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ON A SUBCLASS OF ANALYTIC FUNCTIONS WITH FIXED FINITELY MANY COEFFICIENTS BASED ON SĂLĂGEAN OPERATOR AND MODIFIED SIGMOID FUNCTION

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ABSTRACT. A new family of analytic functions involving sigmoid function defined as $T_\gamma(\lambda,\beta,\alpha,\mu,c_m) \subset T_\gamma(\lambda,\beta,\alpha,\mu)$ are established. Certain geometric properties of the class are obtained.

1. INTRODUCTION

Let T_γ denote the class of functions of the form

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k;\ a_k \geq 0 \tag{1.1}$$

and

$$g_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) b_k z^k;\ b_k \geq 0, \tag{1.2}$$

which are analytic and univalent in the unit disk $U = \{z : z \leq 1\}$.

If for convenience, we set $T_\gamma = T_1$ we see that T_1 is the usual class of the form $f(z) = z - \sum_{k=2}^{\infty} a_k z^k;\ a_k \geq 0$ which is analytic in the open unit disk U.

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We define an identity function as

$$e_\gamma(z) = z\ ;\ a_k = 0\ for\ all\ k \geq\ 2\ but\ \gamma(s)\ \neq 0. \tag{1.3}$$

1.1. **Definition (Convolution or Hadamard Product):** Given two analytic functions $f_\gamma(z)$ and $g_\gamma(z)$ in T_γ where $f_\gamma(z)$ and $g_\gamma(z)$ are given by ((1.1)) and ((1.2)) respectively.

Convolution of $f_\gamma(z)$ and $g_\gamma(z)$ are defined as

$$(f_\gamma * g_\gamma) = z - \sum_{k=2}^{\infty} \gamma(s) a_k b_k z^k = (g_\gamma * f_\gamma);\ a_k b_k \geq 0. \tag{1.4}$$

2. Preliminary Results

Sălăgean Differential Operator Involving Modified Sigmoid Function

Define

$$D^n f_\gamma(z) = \gamma^n(s) z - \sum_{k=2}^{\infty} \gamma^m(s) k^n a_k z^k;\ m = n+1 \tag{2.1}$$

for details see [3].

Definition 2.1. *A function $f_\gamma \in T_\gamma$ defined by (1.1) belongs to the class $T_\gamma(\lambda, \beta, \alpha, \mu)$ if*

$$\left| \frac{\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} - \mu}{\left(\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} + \lambda\right) - 2\alpha\left(\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} - \mu\right)} \right| > \beta\ (z \in U),$$

for $|z| < 1,\ 0 < \lambda \leq 1, \gamma(s) = \frac{2}{1+e^{-s}}\ (i.e. \gamma \neq 0), 0 < \beta \leq 1, \frac{1}{2} \leq \alpha \leq 1,\ \mu \geq 1,$ $n \in N_0 = N \cup \{0\}$. See [3].

Lemma 2.1. *[3] Let $f_\gamma(z) \in T_\gamma$ defined as $f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k$ ($a_k \geq 0$ and $\gamma \neq 0$). If $f_\gamma \in T_\gamma(\lambda, \beta, \alpha, \mu)$, then*

$$\sum_{k=2}^{\infty} \gamma(s) k^n \left[\gamma(s) k [1 - \beta(1-2\alpha) - \beta] \left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$
$$< \gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu.$$

Now we introduce the class $T_\gamma(\lambda,\beta,\alpha,\mu,c_m)$ as a subclass of $T_\gamma(\lambda,\beta,\alpha,\mu)$ with fixed finitely many negative coefficients of the form:

$$f_\gamma(z) = z - \sum_{m=2}^{j} \frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-1]-\mu]c_m}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(2\alpha\mu+\frac{1}{\beta}\right)\right]} z^m - \sum_{k=j+1}^{\infty} a_k z^k.$$

These type of functions were also investigated by various authors. For examples, see [1], [4], [5], [6] and [7] for details. Other authors such as [2] and [3] also investigated some geometric properties of certain subclasses of univalent functions. In order to establish our results in this current work therefore, we use 2.1.

3. Main Results

Theorem 3.1. *Let $f_\gamma(z)$ defined by ((1.1)). Then, $f_\gamma(z)\in T_\gamma(\lambda,\beta,\alpha,\mu,c_m)$ if and only if*

$$\sum_{k=j+1}^{\infty} \frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu} < 1 - \sum_{m=2}^{j+1} c_m.$$

Proof. Let

$$a_m = \frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)m^n\left[\gamma(s)m[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}. \tag{3.1}$$

We say that $T_\gamma(\lambda,\beta,\alpha,\mu,c_m)\in T_\gamma(\lambda,\beta,\alpha,\mu)$, if, and only if,

$$\sum_{m=2}^{j+1} \frac{\gamma(s)m^n\left[\gamma(s)m[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu} a_m$$

$$+ \sum_{k=j+1}^{\infty} \frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu} a_k < 1.$$

So that

$$\sum_{m=2}^{j+1} c_m + \sum_{k=j+1}^{\infty} \frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu} a_k < 1.$$

Thus,

$$\sum_{k=j+1}^{\infty} \frac{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu} a_k < 1 - \sum_{m=2}^{j+1} c_m$$

as required. □

Corollary 3.1. *Let the function $f_\gamma(z)$ defined by ((1.1)) belongs to the class $T_\gamma(\lambda, \beta, \alpha, \mu, c_m)$, then*

$$a_k = \frac{[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]\left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}.$$

Equality holds for the function

$$f_\gamma(z) = z - \frac{[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]\left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k,$$

$|z| < 1$, $0 < \lambda \leq 1$, $\gamma(s) = \frac{2}{1+e^{-s}}$ $0 < \beta \leq 1$, $\frac{1}{2} \leq \alpha \leq 1$, $\mu \geq 1$, $n \in N_0 = N \cup \{0\}$.

Corollary 3.2. *If $f_\gamma(z) \in T_\gamma(1, \beta, \alpha, \mu, c_m)$, then*

$$\sum_{k=j+1}^{\infty} \gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(1 + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$

$$< [\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - 1] - \mu]\left(1 - \sum_{m=2}^{j+1} c_m\right)$$

which implies

$$a_k = \frac{[\gamma(s)(1-\beta) + \beta[1 + 2\alpha(\gamma(s)-\mu) - 1] - \mu]c_m}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(1 + 2\alpha\mu + \frac{1}{\beta}\right)\right]}.$$

The result is sharp with the extremal functions as

$$f_\gamma(z) = z - \frac{[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - 1] - \mu]\left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(1 + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k,$$

$|z| < 1$, $0 < \lambda \leq 1$, $\gamma(s) = \frac{2}{1+e^{-s}}$ $0 < \beta \leq 1$, $\frac{1}{2} \leq \alpha \leq 1$, $\mu \geq 1$, $n \in \mathbb{N}_{\not{\mathscr{V}}} = \mathbb{N} \cup \{0\}$.

Corollary 3.3. *If $f_\gamma(z) \in T_\gamma(\lambda, 1, \alpha, \mu, c_m)$, then*

$$\sum_{k=j+1}^{\infty} \gamma(s)k^n \left[2\alpha k\gamma(s) - (\lambda + 2\alpha\mu + 1)\right] |a_k|$$
$$< \{[2\alpha(\gamma(s) - \mu) - \lambda] - \mu\} \left(1 - \sum_{m=2}^{j+1} c_m\right),$$

which implies

$$a_k = \frac{\{[2\alpha(\gamma(s) - \mu) - \lambda] - \mu\} \left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[2\alpha k\gamma(s) - (\lambda + 2\alpha\mu + 1)\right]}.$$

The result is sharp with the extremal functions as

$$f_\gamma(z) = z - \frac{\{[2\alpha(\gamma(s) - \mu) - \lambda] - \mu\} \left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[2\alpha k\gamma(s) - (\lambda + 2\alpha\mu + 1)\right]} z^k,$$

$|z| < 1$, $0 < \lambda \leq 1$, $\gamma(s) = \frac{2}{1+e^{-s}}$ $0 < \beta \leq 1$, $\frac{1}{2} \leq \alpha \leq 1$, $\mu \geq 1$, $n \in \mathbb{N}_{\not\vdash} = \mathbb{N} \cup \{0\}$.

Corollary 3.4. *If $f_\gamma(z) \in T_\gamma(\lambda, \beta, \alpha, 1, c_m)$, then*

$$\sum_{k=j+1}^{\infty} \gamma(s)k^n \left[\gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta\left(\lambda + 2\alpha + \frac{1}{\beta}\right)\right] |a_k|$$
$$< [\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - 1) - \lambda] - 1] \left(1 - \sum_{m=2}^{j+1} c_m\right),$$

which implies

$$a_k = \frac{[\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - 1) - \lambda] - 1] \left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[\gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta\left(\lambda + 2\alpha + \frac{1}{\beta}\right)\right]}.$$

The result is sharp with the extremal functions as

$$f_\gamma(z) = z - \frac{[\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - 1) - \lambda] - 1] \left(1 - \sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n \left[\gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta\left(\lambda + 2\alpha + \frac{1}{\beta}\right)\right]} z^k,$$

$|z| < 1$, $0 < \lambda \leq 1$, $\gamma(s) = \frac{2}{1+e^{-s}}$ $0 < \beta \leq 1$, $\frac{1}{2} \leq \alpha \leq 1$, $\mu \geq 1$, $n \in \mathbb{N}_{\not\vdash} = \mathbb{N} \cup \{0\}$.

Corollary 3.5. *If* $f_\gamma(z) \in T_\gamma(1, 1, \alpha, 1, c_m)$*, then*

$$\sum_{k=j+1}^{\infty} \gamma(s)k^n(2\alpha k\gamma(s)) - 2(1+\alpha)|a_k| < 2[\alpha(\gamma(s)-1)-1]\left(1 - \sum_{m=2}^{j+1} c_m\right),$$

which implies

$$a_k = \frac{2[\alpha(\gamma(s)-1)-1]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n(2\alpha k\gamma(s)) - 2(1+\alpha)}.$$

The result is sharp with the extremal functions as

$$f_\gamma(z) = z - \frac{2[\alpha(\gamma(s)-1)-1]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n(2\alpha k\gamma(s)) - 2(1+\alpha)} z^k,$$

$|z| < 1,\ 0 < \lambda \le 1,\ \gamma(s) = \frac{2}{1+e^{-s}}\ 0 < \beta \le 1,\ \frac{1}{2} \le \alpha \le 1,\ \mu \ge 1,\ n \in \mathbb{N}_{\not\,} = \mathbb{N} \cup \{0\}$.

Corollary 3.6. *If* $f_\gamma(z) \in T_1(\lambda, \beta, \alpha, \mu, c_m)$*, then*

$$\sum_{k=j+1}^{\infty} k^n \left[k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$

$$< [(1-\beta) + \beta[2\alpha(1-\mu) - \lambda] - \mu]\left(1 - \sum_{m=2}^{j+1} c_3\right),$$

which implies

$$a_k = \frac{[(1-\beta) + \beta[2\alpha(1-\mu) - \lambda] - \mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{k^n\left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}.$$

The result is sharp with the extremal functions as

$$f_\gamma(z) = z - \frac{[(1-\beta) + \beta[2\alpha(1-\mu) - \lambda] - \mu]\left(1-\sum_{m=2}^{j+1} c_3\right)}{k^n\left[k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k,$$

$|z| < 1,\ 0 < \lambda \le 1,\ \gamma(s) = \frac{2}{1+e^{-s}}\ 0 < \beta \le 1,\ \frac{1}{2} \le \alpha \le 1,\ \mu \ge 1,\ n \in \mathbb{N}_{\not\,} = \mathbb{N} \cup \{0\}$.

Corollary 3.7. *If* $f_\gamma(z) \in T_\gamma(\lambda, \beta, \alpha, \mu, c_3)$*, then*

$$\sum_{k=j+1}^{\infty} \gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$

$$< [\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_3\right),$$

which implies

$$a_k=\frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}.$$

Theorem 3.2. *Let $f_\gamma(z)$ be defined by ((1.1)) be in the class $T_\gamma(\lambda,\beta,\alpha,\mu,c_m)$. Then $f_\gamma(z)$ is starlike of order δ $(0\le\delta<1)$ in $|z|<R_1$, where R_1 is the largest value such that*

$$\sum_{m=2}^{j+1}\frac{(m-2-\delta)\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_1^{m-1}$$
$$-\sum_{k=j+1}^{\infty}\frac{(k-2-\delta)\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_1^{k-1}$$
$$<\frac{1}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}.$$

Proof. Let

(3.2)
$$f_\gamma(z)=z-\sum_{m=2}^{j+1}\frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^m$$
$$-\sum_{k=j+1}^{\infty}a_kz^k.$$

We need to show that $Re\left(\frac{zf_\gamma'(z)}{f_\gamma(z)}\right)>\delta$, implying

$$\left|\frac{zf_\gamma'(z)}{f_\gamma(z)}-1\right|<1-\delta. \tag{3.3}$$

Consider the analytic function $f_\gamma(z)$ with finitely many coefficient defined as ((3.2)),

$$zf'_\gamma(z) = z\Bigg(1 - \sum_{m=2}^{j+1} \frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1} - \sum_{k=j+1}^{\infty} k a_k z^{k-1}\Bigg). \tag{3.4}$$

So,

$$\left|\frac{zf'_\gamma(z)}{f_\gamma(z)} - 1\right| = \left|\frac{z - \sum_{m=2}^{j+1} \frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^m - \sum_{k=j+1}^{\infty} k a_k z^k}{z - \sum_{m=2}^{j+1} \frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^m - \sum_{k=j+1}^{\infty} a_k z^k} - 1\right|.$$

By collecting the like terms and dividing each term of the numerator and the denominator by $|z|$, we have

$$\left|\frac{\sum_{m=2}^{j+1} \frac{(m-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1} - \sum_{k=j+1}^{\infty}(k-1) a_k z^{k-1}}{\sum_{m=2}^{j} \frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1} - \sum_{k=j+1}^{\infty} a_k z^{k-1}}\right| < 1-\delta.$$

But $|z| < R_1$. Thus, by substituting a_k in 3.1 in the relation above, we have by using equation ((3.3)) that,

$$\frac{\sum_{m=2}^{j+1} \frac{(m-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{m-1}}{1 - \sum_{m=2}^{j+1} \frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{m-1}}$$

$$\frac{-\sum_{k=j+1}^{\infty} \frac{(k-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{k-1}}{-\sum_{k=j+1}^{\infty} \frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{k-1}} < 1-\delta.$$

Thus,

$$\sum_{m=2}^{j+1} \frac{(m-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{m-1}$$

$$-\sum_{k=j+1}^{\infty} \frac{(k-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{k-1}$$

$$<\sum_{m=2}^{j+1} \frac{(1-\delta)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{m-1}$$

$$-\sum_{k=j+1}^{\infty} \frac{(1-\delta)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{k-1}.$$

By collecting the like terms and dividing through by $[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]$,

$$\sum_{m=2}^{j+1} \frac{[(m-1)-(1-\delta)][\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{m-1}$$

$$-\sum_{k=j+1}^{\infty} \frac{[(k-1)-(1-\delta)][\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{k-1}$$

$$<1.$$

That is,

$$\sum_{m=2}^{j+1} \frac{(m-2-\delta)\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{m-1}$$

$$-\sum_{k=j+1}^{\infty} \frac{(k-2-\delta)\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_1^{k-1}$$

$$<\frac{1}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}.$$

□

Theorem 3.3. *Let $f_\gamma \in T_\gamma(\lambda, \beta, \alpha, \mu, c_m)$. Then $f_\gamma(z)$ is convex of order δ $(0 \leq \delta < 1)$ in $|z| < R_2$ is the largest value such that*

$$\sum_{m=2}^{j+1} \frac{m[(1-\delta)-m]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_2^{m-1}$$
$$-\sum_{k=j+1}^{\infty} \frac{k[(1-\delta)-k]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} R_2^{k-1}$$
$$< \frac{\delta}{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]}.$$

Proof. Let $g_\gamma(z) = f'_\gamma(z)$ given by ((3.2)). It suffices to show that

$$\frac{zg'_\gamma(z)}{g_\gamma(z)} = 1 + \frac{zf''_\gamma(z)}{f'_\gamma(z)}.$$

So,

$$1 + \frac{zf''_\gamma(z)}{f'_\gamma(z)} = 1 -$$
$$\frac{\sum_{m=2}^{j+1} \frac{m(m-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1} + \sum_{k=j+1}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{m=2}^{j+1} \frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1} - \sum_{k=j+1}^{\infty} k a_k z^{k-1}}$$

using a_k given by Theorem 3.1 yields

$$1 - \frac{\sum_{m=2}^{j+1} \frac{m(m-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1}}{1-\sum_{m=2}^{j+1} \frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{m-1}}$$
$$\frac{+\sum_{k=j+1}^{\infty} \frac{k(k-1)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{k-1}}{-\sum_{k=j+1}^{\infty} \frac{k[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1} c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} z^{k-1}}.$$

$$=\frac{1-\sum_{m=2}^{j+1}\frac{m[1+(m-1)][\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{m-1}}{1-\sum_{m=2}^{j+1}\frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{m-1}}$$

$$\frac{-\sum_{k=j+1}^{\infty}\frac{k[1+(k-1)][\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{k-1}}{-\sum_{k=j+1}^{\infty}\frac{k[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{k-1}}$$

$$=\frac{1-\sum_{m=2}^{j+1}\frac{m^2[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{m-1}}{1-\sum_{m=2}^{j+1}\frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{m-1}}$$

$$\frac{-\sum_{k=j+1}^{\infty}\frac{k^2[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{k-1}}{-\sum_{k=j+1}^{\infty}\frac{k[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}z^{k-1}}.$$

But,

$$\text{Re}\left(1+\frac{zf_\gamma''(z)}{f_\gamma'(z)}\right)>\delta \quad \text{implying} \quad \left|\frac{zf_\gamma''(z)}{f_\gamma'(z)}\right|<1-\delta. \tag{3.5}$$

Hence,

$$\left|\frac{zg_\gamma'(z)}{g_\gamma(z)}-1\right|$$
$$=\left|\frac{1-\sum_{m=2}^{j+1}\Phi z^{m-1}-\sum_{k=j+1}^{\infty}\Psi z^{k-1}}{1-\sum_{m=2}^{j+1}\Pi z^{m-1}-\sum_{k=j+1}^{\infty}\Omega z^{k-1}}\right|$$
$$=\frac{1-\sum_{m=2}^{j+1}\Phi|z|^{m-1}-\sum_{k=j+1}^{\infty}\Psi|z|^{k-1}}{1-\sum_{m=2}^{j+1}\Pi|z|^{m-1}-\sum_{k=j+1}^{\infty}\Omega|z|^{k-1}}$$

where

$$\Phi = \frac{m^2[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]},$$

$$\Psi = \frac{k^2[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]},$$

$$\Pi = \frac{m[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]},$$

$$\Omega = \frac{k[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}.$$

From the above values of Φ, Ψ, Π and Ω and the fact that $|z| < R_2$, then

$$\begin{aligned}
1 &- \sum_{m=2}^{j+1}\frac{m^2[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_2^{m-1} \\
&- \sum_{k=j+1}^{\infty}\frac{k^2[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_2^{k-1} \\
< (1-\delta) &- \sum_{m=2}^{j+1}\frac{m(1-\delta)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_2^{m-1} \\
&- \sum_{k=j+1}^{\infty}\frac{k(1-\delta)[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_2^{k-1}.
\end{aligned}$$

Collecting like terms and dividing through by $[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]$, we have

$$1-\sum_{m=2}^{j+1}\frac{m[(1-\delta)-m]\left(1-\sum_{k=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_2^{m-1}$$
$$-\sum_{k=j+1}^{\infty}\frac{k[(1-\delta)-k]\left(1-\sum_{m=2}^{j+1}c_m\right)}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}R_2^{k-1}$$
$$<\frac{\delta}{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]},$$

which completes the proof. □

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REFERENCES

[1] M. ACU Ć, SH. NAJAFZADEH Ć: *Univalent functions with fixed finitely many coefficients involving Sălăgean operator*, Int. J. Nonlinear Anal. Appl. **1** (2010), 1-5.

[2] S. ALTINKAYA Ć, S. Y̌ALCINĆ: *On the Chebyshev polynomial bounds for classes of univalent functions*, Khayyam Journal of Mathematics **2**(1) (2016), 1–5.

[3] O. A. FADIPE-JOSEPHĆ, B. O. M̌OSESĆ, M. O. ǑLUWAYEMIĆ: *Certain New Classes of analytic functions defined by using sigmoid function*, Advances in Mathematics: Scientific Journal, **5** (2016), 83–89.

[4] A. R. S. JUMAĆ, S. R. ǨULKARNIĆ: *Applications of Generalised Ruscheweyh derivatives to univalent functions with finitely many coefficients*, Surveys in Mathematics and its Applications, **4** (2009), 77–88.

[5] SH. NAJAFZADEH Ć: *Application of Sălăgean and Ruscheweyh operators on univalent functions with finitely many coefficients*, Fractional Calculus & Applied Analysis, **13**(5) (2010), 517–520.

[6] S. S. VARMAĆ, T. ŘOSY Ć: *Certain properties of a subclass of univalent functions with finitely many fixed coefficients*, Khayyam Journal Math., **1**(3) (2017), 25–32.

[7] K. V. VIDYASAGARĆ: *Geometric properties of some class of univalent functions by fixing finitely many coefficients*, International Journal of Innovative Science, Engineering and Technology, **1**(7) (2019), 49–56.

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