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EXACT SOLUTION OF VAN DER POL NONLINEAR OSCILLATORS ON FINITE DOMAIN BY PADE APPROXIMANT AND ADOMIAN DECOMPOSITION METHODS

E.U. Agom[1] and F.O. Ogunfiditimi

ABSTRACT. This paper is concerned with a thorough investigation in achieving exact analytical solution for the Van der Pol (VDP) nonlinear oscillators models via Adomian decomposition method (ADM). The models are nonlinear time dependent second order ordinary differential equations. ADM has already been applied, in existing literatures, to obtain approximate results. But, we adapt the method by adjusting the source term; a procedure that is base on the asymptotic Taylor's series expansion on the term that would have resulted to proliferation of terms during the invertible process. Then, the rational Pade Approximant is applied to clarify and get a better understanding of the uniqueness and convergence of our findings. Two models were used as illustrations and their result pictured to indicate their behaviour in the given domains. And, we found that the adaptation on the models yielded exact results which were further displayed in constructed tables.

[1]*corresponding author*

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1. INTRODUCTION

The generalised VDP nonlinear oscillator equation is given as

$$\ddot{x} + x^{\frac{2m+1}{2n+1}} = \epsilon(x - x^2)\dot{x} + \mu, \qquad x(t_0) = x_0, \quad \dot{x}(t_0) = x_1, \tag{1.1}$$

where $x(t)$ is the position coordinate, ϵ is a controlled parameter whose value is crucial to the nonlinear behaviour of the oscillator, $\mu = f(t)$, and $m, n \in \mathbb{N}$. From equation (1.1) if $m = n = f(t) = 0$, we have the unforced VDP oscillator equation which becomes a special case the Lienard equation. During the days of VDP, the equation was studied for $\epsilon > 1$. Nowadays, the original technical problem stipulates that $\epsilon \in (0, 1]$.

The triode oscillator equation, as equation (1.1) was formerly known, represent a simplified equation for the amplitude of an oscillating current driven by a triode. It has been studied by many researchers, just recently [1] investigated and compared numerical results they obtained by the modified ADM and truncated Taylor series method. [2] investigated the third order VDP oscillator equation with implicit Runge-Kutta and Adams-Bashforth methods to obtain numerical results. [3] implemented the integral method to analytically express and obtain periodic solution. The result from the illustrative examples in the paper shows that the theoretical predictions and numerical solution were in agreement with each other. [4] studied the dynamics of the Duffing-VDP driven oscillator and solved the equations for singular points of resonance curved exactly. The bifurcation diagrams computed were in agreement with the theoretical analysis.

Also, [5] proposed a combination of homotopy series and Laplace Transform method to solve the nonlinear Duffing-VDP oscillator equation in conjunction with the Pade approximation. The trio-method gave an alternative framework designed to overcome the difficulty of capturing the behaviour of the solution and it gave an approximation to the considered examples with larger t. [6] implemented the perturbation analysis method on the VDP-Duffing oscillator. [7] presented the first integral of the Duffing-VDP oscillator under certain parametric conditions. This was made possible through inverse transformation without complicated calculations. [8] implemented the perturbation method based on integral vectors to approximate first integral and periodic solution of the generalised Nonlinear Van der Pol oscillator equations. [9] Obtained numerical solutions to cubic-quintic Duffing equation using one-step frequency formulation.

Some similar mathematical studies of equation (1.1) reported in literatures are as follows. [10] implemented the Homotopy perturbation method alongside the Variational iteration method on the nonlinear oscillator problems to obtain numerical result, [11] presented an invariant torus of a weakly coupled autonomous oscillator. A sequence of algorithm were developed and applied and the resulting approximation showed reasonable numerical accuracy. [12] studied analytically the fixed point stability criteria for the VDP oscillator which was subsequently solved numerically. [13] investigated the efficiency of improved Heun's, 4th order Runge-Kutta and Mid-point methods. The numerical illustrations revealed 4th order Runge-kutta method gave a better accuracy to the VDP oscillator. [14] implemented the Laplace decomposition and 4th order Runge-Kutta method methods on the VDP equations. The method was based on ADM and Laplace transform method. The nonlinear terms in the illustrative equations were decomposed by He's polynomials in the article. The approximate result obtained gave better accuracy with Laplace decomposition method than the other method studied.

Several other studies has also been carried out and can be found in literature. So far none of these researches on the nonlinear VDP oscillator equation has produced analytically exact solutions using the decomposition method by Adomian [15]. We propose an adjustment to the ADM on the nonlinear VDP oscillator equation and implement it on this class of equation to obtain optimal, continuous, analytical exact result. We then apply the nonlinear Pade approximant to understand the uniqueness and fast convergence of the obtained solution. The improvement is an alternative frame work designed to overcome the difficulty in obtaining analytical solutions as a result of the present of a noise term reoccurring during the invertible process.

2. Theory of Pade Approximant and ADM on Nonlinear Oscillator

VDP also proposed a nonlinear differential equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = \mu, \tag{2.1}$$

commonly referred to as the unforced VDP equation. Equation (2.1) occurs frequently in applied dynamics where $\mu = f(t)$ is the forced behaviour of the oscillator that contains the amplitude of motion. In operator form, ADM [15] represent (2.1) as

$$Lx = \mu - \epsilon(x^2 - 1)\dot{x} - x, \tag{2.2}$$

where $L(.) = \dfrac{d^2}{dt^2}(.)$, then the inverse of L, L^{-1}, is given as $L^{-1} = \int_0^t \int_0^t (.) ds ds$. Applying L^{-1} on both sides of equation(2.2), we get

$$x(t) = F(t) - L^{-1}(x - \epsilon\dot{x}) - L^{-1}(\epsilon x^2 \dot{x}), \tag{2.3}$$

where $F(t) = \alpha + \int_0^t \int_0^t f(s) ds ds$, $\alpha = x(t_0) + t\dot{x}(t_0)$ and $\epsilon x^2 \dot{x}$ is the nonlinear term $N(x)$ that is to be decomposed using the Adomian polynomials, A_k. That is

$$\sum_{k=0}^{\infty} A_k = N(x), \tag{2.4}$$

where

$$A_k(x_0, x_1, x_2, x_3, \ldots, x_k) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} N\left(\sum_{i=0}^{\infty} x_i \lambda^i\right)\right]_{\lambda=0}. \tag{2.5}$$

The Adomian polynomials by [15] of several categories of functions and how it can be calculated has widely been reported in existing literatures. [16] recently gave those of transcendental hyperbolic function in its original and exponential equivalent form. The description on polynomial nonlinear terms with integer exponent is contain in [17]. See also [18] and [19] with the literatures therein.

From equation (2.1), the ADM writes the position coordinate as

$$x(t) = \sum_{k=0}^{\infty} x_k(t). \tag{2.6}$$

Substituting equations (2.4), (2.5) and (2.6) in equation (2.3), we have

$$\sum_{k=0}^{\infty} x_k(t) = F(t) - L^{-1}\left(\sum_{k=0}^{\infty} (x_k(t) - \epsilon\dot{x}_k(t)) + A_k(x(t))\right). \tag{2.7}$$

Since $f(t)$ contains the amplitude of motion. Thus, it contains some trigonometric functions. In the process of applying L^{-1} on equation (2.7), $f(t)$ would have lead to noise proliferation of terms. To overcome this dangerous phenomena, we replace $f(t)$ with $\sum_{k=0}^{\infty} f_k(t)$. Where $\sum_{k=0}^{\infty} f_k(t)$ is the Taylor's series expansion of $f(t)$ about $t = 0$. Thus equation (2.7) automatically becomes

$$\sum_{k=0}^{\infty} x_k(t) = \alpha + L^{-1}\left(\sum_{k=0}^{\infty}(f_k(t) - x_k(t))\right) + L^{-1}\left(\epsilon \sum_{k=0}^{\infty} \dot{x}_k(t) - A_k(x(t))\right), \tag{2.8}$$

leading to the following recurrence scheme

$x_0 = \alpha,$

$x_{k+1} = L^{-1}\left(f_k(t) - x_k(t) + \epsilon\dot{x}_k(t)\right) - L^{-1}\left(A_k(x_0(t), x_1(t), x_2(t), \cdots, x_k(t))\right).$

From equation (2.1),

$A_0(x_0(t)) = \epsilon x_0^2 \dot{x}$

$A_1(x_0(t), x_1(t)) = \epsilon\left(x_0^2\dot{x}_1 + 2x_0x_1\dot{x}_0\right)$

$A_2(x_0(t), x_1(t), x_2(t)) = \epsilon\left(x_1^2\dot{x}_2 + 2x_0x_1\dot{x}_1 + 2x_0x_2\dot{x}_0 + x_1^2\dot{x}_0\right)$

$A_3(x_0(t), x_1(t), x_2(t), x_3(t)) = \epsilon\left(x_0^2\dot{x}_3 + 2x_0x_1\dot{x}_2 + 2x_0x_2\dot{x}_1\right.$
$\left. +2x_0x_3\dot{x}_0 + x_1^2\dot{x}_1 + 2x_1x_2\dot{x}_0\right)$

$A_4(x_0(t), x_1(t), x_2(t), x_3(t), x_4(t)) = \epsilon(x_0^2\dot{x}_4 + 2x_0x_1\dot{x}_3 + 2x_0x_2\dot{x}_2$
$+2x_0x_3\dot{x}_1 + x_1^2\dot{x}_2 + 2x_1x_2\dot{x}_1 + 2x_0x_4\dot{x}_0 + 2x_1x_3\dot{x}_0 + x_2^2\dot{x}_0)$

...

where we write the position coordinate as

$$x(t) = \lim_{n\to\infty} \sum_{k=0}^{n} x_k(t). \tag{2.9}$$

From known facts available in literatures, especially the vivid exposition as contain in [20] and the literatures therein, the position coordinate can also be expanded asymptotically as

$$x(t) = \lim_{(M+N)\to\infty} \sum_{k=0}^{M+N} c_k t^k \quad M, N \notin \mathbb{Z}^-. \tag{2.10}$$

In accordance with understanding the in-depth behaviour of equation (2.9), we assume that $x(t)$ is smooth and represent as Taylor series expansion. Which, by [21], can be expressed as a ratio of asymptotic expansion of a rational polynomial, $P_N^M(t)$. Known as the $[M, N]\,th$ Pade Approximant. It has been reported to be the most natural analytic continuation method, although it has poles, but it is bounded and free of oscillation. See Baker [21] and the literatures therein.

By definition

$$P_N^M(t) = \frac{\sum_{k=0}^{M} a_k t^k}{1 + \sum_{k=1}^{N} b_k t^k} = x(t). \tag{2.11}$$

Truncating the series on the right side in equation (2.10) by ignoring terms of $\mathcal{O}^{M+N+1}$, we get a finite Taylor series, $T_{M+N}(t)$, of $x(t)$ as

$$T_{M+N}(t) = \sum_{k=0}^{M+N} c_k t^k. \tag{2.12}$$

It implies from equations (2.11) and (2.12) that

$$T_{M+N}(t) = P_N^M(t). \tag{2.13}$$

Simplifying the system in equation (2.13) and equating powers of t, we get an explicitly expanded form relation. Which result to two sets of system of equations; one that contains both $a_k' s$ and $b_k' s$ as the unknowns

$$C_1 B = A, \tag{2.14}$$

and one that contains only the $b_k' s$ as the unknowns,

$$C_2 B = 0 \tag{2.15}$$

Here

$$A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_M \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_N \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} c_0 & & & & \\ c_1 & c_0 & & & \\ c_2 & c_1 & c_0 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} c_{n+1} & c_n & c_{n-1} & \cdots \\ c_{n+2} & c_{n+1} & c_n & \cdots \\ c_{n+3} & c_{n+2} & c_{n+1} & \cdots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ c_{M+N} & c_{M+N-1} & c_{M+N-2} & \cdots \end{bmatrix},$$

with the $c_k's$ known from the exact/ADM solution, M and N specified, the $a_k's$ and $b_k's$ can easily be determined. First, by solving equation (2.15) then equation (2.14).

Theorem 2.1 (Uniqueness convergence of the position coordinate). *The $P_N^M(t)$ Pade Approximant of the position coordinate, $x(t) = \sum_{k=0}^{\infty} c_k t^k$, is unique.*

Proof. From equation (2.9), $x(t)$ is a formal power series and by Baker [21] it is unique, hence converges. □

3. Nonlinear Oscillators problems

In this section we take two nonlinear oscillators differential equations from [10] and [14] as examples in order to demonstrate the robustness and analytical accuracy described in the previous section.

3.1. **Problem 1.** Consider the following

$$\ddot{x} + \dot{x} + x + x^2\dot{x} = 2\cos t - \cos^3 t, \quad x(0) = 0, \quad \dot{x}(0) = 1, \tag{3.1}$$

$0 \leqslant t < 1$. Using the theory of ADM and its adjustment, we quickly identify that

$$f(t) = 2\cos t - \cos^3 t,$$

with

$$\sum_{k=0}^{\infty} f_k(t) = 1 + \frac{1}{2!}t^2 - \frac{19}{4!}t^4 + \frac{181}{6!}t^6 - \frac{1639}{8!}t^8 + \cdots,$$

and the relation

$$\begin{aligned} x_0 &= x(0) + t\dot{x}(0), \\ x_k &= -L^{-1}\{\dot{x}_{k-1} + x_{k-1} + A_{k-1}(x_0, x_1, x_2, \cdots, x_{k-1})\} + L^{-1}\{f_{k-1}(t)\}, \end{aligned}$$

resulting to

$$\sum_{k=0}^{\infty} x_k(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots = \sin t.$$

Hence,

$$x(t) = \sin t$$

from equation (2.9). Which is the theoretical analytical solution of equation (3.1) as contain in Khan et al [14] and [10]. Our result alongside the analytical exact solution is as depicted in figure 1 and, all rational Pade Approximant of degree nine for ADM and analytical result is as given in Table 1.

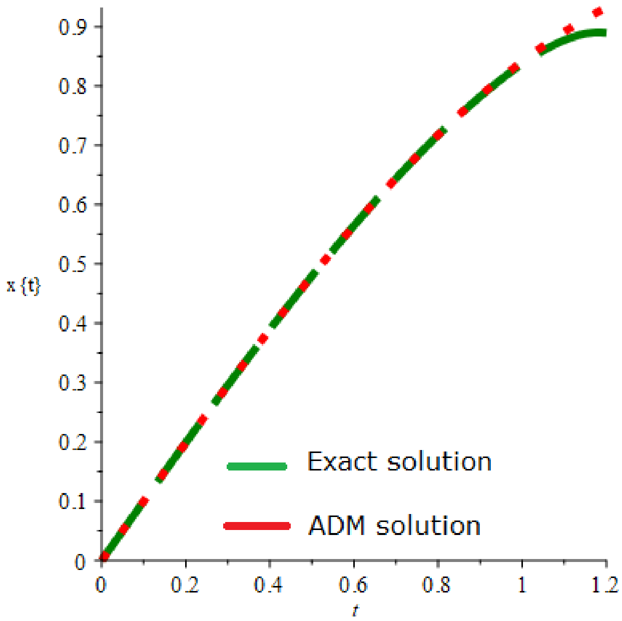

FIGURE 1. Result of Problem 1

3.2. **Problem 2.** Consider the following

$$\ddot{x} + (\dot{x})^2 - x + x^2 - 1 = 0 \quad x(0) = 2, \quad \dot{x}(0) = 1 \tag{3.2}$$

$0 \leqslant t < 1$. Similarly, using the theory of ADM and its adjustment of the previous section in conjunction with [18] and [19] we have

$$A_0(x_0) = x_0^2 + \dot{x}_0^2$$
$$A_1(x_0, x_1) = 2x_0x_1 + 2\dot{x}_0\dot{x}_1$$
$$A_1(x_0, x_1, x_2) = 2x_0x_2 + x_1^2 + 2\dot{x}_0\dot{x}_2 + \dot{x}_1^2$$

Continuing in these order, we eventually have

$$\begin{aligned}\sum_{k=0}^{\infty} x_k(t) &= 2 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots \\ &= 1 + 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots \\ &= 1 + \cos t\end{aligned}$$

TABLE 1. All possible Pade Approximant of degree Nine on the solutions of Problem 1

$[\mathbf{M},\mathbf{N}]$ **th**	**Exact solutioN** $(x(t)=\sin t)$	**ADM solution** (First 6 terms of $x(t)=\sum_{k=0}^{10} x_k(t)$)
$[9,0]th$	$t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7+\frac{1}{9!}t^9$	$t-\frac{1}{6}t^3+\frac{1}{120}t^5-\frac{1}{5040}t^7+\frac{1}{362880}t^9$
$[8,1]th$	$t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7$	$t-\frac{1}{6}t^3+\frac{1}{120}t^5-\frac{1}{5040}t^7$
$[7,2]th$	$\frac{t-\frac{11}{72}t^3+\frac{13}{2160}t^5-\frac{1}{12096}t^7}{1+\frac{1}{72}t^2}$	$\frac{t-\frac{11}{72}t^3+\frac{13}{2160}t^5-\frac{1}{12096}t^7}{1+\frac{1}{72}t^2}$
$[6,3]th$	$\frac{t-\frac{1}{7}t^3+\frac{11}{2520}t^5}{1+\frac{1}{42}t^2}$	$\frac{t-\frac{1}{7}t^3+\frac{11}{2520}t^5}{1+\frac{1}{42}t^2}$
$[5,4]th$	$\frac{t-\frac{53}{396}t^3+\frac{551}{166320}t^5}{1+\frac{13}{396}t^2+\frac{5}{11088}t^4}$	$\frac{t-\frac{53}{396}t^3+\frac{551}{166320}t^5}{1+\frac{13}{396}t^2+\frac{5}{11088}t^4}$
$[4,5]th$	$\frac{t-\frac{31}{294}t^3}{1+\frac{3}{49}t^2+\frac{11}{5880}t^4}$	$\frac{t-\frac{31}{294}t^3}{1+\frac{3}{49}t^2+\frac{11}{5880}t^4}$
$[3,6]th$	$\frac{t-\frac{127}{1240}t^3}{1+\frac{239}{3720}t^2+\frac{53}{22320}t^4+\frac{511}{9374400}t^6}$	$\frac{t-\frac{127}{1240}t^3}{1+\frac{239}{3720}t^2+\frac{53}{22320}t^4+\frac{511}{9374400}t^6}$
$[2,7]th$	$\frac{t}{1+\frac{1}{6}t^2+\frac{7}{360}t^4+\frac{31}{15120}t^6}$	$\frac{t}{1+\frac{1}{6}t^2+\frac{7}{360}t^4+\frac{31}{15120}t^6}$
$[1,8]th$	$\frac{t}{1+\frac{1}{6}t^2+\frac{7}{360}t^4+\frac{31}{15120}t^6+\frac{127}{604800}t^8}$	$\frac{t}{1+\frac{1}{6}t^2+\frac{7}{360}t^4+\frac{31}{15120}t^6+\frac{127}{604800}t^8}$
$[0,9]th$	$\frac{t}{1+\frac{1}{6}t^2+\frac{7}{360}t^4+\frac{31}{15120}t^6+\frac{127}{604800}t^8}$	$\frac{t}{1+\frac{1}{6}t^2+\frac{7}{360}t^4+\frac{31}{15120}t^6+\frac{127}{604800}t^8}$

Resulting to

$$x(t)=1+\cos t$$

from equation (2.9). Which is the theoretical exact solution of equation (3.2) as contains in [10]. Similarly, our results alongside the analytical exact solution and that obtained by ADM is as depicted in figure 2. And, all possible rational Pade Approximant of degree nine for the exact solution and that of ADM is as given in Table 2.

4. CONCLUSION

We have been able to use an asymptotic expansion on terms that would have caused further proliferation of terms in the decomposition process with the defined domain of the independent variable. The asymptotic expansion was based on Taylor 's series expansion about $t = 0$. And, in the illustrative examples we obtained analytical exact solutions that are continuous in the entire given domains but not without alternating terms that tends to cancel out for larger values of k in equation (2.9). These is against numerical solutions obtained in existing literatures with solutions that holds at the discrete grids and or collocation points. The high level accuracy in this paper both in the computation of

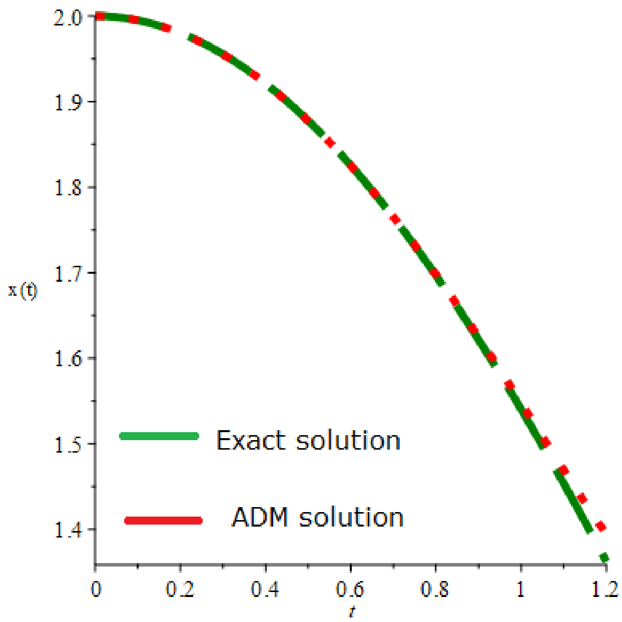

FIGURE 2. Result of Problem 2

TABLE 2. All possible Pade Approximant of degree Nine on the solutions of Problem 2

[M, N] th	**Exact solution** ($x(t) = 1 + \cos t$)	**ADM solution** (First 7 terms of $\sum_{k=0}^{11} x_k(t)$)
$[9,0]th$	$2 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \frac{1}{8!}t^8$	$2 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \frac{1}{40320}t^8$
$[8,1]th$	$2 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \frac{1}{8!}t^8$	$2 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \frac{1}{40320}t^8$
$[7,2]th$	$\frac{2-\frac{13}{28}t^2+\frac{11}{336}t^4-\frac{13}{20160}t^6}{1+\frac{1}{56}t^2}$	$\frac{2-\frac{13}{28}t^2+\frac{11}{336}t^4-\frac{13}{20160}t^6}{1+\frac{1}{56}t^2}$
$[6,3]th$	$\frac{2-\frac{13}{28}t^2+\frac{11}{336}t^4-\frac{13}{20160}t^6}{1+\frac{1}{56}t^2}$	$\frac{2-\frac{13}{28}t^2+\frac{11}{336}t^4-\frac{13}{20160}t^6}{1+\frac{1}{56}t^2}$
$[5,4]th$	$\frac{2-\frac{26}{63}t^2+\frac{163}{7560}t^4}{1+\frac{11}{252}t^2+\frac{13}{15120}t^4}$	$\frac{2-\frac{26}{63}t^2+\frac{163}{7560}t^4}{1+\frac{11}{252}t^2+\frac{13}{15120}t^4}$
$[4,5]th$	$\frac{2-\frac{26}{63}t^2+\frac{163}{7560}t^4}{1+\frac{11}{252}t^2+\frac{13}{15120}t^4}$	$\frac{2-\frac{26}{63}t^2+\frac{163}{7560}t^4}{1+\frac{11}{252}t^2+\frac{13}{15120}t^4}$
$[3,6]th$	$\frac{2-\frac{31}{119}t^2}{1+\frac{57}{476}t^2+\frac{13}{1428}t^4+\frac{163}{342720}t^6}$	$\frac{2-\frac{31}{119}t^2}{1+\frac{57}{476}t^2+\frac{13}{1428}t^4+\frac{163}{342720}t^6}$
$[2,7]th$	$\frac{2-\frac{31}{119}t^2}{1+\frac{57}{476}t^2+\frac{13}{1428}t^4+\frac{163}{342720}t^6}$	$\frac{2-\frac{31}{119}t^2}{1+\frac{57}{476}t^2+\frac{13}{1428}t^4+\frac{163}{342720}t^6}$
$[1,8]th$	$\frac{2}{1+\frac{1}{4}t^2+\frac{1}{24}t^4+\frac{17}{2880}t^6+\frac{31}{40320}t^8}$	$\frac{2}{1+\frac{1}{4}t^2+\frac{1}{24}t^4+\frac{17}{2880}t^6+\frac{31}{40320}t^8}$
$[0,9]th$	$\frac{2}{1+\frac{1}{4}t^2+\frac{1}{24}t^4+\frac{17}{2880}t^6+\frac{31}{40320}t^8}$	$\frac{2}{1+\frac{1}{4}t^2+\frac{1}{24}t^4+\frac{17}{2880}t^6+\frac{31}{40320}t^8}$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALABAR
CALABAR,
NIGERIA.
Email address: agomeunan@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ABUJA
ABUJA,
NIGERIA.