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COINCIDENCE POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN b-METRIC SPACES VIA DIGRAPHS

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ABSTRACT. In this paper, we present the concept of conventional F_G-contraction and prove the results of a new coincidence point for multi-valued in b-metric spaces endowed with a digraph G.

1. INTRODUCTION

Nadler [1] proved that the multi-valued contractions in all metric regions have a fixed point. Since then, many authors, including Pathak [2], Berinde [4], Gordji et al. [5] and others have studied various types of fixed point theories for multi-valued contraction.

Wardowski [6] adopted the concept of F-contraction for single-value mappings and studied the fixed points for such mappings in metric spaces. Using Wardowski's concepts and Nadler's concepts, many authors (see [7–13] and their references) studied fixed points for multi-value mapping.

Bakhtin [14] takes the concept of b-metric spaces as a generalization of metric spaces and makes Banach's famous contraction principle in metric spaces to

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b-metric spaces in a recent investigation. Study of fixed point theory that integrates graphs is a new development in the domain of contractual value multi-value theory.

Motivation by the results in [6,23,24], we present the concept of conventional F_G-contraction in b-metric spaces and achieve coincidence point results for the value match hybrid pair. Multiple values in a b-metric space with a digraph.

2. Some basic concepts

In the sequel, throughout this article, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ denote the set of natural numbers, the set of real numbers and the set of positive real numbers, respectively.

Definition 2.1. [3] *Let Ω be a nonempty set and $s \geq 1$ be a given real number. A function $d : \Omega \times \Omega \to [0, \infty)$ is said to be a b-metric on Ω if the following conditions hold:*

(i) $d(\xi, \zeta) = 0$ *if and only if* $\xi = \zeta$;
(ii) $d(\xi, \zeta) = d(\xi, \zeta)$;
(iii) $d(\xi, \eta) \leq s[d(\xi, \zeta) + d(\zeta, \eta)]$.

The pair (Ω, d) is called a b-metric space.

Example 1. [4] *Let $p \in (0, 1)$. Then the set*

$$l^p(\mathbb{R}) := \left\{ \{\xi_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |\xi_n|^p < \infty \right\}$$

endowed with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ given by

$$d(\{\xi_n\}, \{\zeta_n\}) = \left(\sum_{n=1}^{\infty} |\xi_n - \zeta_n|^p \right)^{\frac{1}{p}}$$

for all $\{\xi_n\}, \{\zeta_n\} \in l^p(\mathbb{R})$ is a b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 2.2. [15] *Let (Ω, d) be a b-metric space, $\xi \in$ and $\{\xi_n\}$ be a sequence in Ω. Then*

(i) $\{\xi_n\}$ *converges to* ξ *if and only if* $\lim d(\xi_n, \xi) = 0$. *We denote this by* $\lim_{n\to\infty} \xi_n = \xi$;
(ii) $\{\xi_n\}$ *is Cauchy if and only if* $\lim_{n,m\to\infty} d(\xi_n, \xi_m) = 0$;
(iii) (Ω, d) *is complete if and only if every Cauchy sequence in* Ω *is convergent.*

Definition 2.3. [16] *Let (Ω, d) be a b-metric space. A subset $A \subseteq \Omega$ is said to be open if and only if for any $a \in A$, there exists $\epsilon > 0$ such that the open ball $B(a, \epsilon) \subseteq A$. The family of all open subsets of Ω will be denoted by τ.*

Theorem 2.1. [16] *Let (Ω, d) be a b-metric space and τ be the topology defined above. Then for any nonempty subset $A \subseteq X$ we have*

(i) *A is closed if and only if for any sequence $\{\xi_n\}$ in A which converges to ξ, we have $\xi \in A$;*

(ii) *if we define A to be the intersection of all closed subsets of Ω which contains A, then for any $\xi \in A$ and for any $\epsilon > 0$, we have $B(\xi, \epsilon) \cap A \neq \emptyset$.*

Let (Ω, d) be a b-metric space and $CB(\Omega)$ be the set of all nonempty closed bounded subsets of Ω. An element $\xi \in \Omega$ is said to be a fixed point of a multi-valued mapping $P : \Omega \to 2^{\Omega}$ if $\xi \in P\xi$, where 2^{Ω} denotes the collection of all nonempty subsets of Ω. For $A, B \in CB(\Omega)$, define

$$H(A, B) = \max\{\sup_{\xi \in A} d(\xi, B), \sup_{\zeta \in B} d(\zeta, A)\},$$

where $d(\xi, B) = \inf\{d(\xi, \zeta) : \zeta \in B\}$. Such a map H is called the Hausdorff b-metric induced by the b-metric d.

Definition 2.4. [23] *Let (Ω, d) be a b-metric space and $P : \Omega \to CB(\Omega)$ and $f : \Omega \to \Omega$ be two mappings. If $\zeta = f\zeta \in P\xi$ for some ξ in X, then ξ is called a coincidence point of P and f and ζ is called a point of coincidence of P and f.*

Lemma 2.1. [18] *Let (Ω, d) be a b-metric space with $s \geq 1$ and $A, B \in CB(\Omega)$. Then, for each $\lambda > 1$ and for each $a \in A$, there exists $b(a) \in B$ such that $d(a, b(a)) \leq \lambda H(A, B)$.*

Lemma 2.2. [19] *Let (Ω, d) be a b-metric space with $s \geq 1$. For any $A, B, C \in CB(\Omega)$ and any $\xi, \zeta \in \Omega$, we have the following:*

(i) $d(\xi, B) \leq d(\xi, b)$ *for any* $b \in B$;

(ii) $d(\xi, B) \leq H(A, B)$ *for any* $\xi \in A$;

(iii) $d(\xi, A) \leq s[d(\xi, \zeta) + d(\zeta, A)]$.

Let (Ω, d) be a b-metric space. We assume that G is a digraph with the set of vertices $V(G) = \Omega$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(\xi, \xi) : \xi \in \Omega\}$. We also assume that G has no parallel edges and obtain a weighted graph by assigning to each edge the distance between its vertices. We can identify G with the pair $(V(G), E(G))$. We denote the conversion of a graph G by G^{-1}, that is, the graph obtained from G by reversing the direction of the edges i.e., $E(G^{-1}) = \{(\xi, \zeta) \in \Omega \times \Omega : (\zeta, \xi) \in E(G)\}$.

Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \tag{2.1}$$

If ξ, ζ are vertices of the digraph G, then a path in G from ξ to ζ of length $n(n \in \mathbb{N})$ is a sequence $\{\xi_j\}_{j=0}^{n}$ of $n+1$ vertices such that $\xi_0 = \xi$, $\xi_n = \zeta$ and $(\xi_{j-1}, \xi_j) \in E(G)$ for $j = 1, 2, \ldots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected (see more detail [9, 20–22]).

Definition 2.5 (see [23]). *Let (Ω, d) be a b-metric space with $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Then the mapping $f : \Omega \to \Omega$ is called edge preserving if*

$$\xi, \zeta \in \Omega,\ (\xi, \zeta) \in E(\tilde{G}) \Rightarrow (f\xi, f\zeta) \in E(\tilde{G}).$$

Definition 2.6 (see [23]). *Let (Ω, d) be a b-metric space with a graph $G = (V(G), E(G))$. Then the mapping $P : \Omega \to CB(\Omega)$ is called edge preserving if*

$$\xi, \zeta \in \Omega,\ \xi \neq \zeta,\ (\xi, \zeta) \in E(\tilde{G}) \Rightarrow (u_1, u_2) \in E(\tilde{G}), \quad \forall\ u_1 \in P\xi,\ u_2 \in P\zeta.$$

Definition 2.7 (see [23]). *Let (Ω, d) be a b-metric space with a graph $G = (V(G), E(G))$. Let $P : \Omega \to CB(\Omega)$ be a multi-valued mapping and $f : \Omega \to \Omega$ be a single-valued mapping. Then P is called edge preserving w.r.t. f if*

$$\xi, \zeta \in \Omega,\ \xi \neq \zeta,\ (f\xi, f\zeta) \in E(\tilde{G}) \Rightarrow (u_1, u_2) \in E(\tilde{G}), \quad \forall\ u_1 \in P\xi,\ u_2 \in P\zeta.$$

Definition 2.8. [24] *Let $s \geq 1$ be a real number. We denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:*

- (F1) *F is strictly increasing*;
- (F2) *for each sequence $\{\psi_n\}$ of positive numbers, $\lim_{n\to\infty} \psi_n = 0$ if and only if $\lim_{n\to\infty} F(\psi_n) = -\infty$;*
- (F3) *for each sequence $\{\psi_n\}$ of positive numbers, $\lim_{n\to\infty} \psi_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} (\psi_n)^k F(\psi_n) = 0$;*

(F4) *such that* $\tau + F(s\psi_n) \leq F(\psi_{n-1}),\ \forall\ n \in \mathbb{N}$ *and some* $\tau > 0$, *then* $\tau + F(s^n\psi_n) \leq F(s^{n-1}\psi_{n-1})\ \forall\ n \in \mathbb{N}$.

Definition 2.9. [24] *Let* (Ω, d) *be a b-metric space with* $s \geq 1$. *A multivalued mapping* $P : \Omega \to CB(\Omega)$ *is called an* F*-contraction of Nadler type if there exist* $F \in \mathcal{F}$, $\tau > 0$ *such that*

$$2\tau + F(sH(P\xi, P\zeta)) \leq F(d(\xi, \zeta)),$$

for all $\xi, \zeta \in \Omega$ *with* $P\xi \neq P\zeta$.

Example 2. [24] *If* $F(\xi) = \ln \xi$, $\xi > 0$, *then* $F \in \mathcal{F}$.

3. Main Results

We support that (Ω, d) is a b-metric space endowed with a reflexive digraph G such that $V(G) = \Omega$ and G has no parallel edges.

Definition 3.1. *Let* (Ω, d) *be a b-metric space with* $s \geq 1$ *and let* $G = (V(G), E(G))$ *be a digraph. Then the pair* (P, f) *of mappings* $P : \Omega \to CB(\Omega)$ *and* $f : \Omega \to \Omega$ *is called a generalized* F_G*-contraction of Nadler type if there exist* $F \in \mathcal{F}$, $\tau > 0$ *and* $L > 0$, *such that*

$$2\tau + F(sH(P\xi, P\zeta)) \leq F(M(f\xi, f\zeta)) + LN(f\xi, f\zeta), \tag{3.1}$$

for all $\xi, \zeta \in X$ *with* $P\xi \neq P\zeta$ *and* $(f\xi, f\zeta) \in E(\tilde{G})$ *where*

$$M(f\xi, f\zeta) = \max\left\{d(f\xi, f\zeta), \frac{d(f\zeta, P\zeta)[1 + d(f\xi, P\xi)]}{1 + d(f\xi, f\zeta)}\right\}$$

and

$$N(f\xi, f\zeta) = \min\left\{d(f\xi, f\zeta), d(f\xi, P\xi), d(f\zeta, P\zeta), d(f\xi, P\zeta), d(f\zeta, P\xi)\right\}.$$

Theorem 3.1. *Let* (Ω, d) *be a b-metric space with* $s \geq 1$ *and let* $G = (V(G), E(G))$ *be a graph. Let* $P : \Omega \to CB(\Omega)$ *and* $f : \Omega \to \Omega$ *be such that* $P(\Omega) \subseteq f(\Omega)$ *and* $f(\Omega)$ *a complete subspace of* Ω. *Support that* P *is edge preserving w.r.t.* f *and there exist a function* $F \in \mathcal{F}$ *which is continuous from right* $\tau > 0$ *and* $L > 0$ *such that* (P, f) *is generalized* F_G*-contraction of Nadler type. Suppose that the triple* (Ω, d, G) *has the following property:*

(i) *If* $\{f\xi_n\}$ *is a sequence in* Ω *such that* $f\xi_n \to \xi$ *and* $(f\xi_n, f\xi_{n+1}) \in E(\tilde{G})$ *for all* $n \in \mathbb{N}$, *then there exists a subsequence* $\{f\xi_{n_k}\}$ *of* $\{f\xi_n\}$ *such that* $(f\xi_{n_k}, \xi) \in E(\tilde{G})$ *for all* $k \in \mathbb{N}$.

(ii) *If there exists* $\xi_0 \in \Omega$ *such that* $(f\xi_0, u) \in E(\tilde{G})$ *for some* $u \in P\xi_0$, *then* f *and* P *have a point of coincidence in* $f(\Omega)$.

Proof. Suppose there exists $\xi_0 \in \Omega$ such that $(f\xi_0, u) \in E(\tilde{G})$ for some $u \in P\xi_0$. If $f\xi_0 \in P\xi_0$, then there is nothing to prove. Then, we support that $f\xi_0 \notin P\xi_0$. So, $d(f\xi_0, P\xi_0) > 0$, from $P\xi_0$ is closed. Hence, $d(f\xi_0, \zeta) > 0$ for all $\zeta \in P\xi_0$. Since $P\xi_0 \subseteq f(\Omega)$ is nonempty, there exists $\xi_1 \in \Omega$ such that $u = f\xi_1 \in P\xi_0$, $d(f\xi_0, f\xi_1) > 0$ and $(f\xi_0, f\xi_1) \in E(\tilde{G})$. If $f\xi_1 \in P\xi_1$, then f and P have a point of coincidence in $f(\Omega)$. Thus, we assume that $f\xi_1 \notin P\xi_1$ and so $P\xi_0 \neq P\xi_1$ which gives that $\xi_0 \neq \xi_1$. From $F \in \mathcal{F}$ is continuous from the right, there exists $\lambda > 1$ such that

$$F(\lambda s H(P\xi_0, P\xi_1)) < F(sH(P\xi_0, P\xi_1)) + \tau. \tag{3.2}$$

From $f\xi_1 \in P\xi_0$ and $\lambda > 1$, using Lemma 2.1, there exists $f\xi_2 \in P\xi_1$ for some $\xi_2 \in$ such that

$$d(f\xi_1, f\xi_2) \leq \lambda H(P\xi_0, P\xi_1). \tag{3.3}$$

Because $f\xi_1 \notin P\xi_1$, we get $d(f\xi_1, P\xi_1) > 0$ and $d(f\xi_1, f\xi_2) > 0$. Using monotonicity property of , from (3.2) and (3.3), we obtain that

$$F(sd(f\xi_1, f\xi_2)) \leq F(\lambda s H(P\xi_0, P\xi_1)) < F(sH(P\xi_0, P\xi_1)) + \tau. \tag{3.4}$$

Using (3.30) and (3.4), we get

$$\begin{aligned} 2\tau + F(sd(f\xi_1, f\xi_2)) &< 2\tau + F(sH(P\xi_0, P\xi_1)) + \tau \\ &\leq F(M(f\xi_0, f\xi_1)) + LN(f\xi_0, f\xi_1)) + \tau. \end{aligned} \tag{3.5}$$

Hence,

$$\tau + F(sd(f\xi_1, f\xi_2)) < F(M(f\xi_0, f\xi_1)) + LN(f\xi_0, f\xi_1)). \tag{3.6}$$

From P is edge preserving w.r.t. f and $\xi_0 \neq \xi 1$, $(f\xi_0, f\xi_1) \in E(\tilde{G})$, $f\xi_1 \in P\xi_0$, $f\xi_2 \in P\xi_1$, it follows that $(f\xi_1, f\xi_2) \in E(\tilde{G})$. If $f\zeta_2 \in P\xi_2$, then the theorem is proved. Thus, we support that $f\xi_2 \notin P\xi_2$. It follows that $P\xi_1 \neq P\xi_2$ and this implies that $\xi_1 \neq \xi_2$. Similarly above, there exists $f\xi_3 \in P\xi_2$ for some $\xi_3 \in \Omega$ and $d(f\xi_2, f\xi_3) > 0$ such that

$$\tau + F(sd(f\xi_2, f\xi_3)) < F(M(f\xi_1, f\xi_2)) + LN(f\xi_1, f\xi_2)). \tag{3.7}$$

From P is edge preserving w.r.t. f and $\xi_1 \neq \xi_2$, $(f\xi_1, f\xi_2) \in E(\tilde{G})$, $f\xi_2 \in P\xi_1$, $f\xi_3 \in P\xi_2$, it follows that $(f\xi_2, f\xi_3) \in E(\tilde{G})$. Continuing this process, we can construct a sequence $\{f\xi_n\}$ in $f(\Omega)$ such that $f\xi_n \in P\xi_{n-1}$, $f\xi_n \notin P\xi_n$, $d(f\xi_n, f\xi_{n+1}) > 0$, $(f\xi_n, f\xi_{n+1}) \in E(\tilde{G})$ for $n = 0, 1, 2, \ldots$ and

$$\tau + F(sd(f\xi_n, f\xi_{n+1})) < F(M(f\xi_{n-1}, f\xi_n)) + LN(f\xi_{n-1}, f\xi_n), \quad \forall\, n \in \mathbb{N}. \tag{3.8}$$

Thus,

$$F(sd(f\xi_n, f\xi_{n+1})) < F(M(f\xi_{n-1}, f\xi_n)) + LN(f\xi_{n-1}, f\xi_n), \quad \forall\, n \in \mathbb{N}. \tag{3.9}$$

Since F is strictly increasing, we get

$$0 < sd(f\xi_n, f\xi_{n+1})) < M(f\xi_{n-1}, f\xi_n) + LN(f\xi_{n-1}, f\xi_n), \quad \forall\, n \in \mathbb{N}. \tag{3.10}$$

So,

$$0 < d(f\xi_n, f\xi_{n+1})) < M(f\xi_{n-1}, f\xi_n) + LN(f\xi_{n-1}, f\xi_n), \quad \forall\, n \in \mathbb{N}, \tag{3.11}$$

where

$$\begin{aligned} M(f\xi_{n-1}, f\xi_n) &= \max\left\{d(f\xi_{n-1}, f\xi_n), \frac{d(f\xi_n, P\xi_n)[1 + d(f\xi_{n-1}, P\xi_{n-1})]}{1 + d(f\xi_{n-1}, f\xi_n)}\right\} \\ &\leq \max\left\{d(f\xi_{n-1}, f\xi_n), \frac{d(f\xi_n, f\xi_{n+1})[1 + d(f\xi_{n-1}, f\xi_n)]}{1 + d(f\xi_{n-1}, f\xi_n)}\right\} \\ &\leq \max\left\{d(f\xi_{n-1}, f\xi_n), d(f\xi_n, f\xi_{n+1})\right\} \end{aligned}$$

and

$$\begin{aligned} &N(f\xi_{n-1}, f\xi_n) \\ &= \min\{d(f\xi_{n-1}, f\xi_n), d(f\xi_{n-1}, P\xi_{n-1}), d(f\xi_n, P\xi_n), d(f\xi_{n-1}, P\xi_n), d(f\xi_n, P\xi_{n-1})\} \\ &\leq \min\{d(f\xi_{n-1}, f\xi_n), d(f\xi_{n-1}, f\xi_n), d(f\xi_n, f\xi_{n+1}), d(f\xi_{n-1}, f\xi_{n+1}), d(f\xi_n, f\xi_n)\} \\ &= 0. \end{aligned}$$

If $d(f\xi_n, f\xi_{n+1}) \geq d(f\xi_{n-1}, f\xi_n)$, it follows from (3.11) that

$$0 < d(f\xi_n, f\xi_{n+1})) < d(f\xi_n, f\xi_{n+1}), \quad \forall\, n \in \mathbb{N}, \tag{3.12}$$

which is a contradiction. Thus, $d(f\xi_n, f\xi_{n+1}) < d(f\xi_{n-1}, f\xi_n)$, and hence

$$\begin{aligned} \tau + F(sd(f\xi_n, f\xi_{n+1})) &< F(M(f\xi_{n-1}, f\xi_n)) + LN(f\xi_{n-1}, f\xi_n) \\ &\le F(d(f\xi_{n-1}, f\xi_n)) + LN(f\xi_{n-1}, f\xi_n) \\ &= F(d(f\xi_{n-1}, f\xi_n)), \quad \forall\, n \in \mathbb{N} \end{aligned} \tag{3.13}$$

Putting $\psi_n = d(f\xi_n, f\xi_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Using (F4) and (3.13), we obtain that

$$\tau + F(s^n \psi_n) \le F(s^{n-1}\psi_{n-1}), \quad \forall\ n \in \mathbb{N}. \tag{3.14}$$

or

$$F(s^n \psi_n) \le F(s^{n-1}\psi_{n-1}) - \tau, \quad \forall\ n \in \mathbb{N}. \tag{3.15}$$

In general, one can get

$$F(s^n \psi_n) \le F(s^{n-1}\psi_{n-1}) - \tau \le \cdots \le F(\psi_0) - n\tau, \quad \forall\ n \in \mathbb{N}. \tag{3.16}$$

Thus,

$$\lim_{n\to\infty} F(s^n \psi_n) = -\infty \tag{3.17}$$

So, from (F2), we have

$$\lim_{n\to\infty} s^n \psi_n = 0. \tag{3.18}$$

Hence, using (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (s^n \psi_n)^k F(s^n \psi_n) = 0. \tag{3.19}$$

Using (3.16), we obtain

$$(s^n \psi_n)^k F(s^n \psi_n) - (s^n \psi_n)^k F(\psi_0) \le -n\tau(s^n \psi_n)^k < 0, \quad \forall\ n \in \mathbb{N}. \tag{3.20}$$

Letting limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} n(s^n \psi_n)^k = 0. \tag{3.21}$$

Using (3.21), there exists $n_1 \in \mathbb{N}$ such that $n(s^n \psi_n)^k \le 1$ for all $n \ge n_1$.

$$s^n \psi_n \le \frac{1}{n^{\frac{1}{k}}}, \quad \forall\ n \ge n_1. \tag{3.22}$$

Consequently, if $m > n > n_1$, then

$$\begin{aligned} d(f\xi_n, f\xi_m) &\le sd(f\xi_n, f\xi_{n+1}) + s^2 d(f\xi_{n+1}, f\xi_{n+2}) \\ &\quad + \cdots + s^{m-n-1} d(f\xi_{m-2}, f\xi_{m-1}) + s^{m-n-1} d(f\xi_{m-1}, f\xi_m) \\ &\le s\psi_n + s^2\psi_{n+1} + \cdots + s^{m-n-1}\psi_{m-2} + s^{m-n}\psi_{m-1} \\ &= \frac{1}{s^{n-1}}[s^n\psi_n + s^{n+1}\psi_{n+1} + \cdots + s^{m-2}\psi_{m-2} + s^{m-1}\psi_{m-1}] \\ &= \frac{1}{s^{n-1}}\sum_{j=1}^{m-1} s^j\psi_j < \frac{1}{s^{n-1}}\sum_{j=1}^{\infty} s^j\psi_j \le \frac{1}{s^{n-1}}\sum_{j=n}^{\infty}\frac{1}{j^{\frac{1}{k}}}. \end{aligned} \tag{3.23}$$

Since $k \in (0,1)$, the series $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}$ is convergent. Hence, $\{f\xi_n\}$ is a Cauchy sequence in $f(\Omega)$. From $f(\Omega)$ is complete, there exists $z \in f(\Omega)$ such that $\lim_{n\to\infty} f\xi_n = z = fu$ for some $u \in \Omega$.

If there exists a subsequence $\{f\xi_{n_k}\}$ of $\{f\xi_n\}$ such that $f\xi_{n_k} \in Pu$ for all $k \in \mathbb{N}$, then $\lim_{k\to\infty} f\xi_{n_k} = fu \in Pu$, and Pu is closed.

Finally, we will show that f and P have a point of coincidence in $f(\Omega)$. We assume that there exists $n_0 \in \mathbb{N}$ such that $f\xi_n \notin Pu$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Then $f\xi_{n+1} \notin Pu$ for all $n \ge n_0$ and so $P\xi_n \ne Pu$ for all $n \ge n_0$. Using property (i), there exists a subsequence $\{f\xi_{n_j}\}$ of $\{f\xi_n\}$ such that $\{f\xi_{n_j}, fu\} \in E(\tilde{G})$ for all $j \in \mathbb{N}$. It follows that $P\xi_{n_j} \ne Pu$ for all $j \ge n_0$. Using (3.30), we get

$$2\tau + F(sH(P\xi_{n_j}, Pu)) \le F(M(f\xi_{n_j}, fu)) + LN(f\xi_{n_j}, fu), \quad \forall\, j \ge n_0. \tag{3.24}$$

Then,

$$\begin{aligned} 2\tau + F(sd(f\xi_{n_j+1}, Pu)) &\le 2\tau + F(sH(P\xi_{n_j}, Pu)) \\ &\le F(M(f\xi_{n_j}, fu)) + LN(f\xi_{n_j}, fu), \quad \forall\, j \ge n_0, \end{aligned} \tag{3.25}$$

From (3.25), we get

$$F(sd(f\xi_{n_j+1}, Pu)) \le F(M(f\xi_{n_j}, fu)) + LN(f\xi_{n_j}, fu), \quad \forall\, j \ge n_0, \tag{3.26}$$

where

$$\begin{aligned} M(f\xi_{n_j}, fu) &= \max\left\{d(f\xi_{n_j}, fu), \frac{d(fu, Pu)[1 + d(f\xi_{n_j}, P\xi_{n_j})]}{1 + d(f\xi_{n_j}, fu)}\right\} \\ &\le \max\left\{d(f\xi_{n_j}, fu), \frac{d(fu, Pu)[1 + d(f\xi_{n_j}, f\xi_{n_j+1})]}{1 + d(f\xi_{n_j}, fu)}\right\} \end{aligned}$$

and

$$
\begin{aligned}
&N(f\xi_{n_j}, fu)\\
&= \min\left\{d(f\xi_{n_j}, fu), d(f\xi_{n_j}, P\xi_{n_j}), d(fu, Pu), d(f\xi_{n_j}, Pu), d(fu, P\xi_{n_j})\right\}\\
&\leq \min\left\{d(f\xi_{n_j}, fu), d(f\xi_{n_j}, f\xi_{n_j+1}), d(fu, Pu), d(f\xi_{n_j}, Pu), d(fu, f\xi_{n_j+1})\right\}.
\end{aligned}
$$

Letting limit as $j \to \infty$ in (3.26), we obtain

$$F(sd(fu, Pu)) \leq F(d(fu, Pu)) + L \cdot 0 \tag{3.27}$$

or

$$F(sd(fu, Pu)) \leq F(d(fu, Pu)), \tag{3.28}$$

which is a contradiction. Hence, $d(fu, Pu) = 0$. From Pu is closed, it follows that $z = fu \in Pu$, i.e., z is a point of coincidence of f and P. □

Corollary 3.1. *Let (Ω, d) be a complete b-metric space with $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Suppose that $P : \Omega \to CB(\Omega)$ is edge preserving and there exist a function $F \in \mathcal{F}$ which is continuous from right and $\tau > 0$ and $L \geq 0$ such that*

$$2\tau + F(sH(P\xi, P\zeta)) \leq F(M(\xi, \zeta)) + LN(\xi, \zeta), \tag{3.29}$$

for all $\xi, \zeta \in X$ with $P\xi \neq P\zeta$ and $(\xi, \zeta) \in E(\tilde{G})$ where

$$M(\xi, \zeta) = \max\left\{d(\xi, \zeta), \frac{d(\zeta, P\zeta)[1 + d(\xi, P\xi)]}{1 + d(\xi, \zeta)}\right\}$$

and

$$N(\xi, \zeta) = \min\left\{d(\xi, \zeta), d(\xi, P\xi), d(\zeta, P\zeta), d(\xi, P\zeta), d(\zeta, P\xi)\right\}.$$

Suppose also that the triple (Ω, d, G) has the following property:

(i) If $\{\xi_n\}$ is a sequence in Ω such that $\xi_n \to \xi$ and $(\xi_n, \xi_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ such that $(\xi_{n_j}, \xi) \in E(\tilde{G})$ for all $j \geq \mathbb{N}$.

(ii) If there exists $\xi_0 \in \Omega$ such that $(\xi_0, u) \in E(\tilde{G})$ for some $u \in P\xi_0$, then P has a fixed point in Ω.

Proof. Letting $f = I$, the identity map on Ω and follows proof from Theorem 3.1. □

Corollary 3.2. *Let (Ω, d) be a b-metric space with $s \geq 1$. Let $P : \Omega \to CB(\Omega)$ and $f : \Omega \to \Omega$ be such that $P(\Omega) \subseteq f(\Omega)$ and $f(\Omega)$ a complete subspace of Ω. Assume that there exist a function $F \in \mathcal{F}$ which is continuous from right $\tau > 0$ and $L \geq 0$ such that*

$$2\tau + F(sH(P\xi, P\zeta)) \leq F(M(\xi,\zeta)) + LN(\xi,\zeta), \tag{3.30}$$

where

$$M(\xi,\zeta) = \max\left\{d(\xi,\zeta), \frac{d(\zeta,P\zeta)[1+d(\xi,P\xi)]}{1+d(\xi,\zeta)}\right\}$$

and

$$N(\xi,\zeta) = \min\left\{d(\xi,\zeta), d(\xi,P\xi), d(\zeta,P\zeta), d(\xi,P\zeta), d(\zeta,P\xi)\right\},$$

for all $\xi, \zeta \in \Omega$ with $P\xi \neq P\zeta$. Then f and P have a point of coincidence in $f(\Omega)$.

Proof. Letting $G = G_0$, where G_0 is the complete graph $(\Omega, \Omega \times \Omega)$. and follows proof from Theorem 3.1. $\square$

Theorem 3.2. *Let (Ω, d) be a b-metric space with $s \geq 1$ and let $P : \Omega \to CB(\Omega)$ and $f : \Omega \to \Omega$ be a hybrid pair of mappings such that $P(\Omega) \subseteq f(\Omega)$ and $f(\Omega)$ a complete subspace of Ω. Assume that there exists $k \in (0,1)$ and $L \geq 0$ such that*

$$sH(P\xi, P\zeta) \leq kM(\xi,\zeta) + LN(\xi,\zeta), \tag{3.31}$$

where

$$M(\xi,\zeta) = \max\left\{d(\xi,\zeta), \frac{d(\zeta,P\zeta)[1+d(\xi,P\xi)]}{1+d(\xi,\zeta)}\right\}$$

and

$$N(\xi,\zeta) = \min\left\{d(\xi,\zeta), d(\xi,P\xi), d(\zeta,P\zeta), d(\xi,P\zeta), d(\zeta,P\xi)\right\},$$

for all $\xi, \zeta \in \Omega$. Then f and P have a point of coincidence in $f(\Omega)$.

Proof. Let $G = G_0 = (\Omega, \Omega \times \Omega)$, $L \geq 0$, and $\tau > 0$ be such that $k = e^{-2\tau}$. For $\xi, \zeta \in \Omega$ with $P\xi \neq P\zeta$, using (3.31), we obtain

$$F(sH(P\xi, P\zeta)) \leq -2\tau + F(M(f\xi, f\zeta)) + LN(\xi,\zeta),$$

which $2\tau + F(sH(P\xi, P\zeta)) \leq F(M(f\xi, f\zeta)) + LN(\xi,\zeta)$, where $F(\xi) = \ln \xi$. Thus, all the hypotheses of Theorem 3.1 hold true. $\square$

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