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A SIMPLE GENERALIZED CONSTRUCTION OF RESOLVABLE BALANCED INCOMPLETE BLOCK DESIGNS WITH PRIME BLOCK SIZES

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ABSTRACT. This paper presents a Simple Generalized Construction of Resolvable Balanced Incomplete Block Designs whose parameter combination is of the form $v = k^2,\ r = k + 1,\ \lambda = k^0 = 1$, where k is prime. The design construction was achieved by using the cyclic subgroup of the symmetric group S_k whose generator is one of the permutations of the 2-permutation generating set of the Dihedral group D_k and 2-permutation generating set of the presentation of S_k. The method is efficient, sufficient and also mitigate against the tediousness encountered in other methods of construction when v is large.

1. INTRODUCTION

A balanced incomplete block design is called a resolvable or referred to as resolvable balanced incomplete block design (RBIBD) if the set of blocks can be partitioned into parallel (distinct) classes called resolution classes, wherein every treatment occurs once in each distinct class. See example in Saka et al. [26].

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Example 1.1. *A Resolvable BIBD with $v = 9, k = 3$ and $b = 12$ in four distinct classes is presented:*

$$\begin{array}{lll}[(4,7,1), & (5,8,2), & (6,9,3)] \\ [(4,5,6), & (7,8,9), & (1,2,3)] \\ [(1,6,8), & (2,4,9), & (3,5,7)] \\ [(1,5,9), & (2,6,7), & (3,4,8)]\end{array}$$

The details of the construction could be seen in Section 2.

Importantly, the early sources for constructing resolvable incomplete block designs with some files are in Yates [24] and [25] for square lattices, Habshbabger [8–10] for rectangular lattices, Kempthorn [14] and Federer [6] for prime power lattices, and David [3] and John et al. [11] on cyclic designs. We remark that there is no absolute feasibility for constructing a complete file of incomplete block designs for all situations. Nevertheless, researchers had cutting edge with algebraic structures to attain constructions usable by experimenters. In view of this, research in the area of Hadamard matrices and their applications has steadily and rapidly grown, especially during the last few decades due to its useful applications in the construction of block designs. On this note, Hedayat and Wallis [7] transformed the Hadamard matrices to produce incomplete block designs, t-designs, Youden designs, orthogonal F-square designs, optimal saturated resolution III designs, optimal weighing designs. Evangelaras et al. [5], also presented a number of applications of Hadamard matrices to signal processing, optical multiplexing, error correction coding, and design and analysis of statistics. Equally, Saka, et al. [26] imposes some algebraic structure on square matrix to construct Zig-zag Matrix for Resolvable Nested Balanced Incomplete Block Designs and also Saka et al. [27] used an algebraic notion, of the left coset type to construct Coset-k^2 Nested Balanced Incomplete Block Designs of Resolvable Type.

The practical importance of, and motivation for resolvability is to gain orthogonality between treatments and nuisance factors of concern. For instance, resolvability in sequential experimentation, with replicates corresponding to time periods, is used to mitigate time effects. Resolvability can likewise be useful in multi-site experiments and in experiments with multiple individuals handling experimental runs. Notably, the use of resolvable designs in agricultural field trials at sometimes received considerable patronage in the United Kingdom; see Patterson and Silvey [17].

This paper shall mitigate the tediousness (computational and combinatorial efforts) encountered in the designs construction processes by Saka, et al. [27] when v is large. Also, the method of construction in this paper does not rest on the generation of a complete set of MOLS of order k as it were in some earlier literatures.

2. METHODS OF CONSTRUCTION

In this section, we describe a simple approach to the construction of RBIBD (Graeco-Latin squares) of order $k \times k$ where k is an odd integer. Let $k = 2s + 1$ and $s \in \mathbb{N}$. The number of resolvable blocks $n = k + 1 = 2(s + 1)$.

Our methods shall employ the use of symmetric groups in algebra.

Definition 2.1. *Let X be a non-empty set. The group of all permutations of X under composition of mappings is called the* ***symmetric group*** *on X and is denoted by S_X. A subgroup of S_X is called a permutation group on X.*

It is easily seen that a bijection $X \simeq Y$ induces in a natural way an isomorphism $S_X \cong S_Y$. If $|X| = n$, S_X is denoted by S_n and called the *symmetric group of degree n*.

A permutation $\sigma \in S_n$ can be exhibited in the form

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

consisting of two rows of integers; the top row has integers $1, 2, \ldots, n$ usually(but not necessarily) in their natural order, and the bottom row has $\sigma(i)$ below i for each $i = 1, 2, \ldots, n$. This is called a two-row notation for a permutation. There is a simpler, one-row notation for a special kind of permutation called *cycle*.

Definition 2.2. *(One-Row Notation) Let $\sigma \in S_n$. If there exists a list of distinct integers $x_1, \ldots, x_r \in \mathbb{N}$ such that*

$$\begin{aligned} &\sigma(x_i) = x_{i+1}, \quad i = 1, \ldots, r-1, \\ &\sigma(x_r) = x_1, \\ &\sigma(x) = x \qquad \text{if } x \notin \{x_1, \ldots, x_r\}, \end{aligned}$$

then σ is called a cycle of length r and denoted by $(x_1 \dots x_r)$.

Remark 2.1. *A cycle of length 2 is called a transposition. In other words , a cycle $(x_1 \dots x_r)$ moves the integers $x_1, \dots, x_r$ one step around a circle and leaves every other integer in $\mathbb{N}$. If $\sigma(x) = x$, we say σ does not move x. Trivially, any cycle of length 1 is the identity mapping I or e. Note that the one-row notation for a cycle does not indicate the degree n, which has to be understood from the context.*

Definition 2.3. *Let X be a set of points in space, so that the distance $d(x, y)$ between points x and y is given for all $x, y \in X$. A permutation σ of X is called a **symmetry** of X if*

$$d(\sigma(x), \sigma(y)) = d(x, y) \; \forall \; x, y \in X.$$

*Let X be the set of points on the vertices of a regular polygon which are labelled $\{1, 2, \dots, n\}$. The group of symmetries of a regular polygon P_n of n sides is called the **dihedral group of degree** n and denoted D_n.*

Remark 2.2. *It must be noted that D_n is a subgroup of S_n i.e $D_n \leq S_n$.*

Theorem 2.1. *(Cayley Theorem) Every group is isomorphic to a permutation group.*

Theorem 2.2. *The dihedral group D_n is a group of order $2n$ generated by two elements σ, τ satisfying $\sigma^n = e = \tau^2$ and $\tau\sigma = \sigma^{n-1}\tau$, where*

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix} = \prod_{2 \leq i \leq \frac{(n+2)}{2}} \begin{pmatrix} i & n+2-1 \end{pmatrix}.$$

Remark 2.3. *Geometrically, σ is a rotation of the regular polygon P_n through an angle $\frac{2\pi}{n}$ in its own plane, and τ is a reflection (or a turning over) in the diameter through the vertex 1. Jaíyéọlá [12] used symmetric groups of degree n to introduce and study palindromic Permutations and generalized Smarandache palindromic permutations. There, it was shown that the dihedral group D_n is generated by a generalized right Smarandache palindromic permutations (RGSPP), σ, and a left generalized Smarandache palindromic permutations (LGSPP), τ of S_n.*

Theorem 2.3. *(Jaíyéọlá [12]) The dihedral group D_n is generated by a RGSPP and a LGSPP i.e $D_n = \langle \sigma, \tau \rangle$ where $\sigma \in GSPP_\rho(S_n)$ and $\tau \in GSPP_\lambda(S_n)$.*

The dihedral group D_n has the presentation

$$D_n = \left\langle r, s \mid r^n = 1, \;\; s^2 = 1, \;\; rsr = s^{-1} \right\rangle \\ = \left\langle r, s \mid r^n = s^2 = (rs)^2 = 1 \right\rangle . \tag{2.1}$$

and so, it is a Coxeter group.

The symmetric group S_n has a presentation

$$S_n = \left\langle s_1, s_2, \ldots, s_{n-1} \mid (s_i s_j)^{m(i,j)} = 1 \right\rangle \tag{2.2}$$

where

$$m(i,j) = \begin{cases} 1, & i = j, \\ 3, & i = j-1 \text{ or } i = j+1, \\ 2, & |i-j| > 1 \end{cases}$$

and so, it is a Coxeter group.

In Bray et al. [2], the authors showed that for every integer $n > 1$, the symmetric group S_n has a presentation on the generators $(1\ 2)$ and $\sigma = (1\ 2\ \cdots\ n)$ in which the number of relations and the presentation length are of some values.

S_3 is the first and smallest non-abelian symmetric group and a Frobenious group. $S_3 \cong D_3$ and $S_3 \equiv D_3$. It has the group representation

$$\left\langle r, a \mid r^3 = 1, \;\; a^2 = 1, \;\; ara = r^{-1} \right\rangle = \left\langle r, a \mid r^3, \;\; a^2, \;\; arar \right\rangle$$

or $\langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle = \langle a, b \mid a^2, \;\; b^2, \;\; (ab)^3\rangle$.

S_5 is the first non-solvable symmetry group. It is the Galois group of general quintic equation.

Theorem 2.4. *Let $M_{k\times k}(N_k)$ be the set of k^2-matrices over $N_k = \{1, 2, \ldots, k\}$. Let $X_0 = ((m-1)k + j)$, $X_0^T = ((j-1)k + m) \in M_{k\times k}(N_k)$ for $1 \le m, j \le k$ and let*

$$X_i = \begin{pmatrix} [b^i] \\ [b^{2i}] \\ [b^{3i}] \\ \cdots \\ \cdots \\ \cdots \\ \left[b^{(k-1)i}\right] \\ \left[b^{ki}\right] \end{pmatrix}, \quad X_i^{-1} = \begin{pmatrix} [b^{-i}] \\ [b^{-2i}] \\ [b^{-3i}] \\ \cdots \\ \cdots \\ \cdots \\ \left[b^{-(k-1)i}\right] \\ \left[b^{-ki}\right] \end{pmatrix} \in M_{k\times k}(N_k)$$

for $i = 1, 2, 3, ..., \frac{k-1}{2}$, *where*

$$< b >= \left\{b^{qi}\right\}_{q=1}^{k} = \left\{b, b^2, b^3, \ldots, b^{ki} = e\right\} \leq D_k \leq S_k$$

with $[b^i], [b^{2i}], [b^{3i}], \cdots, \left[b^{(k-1)i}\right], \left[b^{ki}\right]$ *being the lower rows of the permutations* $b^i, b^{2i}, b^{3i}, \cdots, b^{(k-1)i}, b^{ki}$ *in their* 2*-row formats (similarly for* X_i^{-1} *for* $i = 1, 2, 3, \ldots, \frac{k-1}{2}$*).*

Define a mapping $f : M_{k\times k}(N_k) \to M_{k\times k}(N^{k^2})$ *as follows* $f(X) = f\left((x_{m_j})\right) = \left(x_{m_j}^{(m-1)k+j}\right)$ *for any* $X = (x_{m_j}) \in M_{k\times k}(N_k)$ *where* $N^{k^2} = \{x^l \ : \ 1 \leq l \leq k^2, \ x \in N_k\}$. *For* $X = (x_{m_j}) = X_i$ *or* $X^{-1} = (x'_{m_j}) = X_i^{-1}, \ i = 1, 2, 3, \ldots, \frac{k-1}{2}$, *let* $\Pi : f(X_i) = f\left((x_{m_j}\right) \in M_{k\times k}(N^{k^2}) \to Z \in M_{k\times k}(N_{k^2})$ *be defined as* $Z = \Pi\big(f(X_i)\big) = \Pi\big((x_{m_j})\big) = ((m-1)k + j) = (\pi_{mj}(r)), \ r = 1, 2, \cdots k$ *so that* $Y_i = \left[f(X_i)\right]$ *and* $Y_i^{-1} = \left[f\left(X_i^{-1}\right)\right], \ i = 1, 2, 3, \ldots, \frac{k-1}{2}$. *Then,* $L = \{X_0, X_0^T, Y_1, Y_1^{-1}, \cdots Y_s^{-1}\}$, $s = \frac{k-1}{2}$ *is an RBIBD whose parameter combinations satisfy:* $v = k^2, \ r = k+1, \ \lambda = k^0 = 1$ *and* $L = \{X_1, X_1^{-1}, \cdots X_s^{-1}\}$, $s = \frac{k-1}{2}$ *forms a set of mutually orthogonal Latin squares.*

Proof. The proof will be divided into three steps.

Step 1 The first two blocks are determined by $X_0 = ((m-1)k + j)$

(2.3)
$$X_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-1 & k \\ k+1 & k+2 & k+3 & \cdots & 2k-1 & 2k \\ 2k+1 & 2k+2 & 2k+3 & \cdots & 3k-1 & 3k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k(k-2)+1 & k(k-2)+2 & k(k-2)+3 & \cdots & k(k-2)+k-1 & k^2-k \\ k(k-1)+1 & k(k-1)+2 & k(k-1)+3 & \cdots & k^2-1 & k^2 \end{pmatrix}$$

where the transpose of X_0 denoted by X_0^T is given by $X_0^T = ((j-1)k + m)$

$$X_0^T = \begin{pmatrix} 1 & k+1 & 2k+1 & \cdots & k(k-2)+1 & k(k-1)+1 \\ 2 & k+2 & 2k+2 & \cdots & k(k-2)+2 & k(k-1)+2 \\ 3 & k+3 & 2k+3 & \cdots & k(k-3)+3 & k(k-1)+3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k-1 & 2k-1 & 3k-1 & \cdots & k(k-2)+k-1 & k^2-1 \\ k & 2k & 3k & \cdots & k^2-k & k^2 \end{pmatrix} \quad (2.4)$$

Step 2. Consider the first row of X_0 as a permutation $b \in S_k$ such that $\mid b \mid = k$ (i.e. $b^k = I$ (Identity permutation)), permutation $b = (1\ \ 2\ \ 3\ \ \dots\ \ k)$ as a cycle. Let

$$X_i = \begin{pmatrix} [b^i] \\ [b^{2i}] \\ [b^{3i}] \\ \dots \\ \dots \\ \dots \\ \left[b^{(k-1)i}\right] \\ \left[b^{ki}\right] \end{pmatrix}, \quad X_i^{-1} = \begin{pmatrix} [b^{-i}] \\ [b^{-2i}] \\ [b^{-3i}] \\ \dots \\ \dots \\ \dots \\ \left[b^{-(k-1)i}\right] \\ \left[b^{-ki}\right] \end{pmatrix} \in M_{k\times k}(N_k)$$

for $i = 1, 2, 3, \dots, \frac{k-1}{2}$, where $M_{k\times k}(N_k)$ is the set of k^2-matrices over $N_k = \{1, 2, \dots, k\}$ and $< b >= \{b^{qi}\}_{q=1}^{k} = \left\{b, b^2, b^3, \dots, b^{ki} = e\right\} \leq D_k \leq S_k$ with $[b^i], [b^{2i}], [b^{3i}], \dots, \left[b^{(k-1)i}\right], \left[b^{ki}\right]$ being the lower rows of the permutations $b^i, b^{2i}, b^{3i}, \dots, b^{(k-1)i}, b^{ki}$ in their 2-row formats. This explains how X_i is gotten and similarly, X_i^{-1} for $i = 1, 2, 3, \dots, \frac{k-1}{2}$.

Step 3. Define a mapping $f : M_{k\times k}(N_k) \to M_{k\times k}(N^{k^2})$ as follows $f(X) = f\left((x_{m_j})\right) = \left(x_{m_j}^{(m-1)k+j}\right)$ for any $X = (x_{m_j}) \in M_{k\times k}(N_k)$ where $N^{k^2} = \{x^l : 1 \leq l \leq k^2,\ x \in N_k\}$.

Let $X = (x_{m_j}) = X_i,\ i = 1, 2, 3, \dots, \frac{k-1}{2}$, so that $f(X_i) = f\left((x_{m_j})\right) = \left(x_{m_j}^{(m-1)k+j}\right), i = 1, 2, 3, \dots, \frac{k-1}{2}$. Similarly, let $X^{-1} = (x'_{m_j}) = X_i^{-1}, i = 1, 2, 3, \dots, \frac{k-1}{2}$ so that $f\left(X_i^{-1}\right) = f\left((x'_{m_j})\right) = \left(x'^{(m-1)k+j}_{m_j}\right), i = 1, 2, 3, \dots, \frac{k-1}{2}$.

Let $\Pi : f(X_i) = f\left((x_{m_j}\right) \in M_{k\times k}(N^{k^2}) \to Z \in M_{k\times k}(N_{k^2})$ be defined as $Z = \Pi\left(f(X_i)\right) = \Pi\left((x_{m_j})\right) = ((m-1)k + j) = (\pi_{mj}(r)),\ r = 1, 2, \dots k$. Then,

(2.5)

$$Y_i = \left[f(X_i)\right] = \begin{pmatrix} \pi_{1j}(1) & \pi_{2j}(1) & \cdots & \cdots & \pi_{(k-1)j}(1) & \pi_{kj}(1) \\ \pi_{1j}(2) & \pi_{2j}(2) & \cdots & \cdots & \pi_{(k-1)j}(2) & \pi_{kj}(2) \\ \pi_{1j}(3) & \pi_{2j}(3) & \cdots & \cdots & \pi_{(k-1)j}(3) & \pi_{kj}(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \pi_{1j}(k-1) & \pi_{2j}(k-1) & \cdots & \cdots & \pi_{(k-1)j}(k-1) & \pi_{kj}(k-1) \\ \pi_{1j}(k) & \pi_{2j}(k) & \cdots & \cdots & \pi_{(k-1)j}(k) & \pi_{kj}(k) \end{pmatrix}$$

for $i = 1, 2, 3, \ldots, \frac{k-1}{2}$ where $j = 1, \ldots, k$.

Similarly, the Y_i^{-1} corresponding to X_i^{-1} can be found for $i = 1, 2, 3, \ldots, \frac{k-1}{2}$ using the same approach. Hence, for a any $n = 2k + 1$, where k is a natural number, RBIBD is

$$L = \{X_0, X_0^T, Y_1, Y_1^{-1}, \ldots Y_s^{-1}\}, \quad s = \frac{k-1}{2}, \tag{2.6}$$

and elements of the set

$$L = \{X_1, X_1^{-1}, \ldots X_s^{-1}\}, \quad s = \frac{k-1}{2}, \tag{2.7}$$

are mutually orthogonal Latin squares. □

Remark 2.4. *It is interesting to note that the construction in Theorem 2.4 is principally based on a cyclic subgroup of the symmetric group S_n whose generator is one of the permutations of the 2-permutation generating set of the dihedral group D_n and 2-permutation generating set of the presentation of S_n.*

3. Construction of Designs

In this section, we shall use the methodology presented in Section 2 to construct RBIBD (Graeco-Latin squares) for cases $k = 3, 5$ and 11 of order $k \times k$ where k is an odd integer. Also, let $k = 2s + 1$ and $s \in \mathbb{N}$ be the number of resolvable blocks $n = k + 1 = 2(s + 1)$. Then, the constructions are as follow:

Case 1: For k=3.
When $v = 9$, $k = 3$, $s = 1$. By equation (2.3),

$$X_0 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}, \quad X_0^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 & 9 \\ \hline \end{array}.$$

Let the permutation

$$b = (1, 2, 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = [2, 3, 1].$$

Note that $\langle b\rangle = \{b, b^2, b^3 = e\} = \langle(123)\rangle \leq S_3$. For $i = 1$, we have

$$X_1 = \begin{pmatrix} b \\ b^2 \\ b^3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad X_1^{-1} = \begin{pmatrix} b^{-1} \\ b^{-2} \\ b^{-3} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

These gives

$$f(X_1) = \begin{pmatrix} 2^1 & 3^2 & 1^3 \\ 3^4 & 1^5 & 2^6 \\ 1^7 & 2^8 & 3^9 \end{pmatrix}, \quad \text{and} \quad f(X_1^{-1}) = \begin{pmatrix} 3^1 & 1^2 & 2^3 \\ 2^4 & 3^5 & 1^6 \\ 1^7 & 2^8 & 3^9 \end{pmatrix}, \tag{3.1}$$

where the superscripts are the corresponding $f(q_m)$ e.g for X_1 in equation (3.1), $f(1_1) = 3$, $f(1_2) = 5$ and $f(1_3) = 7$. Hence, we get

$$Y_1 = \begin{pmatrix} 3 & 5 & 7 \\ 1 & 6 & 8 \\ 2 & 4 & 9 \end{pmatrix} = \begin{array}{|c|c|c|}\hline 3 & 5 & 7 \\ \hline 1 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \hline \end{array} \quad \text{and} \quad Y_1^{-1} = \begin{pmatrix} 2 & 6 & 7 \\ 3 & 4 & 8 \\ 1 & 5 & 9 \end{pmatrix} = \begin{array}{|c|c|c|}\hline 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \hline 1 & 5 & 9 \\ \hline \end{array}$$

Therefore, the blocks for RBIBD of Latin square of order $k = 3$ after computing $f(X_j),\ f(X_j^{-1}),\quad j = 1$ and $Y_j,\ Y_j^{-1}$ are:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 3 & 5 & 7 \\ \hline 1 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \hline \end{array}, \text{ and } \begin{array}{|c|c|c|}\hline 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \hline 1 & 5 & 9 \\ \hline \end{array}.$$

Case 2: For n=5.

When $v = 25$, $k = 5$, $s = 2$. By equation (2.3),

$$X_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & 18 & 19 & 20 \\ \hline 21 & 22 & 23 & 24 & 25 \\ \hline \end{array},$$

$$X_0^T = \begin{pmatrix} 1 & 6 & 11 & 16 & 21 \\ 2 & 7 & 12 & 17 & 22 \\ 3 & 8 & 13 & 18 & 23 \\ 4 & 9 & 14 & 19 & 24 \\ 5 & 10 & 15 & 20 & 25 \end{pmatrix} = \begin{array}{|c|c|c|c|c|}\hline 1 & 6 & 11 & 16 & 21 \\ \hline 2 & 7 & 12 & 17 & 22 \\ \hline 3 & 8 & 13 & 18 & 23 \\ \hline 4 & 9 & 14 & 19 & 24 \\ \hline 5 & 10 & 15 & 20 & 25 \\ \hline \end{array}$$

Let the permutation $b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (1, 2, 3, 4, 5) = [2, 3, 4, 5, 1]$

Note that $< b >= \{b, b^2, b^3, b^4, b^5 = e\} \leq D_{10} \leq S_5$. For $i = 1, 2$, we have

$$X_1 = \begin{pmatrix} b \\ b^2 \\ b^3 \\ b^4 \\ b^5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad X_1^{-1} = \begin{pmatrix} b^{-1} \\ b^{-2} \\ b^{-3} \\ b^{-4} \\ b^{-5} \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} b^2 \\ b^4 \\ b^6 \\ b^8 \\ b^{10} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad X_2^{-1} = \begin{pmatrix} b^{-2} \\ b^{-4} \\ b^{-6} \\ b^{-8} \\ b^{-10} \end{pmatrix} = \begin{pmatrix} 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

This gives

$$f(X_1) = \begin{pmatrix} 2^1 & 3^2 & 4^3 & 5^4 & 1^5 \\ 3^6 & 4^7 & 5^8 & 1^9 & 2^{10} \\ 4^{11} & 5^{12} & 1^{13} & 2^{14} & 3^{15} \\ 5^{16} & 1^{17} & 2^{18} & 3^{19} & 4^{20} \\ 1^{21} & 2^{22} & 3^{23} & 4^{24} & 5^{25} \end{pmatrix} \implies Y_1 =$$

5	9	13	17	21
1	10	14	18	22
2	6	15	19	23
3	7	11	20	24
4	8	12	16	25

and so on.

Therefore, the blocks for RBIBD of Latin square of order $k = 5$ after computing $f(X_j),\ f(X_j^{-1}), \quad j = 1, \ldots, 2$ and $Y_j,\ Y_j^{-1}$ are:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

,

1	6	11	16	21
2	7	12	17	22
3	8	13	18	23
4	9	14	19	24
5	10	15	20	25

,

5	9	13	17	21
1	10	14	18	22
2	6	15	19	23
3	7	11	20	24
4	8	12	16	25

,

2	8	14	20	21
3	9	15	16	22
4	10	11	17	23
5	6	12	18	24
1	7	13	19	25

4	7	15	18	21
5	8	11	19	22
1	9	12	20	23
2	10	13	16	24
3	6	14	17	25

3	10	12	19	21
4	6	13	20	22
5	7	14	16	23
1	8	15	17	24
2	9	11	18	25

Case 3: For k=11.

When $v = 121$, $k = 11$, $s = 5$. By equation (2.3),

$$X_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 \\ 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 \\ 45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 \\ 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 & 65 & 66 \\ 67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 & 76 & 77 \\ 78 & 79 & 80 & 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 \\ 89 & 90 & 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 \\ 100 & 101 & 102 & 103 & 104 & 105 & 106 & 107 & 108 & 109 & 110 \\ 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 & 120 & 121 \end{pmatrix}$$

$Y_0 =$

1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22
23	24	25	26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63	64	65	66
67	68	69	70	71	72	73	74	75	76	77
78	79	80	81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120	121

$$X_1 = \begin{pmatrix} b \\ b^2 \\ b^3 \\ b^4 \\ b^5 \\ b^6 \\ b^7 \\ b^8 \\ b^9 \\ b^10 \\ b^11 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 \\ 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}$$

$Y_1 =$

11	21	31	41	51	61	71	81	91	101	111
1	22	32	42	52	62	72	82	92	102	112
2	12	33	43	53	63	73	83	93	103	113
3	13	23	44	54	64	74	84	94	104	114
4	14	24	34	55	65	75	85	95	105	115
5	15	25	35	45	66	76	86	96	106	116
6	16	26	36	46	56	77	87	97	107	117
7	17	27	37	47	57	67	88	98	108	118
8	18	28	38	48	58	68	78	99	109	119
9	19	29	39	49	59	69	79	89	110	120
10	20	30	40	50	60	70	80	90	100	121

and so on.

Therefore, the blocks for RBIBD of Latin square of order $k = 11$ after computing $f(X_j),\ f(X_j^{-1}),\quad j = 1, \ldots, 5$ and $Y_j,\ Y_j^{-1}$ are:

1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22
23	24	25	26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63	64	65	66
67	68	69	70	71	72	73	74	75	76	77
78	79	80	81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120	121

1	12	23	34	45	56	67	78	89	100	111
2	13	24	35	46	57	68	79	90	101	112
3	14	25	36	47	58	69	80	91	102	113
4	15	26	37	48	59	70	81	92	103	114
5	16	27	38	49	60	71	82	93	104	115
6	17	28	39	50	61	72	83	94	105	116
7	18	29	40	51	62	73	84	95	106	117
8	19	30	41	52	63	74	85	96	107	118
9	20	31	42	53	64	75	86	97	108	119
10	21	32	43	54	65	76	87	98	109	120
11	22	33	44	55	66	77	88	99	110	121

11	21	31	41	51	61	71	81	91	101	111
1	22	32	42	52	62	72	82	92	102	112
2	12	33	43	53	63	73	83	93	103	113
3	13	23	44	54	64	74	84	94	104	114
4	14	24	34	55	65	75	85	95	105	115
5	15	25	35	45	66	76	86	96	106	116
6	16	26	36	46	56	77	87	97	107	117
7	17	27	37	47	57	67	88	98	108	118
8	18	28	38	48	58	68	78	99	109	119
9	19	29	39	49	59	69	79	89	110	120
10	20	30	40	50	60	70	80	90	100	121

2	14	26	38	50	62	74	86	98	110	111
3	15	27	39	51	63	75	87	99	100	112
4	16	28	40	52	64	76	88	89	101	113
5	17	29	41	53	65	77	78	90	102	114
6	18	30	42	54	66	67	79	91	103	115
7	19	31	43	55	56	68	80	92	104	116
8	20	32	44	45	57	69	81	93	105	117
9	21	33	34	46	58	70	82	94	106	118
10	22	23	35	47	59	71	83	95	107	119
11	12	24	36	48	60	72	84	96	108	120
1	13	25	37	49	61	73	85	97	109	121

10	19	28	37	46	66	75	84	93	102	111
11	20	29	38	47	56	76	85	94	103	112
1	21	30	39	48	57	77	86	95	104	113
2	22	31	40	49	58	67	87	96	105	114
3	12	32	41	50	59	68	88	97	106	115
4	13	33	42	51	60	69	78	98	107	116
5	14	23	43	52	61	70	79	99	108	117
6	15	24	44	53	62	71	80	89	109	118
7	16	25	34	54	63	72	81	90	110	119
8	17	26	35	55	64	73	82	91	100	120
9	18	27	36	45	65	74	83	92	101	121

3	16	29	42	55	57	70	83	96	109	111
4	17	30	43	45	58	71	84	97	110	112
5	18	31	44	46	59	72	85	98	100	113
6	19	32	34	47	60	73	86	99	101	114
7	20	33	35	48	61	74	87	89	102	115
8	21	23	36	49	62	75	88	90	103	116
9	22	24	37	50	63	76	78	91	104	117
10	12	25	38	51	64	77	79	92	105	118
11	13	26	39	52	65	67	80	93	106	119
1	14	27	40	53	66	68	81	94	107	120
2	15	28	41	54	56	69	82	95	108	121

9	17	25	44	52	60	68	87	95	103	111
10	18	26	34	53	61	69	88	96	104	112
11	19	27	35	54	62	70	78	97	105	113
1	20	28	36	55	63	71	79	98	106	114
2	21	29	37	45	64	72	80	99	107	115
3	22	30	38	46	65	73	81	89	108	116
4	12	31	39	47	66	74	82	90	109	117
5	13	32	40	48	56	75	83	91	110	118
6	14	33	41	49	57	76	84	92	100	119
7	15	23	42	50	58	77	85	93	101	120
8	16	24	43	51	59	67	86	94	102	121

4	18	32	35	49	63	77	80	94	108	111
5	19	33	36	50	64	67	81	95	109	112
6	20	23	37	51	65	68	82	96	110	113
7	21	24	38	52	66	69	83	97	100	114
8	22	25	39	53	56	70	84	98	101	115
9	12	26	40	54	57	71	85	99	102	116
10	13	27	41	55	58	72	86	89	103	117
11	14	28	42	45	59	73	87	90	104	118
1	15	29	43	46	60	74	88	91	105	119
2	16	30	44	47	61	75	78	92	106	120
3	17	31	34	48	62	76	79	93	107	121

8	15	33	40	47	65	72	79	97	104	111
9	16	23	41	48	66	73	80	98	105	112
10	17	24	42	49	56	74	81	99	106	113
11	18	25	43	50	57	75	82	89	107	114
1	19	26	44	51	58	76	83	90	108	115
2	20	27	34	52	59	77	84	91	109	116
3	21	28	35	53	60	67	85	92	110	117
4	22	29	36	54	61	68	86	93	100	118
5	12	30	37	55	62	69	87	94	101	119
6	13	31	38	45	63	70	88	95	102	120
7	14	32	39	46	64	71	78	96	103	121

5	20	24	39	54	58	73	88	92	107	111
6	21	25	40	55	59	74	78	93	108	112
7	22	26	41	45	60	75	79	94	109	113
8	12	27	42	46	61	76	80	95	110	114
9	13	28	43	47	62	77	81	96	100	115
10	14	29	44	48	63	67	82	97	101	116
11	15	30	34	49	64	68	83	98	102	117
1	16	31	35	50	65	69	84	99	103	118
2	17	32	36	51	66	70	85	89	104	119
3	18	33	37	52	56	71	86	90	105	120
4	19	23	38	53	57	72	87	91	106	121

7	13	30	36	53	59	76	82	99	105	111
8	14	31	37	54	60	77	83	89	106	112
9	15	32	38	55	61	67	84	90	107	113
10	16	33	39	45	62	68	85	91	108	114
11	17	23	40	46	63	69	86	92	109	115
1	18	24	41	47	64	70	87	93	110	116
2	19	25	42	48	65	71	88	94	100	117
3	20	26	43	49	66	72	78	95	101	118
4	21	27	44	50	56	73	79	96	102	119
5	22	28	34	51	57	74	80	97	103	120
6	12	29	35	52	58	75	81	98	104	121

6	22	27	43	48	64	69	85	90	106	111
7	12	28	44	49	65	70	86	91	107	112
8	13	29	34	50	66	71	87	92	108	113
9	14	30	35	51	56	72	88	93	109	114
10	15	31	36	52	57	73	78	94	110	115
11	16	32	37	53	58	74	79	95	100	116
1	17	33	38	54	59	75	80	96	101	117
2	18	23	39	55	60	76	81	97	102	118
3	19	24	40	45	61	77	82	98	103	119
4	20	25	41	46	62	67	83	99	104	120
5	21	26	42	47	63	68	84	89	105	121

4. Conclusion

A Simple Generalized Construction of Resolvable for k being a Prime Number were constructed, in which algebric groupings were capitalized on in the constructions. The method is efficient, sufficient and also mitigate against the tediousness encountered in other methods of construction when v is large.

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