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THE (CLR)-PROPERTY AND COMMON FIXED POINT THEOREMS IN b-METRIC SPACES

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ABSTRACT. In this paper, we prove coincidence point for two pair of mappings satisfying (CLR)-property in b-metric spaces. Moreover, we also attain unique common fixed point for two weakly compatible pairs.

1. INTRODUCTION

In 1989, Bakhtin [2] proposed the concept of b-metric spaces, which was used to prove the generalization of Banach's principle of contraction in this spaces. Since then, many authors have used the concept to create different fixed point theorems. Roshan et al. [3] used the concept of contract mapping generally in a b-metric space, established some fixed and common fixed point results. After that, several interesting results were obtained regarding the existence of the point fixed for operators with single and multiple values in the b-metric space.

In 2011, Sintunavarat et al. [6] introduced the notion of (CLR)-property in the context of metric space. Later, some authors employed this concept to obtain some new fixed point results ([7–12]).

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Motivation by above, in this paper, we prove a common fixed point theorem for two pairs of mappings which satisfy the (CLR_P) and (CLR_Q)-property in b-metric spaces.

2. Preliminaries

Definition 2.1. [2] *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $u, v, w \in X$ the following conditions are satisfied:*

- (b1) $d(u, v) = 0$ *if and only if* $u = v$;
- (b2) $d(u, v) = d(v, u)$;
- (b3) $d(u, w) \leq s[d(u, v) + d(v, w)]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, every metric is a b-metric with $s = 1$.

Definition 2.2. [1] *Let $\{u_n\}$ be a sequence in a b-metric space (X, d).*

- (i) *$\{u_n\}$ is called b-convergent if and only if there is $u \in X$ such that $d(u_n, u) \to 0$ as $n \to \infty$.*
- (ii) *$\{u_n\}$ is a b-Cauchy sequence if and only if $d(u_n, u_m) \to 0$ as $n, m \to \infty$.*

A b-metric space is said to be complete if and only if each b-Cauchy sequence in this space is b-convergent.

Proposition 2.1. [1] *In a b-metric space (X, d) ; the following assertions hold:*

- (i) *A b-convergent sequence has a unique limit.*
- (ii) *Each b-convergent sequence is b-Cauchy.*
- (iii) *In general, a b-metric is not continuous.*

Definition 2.3. [4] *Let (X, d) be a b-metric space. A subset $Y \subset X$ is called closed if and only if for each sequence $\{u_n\}$ in Y which b-converges to an element u, we have $u \in Y$.*

Definition 2.4. [5] *g and h be given selfmappings on a set X. The pair (g, h) is said to be weakly compatible if g and h commute at their coincidence points (i.e. $ghu = hgu$ whenever $gu = hu$).*

Definition 2.5. [6] *Two self-mappings g and h of a metric space (X, d) are said to satisfy (CLR$_h$)-property if there exists a sequence $\{u_n\}$ in X such that*

$$\lim_{n\to\infty} gu_n = \lim_{n\to\infty} hu_n = gu,$$

for some u in X.

3. Main result

During this paper, we assume the control functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ are continuous, nondecreasing functions with $\psi(t) = 0$ if and only if $t = 0$.

Theorem 3.1. *Let (X, d) be a b-metric space and $g, h, P, Q : X \to X$ be mappings with $g(X) \subseteq Q(X)$ and $h(X) \subseteq P(X)$ such that for all $u, v \in X$,*

$$\psi(s^2 d(gu, hv)) \leq \psi(M_s(u, v)) - \varphi(M_s(u, v)), \tag{3.1}$$

where,

$$M_s(u, v) = \max\left\{d(Pu, Qv), \frac{d(gu, Pu)}{1 + d(gu, Pu)}, \frac{d(hv, Qv)}{1 + d(hv, Qv)}, \frac{d(gu, Qv) + d(Pu, hv)}{2s}\right\}.$$

Suppose that the pairs (g, P) and (h, Q) satisfy the (CLR$_P$) and (CLR$_Q$)-properties respectively and that one of the subspaces $g(X), h(X), P(X)$ and $Q(X)$ is closed in X, then the pairs (g, P) and (h, Q) have a coincidence point in X. Moreover, if the pairs (g, P) and (h, Q) are weakly compatible then g, h, P, Q have a unique common fixed point.

Proof. If the pair (g, P) satisfy the (CLR$_P$) property, then there exists a sequence $\{u_n\}$ in X such that

$$\lim_{n\to\infty} gu_n = \lim_{n\to\infty} Pu_n = u,$$

for some $u \in P(X)$. As $g(X) \subseteq Q(X)$ there exists a sequence $\{v_n\}$ in X such that $gu_n = Qv_n$. Hence $\lim_{n\to\infty} Qv_n = u$. Let us show that $\lim_{n\to\infty} hv_n = u$. Letting $u = u_n$ and $v = v_n$ in (3.1), we obtain

$$\begin{aligned} \psi(d(gu_n, hv_n)) \leq \psi(s^2 d(gu_n, hv_n)) &\leq \psi(M_s(u_n, v_n)) - \varphi(M_s(u_n, v_n)) \\ &\leq \psi(M_s(u_n, v_n)), \end{aligned} \tag{3.2}$$

where

$$M_s(u_n, v_n) = \max\left\{d(Pu_n, Qv_n), \frac{d(gu_n, Pu_n)}{1 + d(gu_n, Pu_n)}, \frac{d(hv_n, Qv_n)}{1 + d(hv_n, Qv_n)}, \right.$$
$$\left.\frac{d(gu_n, Qv_n) + d(Pu_n, hv_n)}{2s}\right\}$$
$$= \max\left\{d(Pu_n, gu_n), \frac{d(gu_n, Pu_n)}{1 + d(gu_n, Pu_n)}, \frac{d(hv_n, gu_n)}{1 + d(hv_n, gu_n)}, \right.$$
$$\left.\frac{d(gu_n, gu_n) + d(Pu_n, hv_n)}{2s}\right\}$$
$$\leq \max\left\{d(Pu_n, gu_n), d(gu_n, Pu_n), d(hv_n, gu_n), \frac{d(Pu_n, hv_n)}{2s}\right\}$$
$$\leq \max\left\{d(Pu_n, gu_n), d(gu_n, Pu_n), d(hv_n, gu_n), \frac{s[d(Pu_n, gu_n) + d(gu_n, hv_n)]}{2s}\right\}$$
$$= \max\left\{d(Pu_n, gu_n), d(gu_n, Pu_n), d(hv_n, gu_n), \frac{d(Pu_n, gu_n) + d(gu_n, hv_n)}{2}\right\}.$$

In (3.2)), taking $n \to \infty$, we obtain

$$\psi(s^2 \lim_{n\to\infty} d(u, hv_n)) \leq \psi(\lim_{n\to\infty} d(u, hv_n)).$$

Using the definition of ψ,

$$s^2 \lim_{n\to\infty} d(u, hv_n) \leq \lim_{n\to\infty} d(u, hv_n).$$

Hence, $\lim_{n\to\infty} d(u, hv_n) = 0$. Hence $\lim_{n\to\infty} hv_n = u$.

Since the pairs (g, P) and (h, Q) satisfying the (CLR$_{PQ}$)-property, so there exists sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} gu_n = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty} hv_n = \lim_{n\to\infty} Qv_n = u,$$

where $u \in P(X) \cap Q(X)$, If $P(X)$ is closed subspace of X, then there exists a $t \in X$, such that $Pt = u$. Next, we will show that $gt = Pt$. Letting $u = t$ and $v = v_n$ in (3.1), we obtain

$$\psi(s^2 d(gt, hv_n)) \leq \psi(M_s(t, v_n)) - \varphi(M_s(t, v_n)), \tag{3.3}$$

where

$$\begin{aligned} &M_s(t,v_n) \\ &= \max\left\{d(Pt,Qv_n), \frac{d(gt,Pt)}{1+d(gt,Pt)}, \frac{d(hv_n,Qv_n)}{1+d(hv_n,Qv_n)}, \frac{d(gt,Qv_n)+d(Pt,hv_n)}{2s}\right\} \\ &= \max\left\{d(u,Qv_n), \frac{d(gt,u)}{1+d(gt,u)}, \frac{d(hv_n,Qv_n)}{1+d(hv_n,Qv_n)}, \frac{d(gt,Qv_n)+d(u,hv_n)}{2s}\right\} \\ &\le \max\left\{d(u,Qv_n), d(gt,u), d(hv_n,Qv_n), \frac{d(gt,Qv_n)+d(u,hv_n)}{2s}\right\}. \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} M_s(t,v_n) &\le \max\left\{d(u,u), d(gt,u), d(u,u), \frac{d(gt,u)+d(u,u)}{2s}\right\} \\ &= d(gt,u). \end{aligned}$$

Now, by (3.3) and the definitions of ψ and φ, as $n \to \infty$,

$$\psi(d(gt,u)) \le \psi(s^2 d(gt,u)) \le \psi(d(gt,u)) - \varphi(d(gt,u)),$$

which implies $\varphi(d(gt,u)) \le 0$ giving $gt = u = Pt$. Hence, t is a coincidence point of the pair (g,P). As $u \in Q(X)$, there exists a point $r \in X$ such that $Qr = u$. We claim that $hr = Qr$. Letting $u = t$ and $v = r$ in (3.1), we have

$$\psi(s^2 d(gt,hr)) \le \psi(M_s(t,r)) - \varphi(M_s(t,r)), \tag{3.4}$$

where

$$\begin{aligned} &M_s(t,r) \\ &= \max\left\{d(Pt,Qr), \frac{d(gt,Pt)}{1+d(gt,Pt)}, \frac{d(hr,Qr)}{1+d(hr,Qr)}, \frac{d(gt,Qr)+d(Pt,hr)}{2s}\right\} \\ &\le \max\left\{d(Pt,Qr), d(gt,Pt), d(hr,Qr), \frac{d(gt,Qr)+d(Pt,hr)}{2s}\right\} \\ &= \max\left\{d(u,u), d(u,u), d(hr,u), \frac{d(u,u)+d(u,hr)}{2s}\right\} \\ &= d(hr,u). \end{aligned}$$

From (3.4), we obtain

$$\psi(s^2 d(gt,hr)) = \psi(s^2 d(u,hr)) \le \psi(d(u,hr)) - \varphi(d(u,hr))$$

implies that $\varphi(d(u,hr)) \le 0$. Hence, $hr = u = Qr$. Hence, r is a coincidence point of the pair (h,Q). Since, the pair $(g,P)$is weakly compatible, then $gu =$

Pu. We will show that u is a common fixed point of g and P. Letting $x = u,\ y = r$ in (3.1), we obtain

$$\psi(s^2 d(gu, hr)) \leq \psi(M_s(u, r)) - \varphi(M_s(u, r)), \tag{3.5}$$

where

$$\begin{aligned} &M_s(u,r)\\ &= \max\left\{d(Pu, Qr), \frac{d(gu, Pu)}{1 + d(gu, Pu)}, \frac{d(hr, Qr)}{1 + d(hr, Qr)}, \frac{d(gu, Qr) + d(Pu, hr)}{2s}\right\}\\ &= \max\left\{d(gu, u), \frac{d(gu, gu)}{1 + d(gu, gu)}, \frac{d(u, u)}{1 + d(u, u)}, \frac{d(gu, u) + d(gu, u)}{2s}\right\}\\ &= \max\left\{d(gu, u), \frac{d(gu, u)}{2s}\right\}\\ &= d(gu, u). \end{aligned}$$

From (3.5)

$$\psi(s^2 d(gu, hr)) = \psi(s^2 d(gu, u)) \leq \psi(d(gu, u)) - \varphi(d(gu, u))$$

implies that $\varphi(d(gu, u)) \leq 0$. So $gu = u = Pu$. Hence, u is a common fixed point of the pair (g, P). Similarly, it can be shown $hu = u = Qu$. Hence, u is a common fixed point of g, h, P and Q.

Finally, we prove the uniqueness of fixed point, suppose u^* is another fixed point of g, h, P and Q. Letting $x = u,\ y = u^*$ in (3.1), we obtain

$$\psi(d(u, u^*)) = \psi(d(gu, hu^*)) \leq \psi(s^2 d(gu, hu^*)) \leq \psi(M_s(u, u^*)) - \varphi(M_s(u, u^*))$$

and

$$\begin{aligned} &M(u, u^*)\\ &= \max\left\{d(Pu, Qu^*), \frac{d(gu, Pu)}{1 + d(gu, Pu)}, \frac{d(hu^*, Qu^*)}{1 + d(hu^*, Qu^*)}, \frac{d(gu, Qu^*) + d(Pu, hu^*)}{2s}\right\}\\ &= \max\left\{d(u, u^*), \frac{d(u, u)}{1 + d(u, u)}, \frac{d(u^*, u^*)}{1 + d(u^*, u^*)}, \frac{d(u, u^*) + d(u, u^*)}{2s}\right\}\\ &= \max\left\{d(u, u^*), \frac{d(u, u^*)}{s}\right\}\\ &= d(u, u^*). \end{aligned}$$

Then, we have

$$\psi(d(u, u^*)) \leq \psi(d(u, u^*)) - \varphi(d(u, u^*)),$$

which implies that $\varphi(d(u,u^*)) = 0$. So $u = u^*$. Hence, u is the unique common fixed point of g, h, P and Q. □

Corollary 3.1. *Let (X,d) be a b-metric space and $g, Q : X \to X$ be mappings with $g(X) \subseteq Q(X)$ such that for all $u, v \in X$,*

$$\psi(s^2 d(gu, gv)) \leq \psi(M_s(u,v)) - \varphi(M_s(u,v)),$$

where

$$M_s(x,y) = \max\left\{d(Qu,Qv), \frac{d(gu,Qu)}{1+d(gu,Qu)}, \frac{d(gv,Qv)}{1+d(gv,Qv)}, \frac{d(gu,Qv)+d(Qu,gv)}{2s}\right\}.$$

Suppose that the pair (g,Q) satisfies the (CLR)-property and $Q(X)$ is closed in X. Then the pair (g,Q) has a common point of coincidence in X. Moreover, if the pair (g,Q) is weakly compatible, then g and Q have a unique common fixed point.

Proof. By letting $h = g$ and $P = Q$ in Theorem 3.1, the result can be proved. □

Corollary 3.2. *Let (X,d) be a b-metric space and $f, g, S, T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that for all $u, v \in X$,*

$$\psi(d(gu, hv)) \leq \psi(M_s(u,v)) - \varphi(M_s(u,v)),$$

where

$$M_s(u,v) = \max\left\{d(Pu,Qv), \frac{d(gu,Pu)}{1+d(gu,Pu)}, \frac{d(hv,Qv)}{1+d(hv,Qv)}, \frac{d(gu,Qv)+d(Pu,hv)}{2s}\right\}.$$

Suppose that the pairs (g,P) and (h,Q) satisfy the (CLR$_P$) and (CLR$_Q$)-properties respectively and that one of the subspaces $g(X), h(X), P(X)$ and $Q(X)$ is closed in X. Then the pairs (g,P) and (h,Q) have a point of coincidence in X. Moreover, if the pairs (g,P) and (h,Q) are weakly compatible, then g, h, p and Q have a unique common fixed point.

4. Conclusion

In this work it was proved using (CLR)-property for four self-maps in the framework of b-metric spaces. Furthermore, as perspectives, we suggest extending more results to find (common) fixed point results in generalized metric spaces by using the aid of altering and ultra altering distance mapping.

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