). Assume that  $w \in C^1([0, T]; H_E^2(0, 1)) \cap C^2((0, T); H_E^2(0, 1))$  and  $w_{tt} \in L^2([0, T]; H^4(0, 1) \cap H_E^2(0, 1))$  for  $t \geq 0$ . Assume also that Corollary 4.3 holds. Then,*

$$\begin{aligned}
 \|e_h(t)\|_{H_E^2(0,1)} + \|(e_h)_t(t)\|_{L^2(0,1)} &\leq \hat{K}_b h^2 (|w(t)|_4 + |w_t(t)|_4) \\
 &+ \sqrt{24e^{3t}} \hat{K}_b h^2 \left[ C_b \max_{s \in [0, T]} |w_{tt}(s)|_4 \right. \\
 &+ 3C_K \max |w_t(t)|_4 + 3C_K \max_{s \in [0, T]} |w_{tt}(s)|_4 \\
 (4.5) \quad &+ \left. C_b |w_1|_4 + \sqrt{1+C_K} |w_0|_4 + \sqrt{C_K} |w_1|_4 \right]
 \end{aligned}$$

for  $t \in (0, T)$ .

*Proof.* The result follows from Lemma 4.3, Corollary 4.4 and note that:

$$\begin{aligned} \|Pw_0 - \Pi w_0\|_{H_E^2(0,1)} &\leq \widehat{K}_b h^2 |w_0|_4 \\ \|Pw_1 - \Pi w_1\|_{L^2(0,1)} &\leq C_b \|Pw_1 - \Pi w_1\|_{H_E^2(0,1)} \leq \widehat{K}_b C_b h^2 |w_1|_4 \\ \int_0^T \|(e_p)_{tt}(s)\|_{L^2(0,1)} ds &\leq C_b \widehat{K}_b h^2 \max_{s \in [0,T]} |w_{tt}(s)|_4. \end{aligned}$$

□

## 5. NUMERICAL RESULT

In this section, we implement results of numerical schemes developed in subsections above.

**5.1. Representations.** In this subsection, we show the simulation results of the numerical methods. We consider the time step  $k = 0.01$  and the spatial discretization  $h = 0.01$ . The initial conditions are taken as follows:

$$w_0(x) = 0.4x^3 - 0.6x^2 \text{ and } v_0 \equiv 0.$$

**5.1.1. Representation of the deflection and energy.** Here, we give for different values of  $\alpha$ ,  $\beta$  and  $m = 0.1$  the graphic representation of the deflection.

Figures 1, 2 and 3 represent the deflection of beam  $w(x, t)$  when  $\alpha = 2$ ,  $\beta = 10^{-8}$ ;  $\alpha = \beta = 10^{-8}$  and  $\alpha = 10^{-8}$ ,  $\beta = 10^{-6}$ .

Figure 4 represents the decay of Lyapunov function (2.5) on the time interval  $[0, 50]$  for  $\alpha = 2$ ,  $\beta = 10^{-8}$ ;  $\alpha = \beta = 10^{-8}$  and  $\alpha = 10^{-8}$ ,  $\beta = 10^{-6}$ . We can notice that when  $\alpha = \beta = 10^{-8}$ , the energy decreases rapidly. In the other two cases the decrease is rather slow.

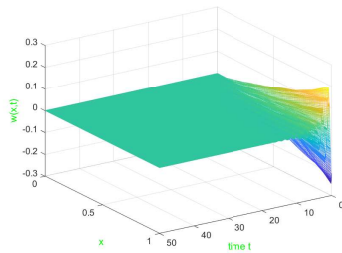


FIGURE 1. Deflection  $w(x, t)$

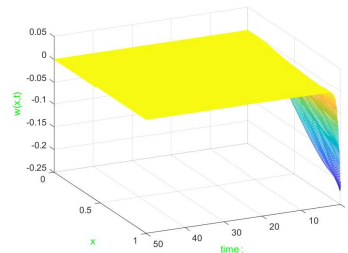
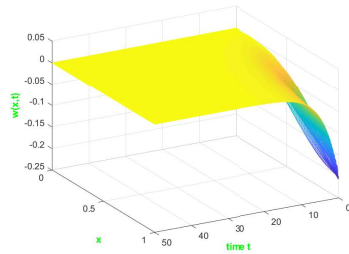
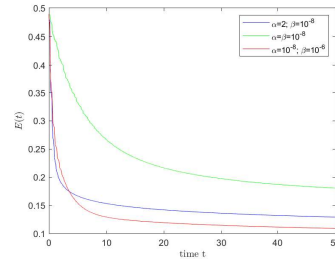


FIGURE 2. Deflection  $w(x, t)$

FIGURE 3. Deflection  $w(x, t)$ FIGURE 4. Energy for different values of  $\alpha$  and  $\beta$ 

5.1.2. *Representation of Tip position and Tip angle.* Here, the results are compared to the simulation results in the cases where  $\alpha = 2, \beta = 10^{-8}$ ;  $\alpha = \beta = 10^{-8}$  and  $\alpha = 10^{-8}, \beta = 10^{-6}$ .

In the figures 5 and 6, the tip position  $w(1, t)$  and the tip angle  $w_x(1, t)$  of the beam are compared on time interval  $[0, 50]$ .

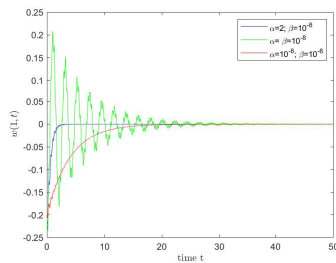


FIGURE 5. Tip position comparison

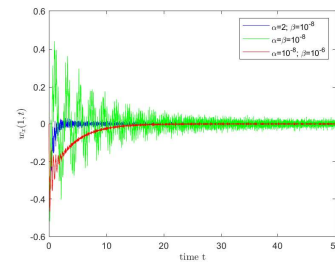


FIGURE 6. Tip angle comparison

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