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FIXED POINT THEOREMS IN *MR*-METRIC SPACE THROUGH SEMI-COMPATIBILITY

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ABSTRACT. In this paper, we interpret the concept of $MR-$semi-compatible maps in $MR-$metric spaces and in the view of orbital concept we deduce some fixed point theorems through $MR-$semi-compatibly for the pair (U,V) of self-mappings on the set $\mathbb{X}$ under a set of conditions.

1. INTRODUCTION

In 2021, the concept of $MR-$metric is defined by A. Malkawi et. al [10] which is a generalization of a $D-$metric space. Dhage [3] presented the concept of a $D-$metric space which is introduced and proved the existence a unique fixed point for a self-mapping satisfied a contractive condition.

Latter, Cho et. al [2] initiated the notion of semi-compatible maps in d-topological spaces.

Definition 1.1. *[2] A pair of self-maps (U,V) to be semi-compatible if the following two conditions are satisfied.*

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(1) $U\eta = V\eta$ *implies* $UV\eta = VU\eta$*;*
(2) $U\zeta_n \to \zeta$ *and* $V\zeta_n \to \zeta$ *implies* $UV\zeta_n \to V\zeta$*, as* $n \to \infty$.

In the above definition, note that (2) gives (1), set $\zeta_n = \eta$ and $\zeta = V\eta = U\eta$. Thus, by condition (2), we define the $MR-$semi-compatibly of the pair (U, V) in an $MR-$ metric space.

On the other hand, we devise the definition of an $MR-$semi-compatible pair of self-mappings in an $MR-$metric space and introduce its relationship with an $MR-$compatible pair of self-maps with an example.

Additionally, if V is continuous, then (U, V) in $MR-$compatible implies (U, V) is MR-semi compatible. Therefore, the semi-compatibility of the pair (U, V) does not imply its $MR-$compatibility, even if V is continuous see Example 1.

2. Preliminaries

In every part of this paper $\mathbb{N}$ stands for all natural numbers and $(\mathbb{X}, M)$ denote an $MR-$metric space.

In 1994, Dhage [6] defined a generalization of metric space it is called a $D - metric$ space.

Definition 2.1. *[3] Let* $\mathbb{X} \neq \phi$ *be a set. A function* $D : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ *is called a* $D - metric$*, if the following properties are satisfied for each* $\zeta, \eta, \xi \in \mathbb{X}$.

$(D1) : D(\zeta, \eta, \xi) \geq 0$.
$(D2) : D(\zeta, \eta, \xi) = 0$ *iff* $\zeta = \eta = \xi$.
$(D3) : D(\zeta, \eta, \xi) = D(p(\zeta, \eta, \xi))$; *for any permutation* $p(\zeta, \eta, \xi)$ *of* ζ, η, ξ.
$(D4) : D(\zeta, \eta, \xi) \leq D(\zeta, \eta, \ell) + D(\zeta, \ell, \xi) + D(\ell, \eta, \xi)$.
A pair $(\mathbb{X}, D)$ *is called a* $D - metric$ *space.*

The following definition is an $MR-$metric space.

Definition 2.2. *[10] Let* $\mathbb{X} \neq \phi$ *be a set and* $R > 1$ *be a real number. A function* $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ *is called an* $MR - metric$*, if it satisfies the following properties for each* $\zeta, \eta, \xi \in \mathbb{X}$.

$(M1) : M(\zeta, \eta, \xi) \geq 0$.
$(M2) : M(\zeta, \eta, \xi) = 0$ *iff* $\zeta = \eta = \xi$.
$(M3) : M(\zeta, \eta, \xi) = M(p(\zeta, \eta, \xi))$; *for any permutation* $p(\zeta, \eta, \xi)$ *of* ζ, η, ξ.
$(M4) : M(\zeta, \eta, \xi) \leq R\left[M(\zeta, \eta, \ell_1) + M(\zeta, \ell_1, \xi) + M(\ell_1, \eta, \xi)\right]$.

A pair $(\mathbb{X}, M)$ *is called an* $MR - metric$ *space.*

In the following, we present two definitions of $MR - convergent$ and $MR - Cauchy$ defined by Malkawi et. al [10].

Definition 2.3. *[10] A sequence* $\{\zeta_{1_n}\}$ *in an* $MR - metric$ *space* $(\mathbb{X}, M)$ *is called* $MR - convergent$ *if there exists* ζ_1 *in* $\mathbb{X}$ *such that for* $\epsilon > 0$*, there exists a* $N > 0$ *integer number such that* $M(\zeta_{1_n}, \zeta_{1_m}, \zeta_1) < \epsilon$ *for all* $m \geq N$*,* $n \geq N$*. So we called* $\{\zeta_{1_n}\}$ $MR - convergent$ *to* ζ_1 *and* ζ_1 *is a limit of* $\{\zeta_{1_n}\}$ *.*

Definition 2.4. *[10] A sequence* $\{\zeta_{1_n}\}$ *is a sequence in* $MR - metric$ *space* $(\mathbb{X}, M)$ *is called* $MR - Cauchy$ *if for a given* $\epsilon > 0$*, there exists a positive integer* N *such that* $M(\zeta_{1_n}, \zeta_{1_m}, \zeta_{1_p}) < \epsilon$ *for all* $m, n, p \geq N$*.*

Definition 2.5. *Let* $(\mathbb{X}, M)$ *be an* $MR - metric$ *space and* $\phi \neq U \subseteq \mathbb{X}$*. We define the diameter of* U *as:*

$$\delta_M(U) = Sup\{M(\zeta, \eta, \xi) : \zeta, \eta, \xi \in U\}.$$

Definition 2.6. *Let* V *be a multi-valued map on* $MR-$*metric space* $(\mathbb{X}, M)$*. Let* $\zeta_0 \in \mathbb{X}$*.* ζ_n *be a sequence in* $\mathbb{X}$ *called be an orbit of* V *at* ζ_0 *denoted by* $O(V, \zeta_0)$ *if* $\zeta_{n-1} \in V^{n-1}(\zeta_0)$*; That is* $\zeta_n \in V\zeta_{n-1}$ *for all* $n \in \mathbb{N}$*.*

An orbit $O(V, \zeta_0)$ *is said to be* $MR-$*bounded if its diameter is finite. It is said to be complete if every* $MR-$*Cauchy sequence in it* $MR-$*converges to a point of* $\mathbb{X}$*.*

Definition 2.7. *A pair* (U, V) *of self-maps on* $MR - metric$ *space* $(\mathbb{X}, M)$ *is called* $MR-$*compatible if for all* ζ, η *and* $\xi \in \mathbb{X}$ *for some* $\beta \in (0, \infty)$

$$M(UV\zeta, UV\eta, VU\xi) \leq \beta M(V\zeta, V\eta, U\xi) \tag{2.1}$$

Definition 2.8. *A pair* (U, V) *of self-maps on* $MR - metric$ *space* $(\mathbb{X}, M)$ *is called an* $MR-$ *semi-compatible if* $\lim_{n\to\infty} UV\zeta_n = V\zeta$*, such that* ζ_n *is a sequence in* $\mathbb{X}$ *such that* $\lim_{n\to\infty} V\zeta_n = \lim_{n\to\infty} U\zeta_n = \zeta$*; That is, A pair* (U, V) *of self-maps on* $MR - metric$ *space is called* $MR-$ *semi-compatible if* $\lim_{n\to\infty} M(U\zeta_n, U\zeta_{n+p}, \zeta) = 0$ *and* $\lim_{n\to\infty} M(V\zeta_n, V\zeta_{n+p}, \zeta) = 0$ *imply* $\lim_{n\to\infty} M(UV\zeta_n, UV\zeta_{n+p}, V\zeta) = 0$*.*

Definition 2.9. *Let* $(\mathbb{X}, M)$ *be an* $MR-$*metric space and* U *and* V *be two self - mappings of* $\mathbb{X}$ *and* $\{\zeta_n\}$ *be a sequence in* $\mathbb{X}$ *such that* $U\zeta_{n-1} = V\eta_n$ *for* $n \in \mathbb{N}$*. Then we define* $O_U(V\zeta_n) = \{V\zeta_n : p \geq n\}$ *for* $n \in \mathbb{N}$ *,where p is a unique fixed point in* $\mathbb{X}$ *of* V*.*

Proposition 2.1. *Let (U, V) be a $MR-$compatible pair of self-maps on an $MR-metric$ space $(\mathbb{X}, M)$ and V be a continuous. Then the pair (U, V) is $MR-$ semi-compatible.*

Proof. Let $U\zeta_n \to s, V\zeta_n \to s$. We have to show that $UV\zeta_n \to Vs$. Since V is continuous implies $VU\zeta_n \to Vs$. Since (U, V) is $MR-$ compatible, for some $\beta \in (0, \infty)$

$$M(UV\zeta, UV\eta, VU\xi) \leq \beta M(V\zeta, V\eta, U\xi),$$

for all ζ, η and $\xi \in \mathbb{X}$.

Setting $\zeta = \zeta_n, \eta = \zeta_{n+p}$ and $\xi = \zeta_n$ in the above condition, we have

$$M(UV\zeta_n, UV\zeta_{n+p}, VU\zeta_n) \leq \beta M(V\zeta_n, V\zeta_{n+p}, U\zeta_n),$$

implies $\lim_{n\to\infty} M(UV\zeta_n, UV\zeta_{n+p}, Vs) = 0$. Thus $\lim_{n\to\infty} UV\zeta_n = Vs$. Therefore the pair (U, V) is $MR-$ semi-compatible. □

Remark 2.1. *By the next example we note that,*

(1) *The pair of self-maps (U, V) is $MR-$ semi-compatible yet it is not $MR-$ compatible even though V is continuous.*
(2) *The pair (U, V) is $MR-$ semi-compatible but (V, U) is not $MR-$ semi-compatible.*
(3) *$UV = VU$, still (V, U) is not $MR-$ semi-compatible.*

Example 1. *Let (R^+, M) be an $MR - metric$ space. Define a function $M : R^+ \times R^+ \times R^+ \to [0, \infty)$ as*

$$M_\infty(\zeta, \eta, \xi) = \frac{1}{R} \max\{|\zeta - \eta|, |\eta - \xi|, |\xi - \zeta|\},$$

for all ζ, η and $\xi \in R^+$.

Now, U and V on R^+ are defined as:

$$U(\zeta) = \begin{cases} 0 & \text{if } \zeta > 0 \\ 1 & \text{otherwise} \end{cases}$$

Also, $V\zeta = \zeta$ for all $\zeta \in R^+$. Let $\zeta_n = \frac{1}{n}$. Then $U\zeta_n, V\zeta_n \to 0$ as $n \to \infty$.

(1) *$UV\zeta_n = U\zeta_n \to 0 = V(0)$; That is $UV\zeta_n \to V(0)$. Moreover, if we take V as the identity function I, for any sequence $\{\zeta_n\}$ such that $\{U\zeta_n\} \to s$ and $\{V\zeta_n\} \to s$, as $n \to \infty$, $UV\zeta_n = U\zeta_n \to s(= Vs)$ i.e. $UV\zeta_n \to Vs$. Thus (U, V) is $MR-$ semi-compatible. Also $V = I$ and V is continuous.*

Set $\zeta = 0, \eta = 0$ and $\xi = 1$ in (2.1) we have, $M(1,1,1) \leq \beta M(0,0,0)$, for all $\beta \in (0,\infty)$, which is not true. Thus (U,V) is not $MR-$ compatible.

(2) *Also, $U\zeta_n, V\zeta_n \to 0$ as $n \to \infty$, $VU\zeta_n = V(0) \to 0 \neq U(0)$. So (V,U) is not $MR-$ semi-compatible. From (1), $UV\zeta_n \to V(0)$. Thus (U,V) is $MR-$ semi- compatible.*

(3) *Additionally, we observe that as $V = I, UV = VU$. Therefore (U,V) is commuting yet (V,U) is not $MR-$ semi-compatible.*

Proposition 2.2. *Let U and V be two self-mappings of an $MR-$metric space $(\mathbb{X}, M)$ such that $U(\mathbb{X}) \subseteq V(\mathbb{X})$. For $\zeta_0 \in \mathbb{X}$ define sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\mathbb{X}$ by $U\zeta_{n-1} = V\zeta_n = \eta_n$, for all $n \in \mathbb{N}$. Then*

(1) $O(V^{-1}U, \zeta_0) = \{\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n, \ldots\}$,
(2) $O(UV^{-1}U, U\zeta_0) = \{\eta_0, \eta_1, \eta_2, \ldots, \eta_n, \ldots\}$.

Proof. $U\zeta_0 = V\zeta_1$ implies $\zeta_1 \in V^{-1}U\zeta_0$ and $U\zeta_1 = V\zeta_2$ implies $\zeta_2 \in V^{-1}U\zeta_1 = (V^{-1}U)^2\zeta_0$.

Similarly, $U\zeta_{n-1} = V\zeta_n$ gives $\zeta_n \in V^{-1}U\zeta_{n-1} = (V^{-1}U)^n\zeta_0$. Once more

$$\begin{aligned}
\eta_1 &= U\zeta_0, \eta_2 = U\zeta_1 \in U(V^{-1}U\zeta_0) = (UV^{-1})U\zeta_0, \\
\eta_3 &= U\zeta_2 \in U(V^{-1}UV^{-1}U\zeta_0) = (UV^{-1})^2U\zeta_0, \\
&\vdots \\
\eta_n &\in (UV^{-1})^{n-1}U\zeta_0.
\end{aligned}$$

□

According to the definition of (c)-comparison function with base R defined by Shatanawi [11] and Dhage [5], we introduce the following family of functions:

Definition 2.10. *[5,11] Let R be a constant $R \geq 1$. A map $\Psi : [0,+\infty) \to [0,+\infty)$ is called a $(c)-comparison$ function with base R if Ψ satisfies the following:*

(i) *Ψ is continuous,*
(ii) *Ψ is non-decreasing,*
(iii) *$\sum_{n=1}^{\infty} R^n\Psi^n(Rt)$ converges for all $t \geq 0$.*

If ψ is a (c)-comparison function, then for all $t > 0$ we have $\psi(t) < t$ and $\psi(0) = 0$.

Definition 2.11. *A set $S \subseteq \mathbb{X}$ is called an $MR-$bounded if there exists a constant $K > 0$ such that $M(\zeta, \eta, \xi) \leq K$ for all $\zeta, \eta, \xi \in S$ and the constant K is called an $MR-$bound of S.*

Lemma 2.1. *Let $\{\zeta_n\} \subseteq \mathbb{X}$ be $MR-$bounded with $MR-$bound K satisfying*

$$M(\zeta_n, \zeta_{n+1}, \zeta_m) \leq R^n \Psi^n(RM), \quad \forall m > n+1,$$

Then $\{\zeta_n\}$ is an $MR-$Cauchy in $\mathbb{X}$.

Proof. Since $\sum_{j=1}^{\infty} R^j \Psi^j(Rt)$ converges series of nonnegative real number for all $t \geq 0$, we have $\lim_{n\to\infty} R^n \Psi^n(Rt) = 0$ and $\lim_{m>n} \sum_{j=n+1}^{\infty} R^j \Psi^j(Rt) = 0$. For $p, t \in \mathbb{N}$, we have

$$M(\zeta_n, \zeta_{n+1}, \zeta_{n+p}) \leq R^n \Psi^n(RK),$$

and

$$M(\zeta_n, \zeta_{n+1}, \zeta_{n+p+t}) \leq R^n \Psi^n(RK).$$

By continuing this process of the tetrahedral inequality we obtain

$$\begin{aligned}
& M(\zeta_n, \zeta_{n+p}, \zeta_{n+p+t}) \\
\leq\ & RM(\zeta_n, \zeta_{n+1}, \zeta_{n+p+t}) + RM(\zeta_n, \zeta_{n+p}, \zeta_{n+1}) + RM(\zeta_{n+1}, \zeta_{n+p}, \zeta_{n+p+t}) \\
\leq\ & 2R^n \Psi^n(RK) + RM(\zeta_{n+1}, \zeta_{n+p}, \zeta_{n+p+t}) \\
\leq\ & 2R^n \Psi^n(RK) + RM(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+p+t}) + RM(\zeta_{n+1}, \zeta_{n+p}, \zeta_{n+2}) \\
& \quad + RM(\zeta_{n+2}, \zeta_{n+p}, \zeta_{n+p+t}) \\
\leq\ & 2[R^n \Psi^n(RK) + R^{n+1} \Psi^{n+1}(RK)] + RM(\zeta_{n+2}, \zeta_{n+p}, \zeta_{n+p+t}) \leq \dots \\
\leq\ & 2 \sum_{j=n}^{n+p-1} R^j \Psi^j(RK) + RM(\zeta_{n+p}, \zeta_{n+p-1}, \zeta_{n+p+t}) \\
\leq\ & 2 \sum_{j=n}^{n+p-1} R^j \Psi^j(RK) + 2 \sum_{j=n+p}^{n+p+t-1} R^j \Psi^j(RK) + RM(\zeta_{n+1+t}, \zeta_{n+p}, \zeta_{n+p+t}) \\
\leq\ & 2 \sum_{j=n}^{n+p+t} R^j \Psi^j(RK) R^{n+p-1} \Psi^{n+p-1}(RK) \\
\to\ & 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\{\zeta_n\}$ is an $MR-$Cauchy. □

Lemma 2.2. *Let U and V be two self-mappings of an $MR-$metric space $(\mathbb{X}, M)$ such that:*

(i) $U(\mathbb{X}) \subseteq V(\mathbb{X})$;

(ii) *Some orbit* $\{\eta_n\} = O(UV^{-1}, U\zeta_0)$ *is bounded;*

(iii) *For all* $\zeta, \eta, \xi \in O(V^{-1}U, \zeta_0)$ *and for some* Ψ*, where* Ψ *is a* $(c)-comparison$ *function with base* R*,*

$$M(U\zeta, U\eta, U\xi) \leq \frac{1}{R}\Psi \max \left\{ \begin{array}{c} RM(V\zeta, V\eta, V\xi), RM(U\zeta, V\zeta, V\xi), \\ RM(U\eta, V\eta, V\xi), RM(U\zeta, V\eta, V\xi), \\ RM(U\eta, V\zeta, V\xi) \end{array} \right\}.$$

Then $\{\eta_n\}$ *is an* $MR-$*Cauchy sequence in* $O(UV^{-1}, U\zeta_0)$.

Proof. Let $\zeta_0 \in \mathbb{X}$. As $U(\mathbb{X}) \subseteq V(\mathbb{X})$, we define two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\mathbb{X}$ by $U\zeta_{n-1} = V\zeta_n = \eta_n, \forall n \in \mathbb{N}$. Then

$$\begin{aligned} M(\eta_n, \eta_{n+1}, \eta_{n+p}) &= M(U\zeta_{n-1}, U\zeta_n, U\zeta_{n+p-1}) \\ &\leq \frac{1}{R}\Psi \max \left\{ \begin{array}{c} RM(\eta_n, \eta_{n-1}, \eta_{n+p-1}), RM(\eta_{n-1}, \eta_n, \eta_{n+p-1}), \\ RM(\eta_{n+1}, \eta_n, \eta_{n+p-1}), RM(\eta_n, \eta_n, \eta_{n+p-1}), \\ RM(\eta_{n-1}, \eta_{n+1}, \eta_{n+p-1}) \end{array} \right\}. \end{aligned}$$

That is

$$\begin{aligned} M(\eta_n, \eta_{n+1}, \eta_{n+p}) \leq \frac{1}{R}\Psi \max \{&RM(\eta_n, \eta_{n-1}, \eta_{n+p-1}), \\ RM(\eta_{n+1}, \eta_n, \eta_{n+p-1}), RM(\eta_n, \eta_n, \eta_{n+p-1}), &RM(\eta_{n-1}, \eta_{n+1}, \eta_{n+p-1})\}. \end{aligned} \tag{2.2}$$

Once more

$$\begin{aligned} M(\eta_{n-1}, \eta_n, \eta_{n+p-1}) \leq \frac{1}{R}\Psi \max \{&RM(\eta_{n-2}, \eta_{n-1}, \eta_{n+p-2}), \\ RM(\eta_{n-1}, \eta_n, \eta_{n+p-2}), RM(\eta_{n-1}, \eta_{n-1}, \eta_{n+p-2}), &RM(\eta_n, \eta_{n-2}, \eta_{n+p-2})\}. \end{aligned} \tag{2.3}$$

$$\begin{aligned} M(\eta_{n+1}, \eta_n, \eta_{n+p-1}) \leq \frac{1}{R}\Psi \max \{&RM(\eta_n, \eta_{n-1}, \eta_{n+p-2}), \\ RM(\eta_{n+1}, \eta_n, \eta_{n+p-2}), RM(\eta_n, \eta_{n-1}, \eta_{n+p-2}), &RM(\eta_{n-1}, \eta_{n+1}, \eta_{n+p-2}), \\ RM(\eta_n, \eta_n, \eta_{n+p-2})\}. \end{aligned} \tag{2.4}$$

$$\begin{aligned} M(\eta_n, \eta_n, \eta_{n+p-1}) \leq \frac{1}{R}\Psi \max \{&RM(\eta_{n-1}, \eta_{n-1}, \eta_{n+p-2}), \\ &RM(\eta_n, \eta_{n-1}, \eta_{n+p-2})\}. \end{aligned} \tag{2.5}$$

$$\begin{aligned}
&M(\eta_{n-1},\eta_{n+1},\eta_{n+p-1}) \leq \frac{1}{R}\Psi \max\{RM(\eta_{n-2},\eta_n,\eta_{n+p-2}),\\
&RM(\eta_{n-1},\eta_{n-2},\eta_{n+p-2}), RM(\eta_{n+1},\eta_n,\eta_{n+p-2}), RM(\eta_{n-1},\eta_n,\eta_{n+p-2}),\\
&RM(\eta_{n-2},\eta_{n+1},\eta_{n+p-2})\}.
\end{aligned} \tag{2.6}$$

Substituting (2.3) to (2.6) into (2.2) we attain,

$$M(\eta_n,\eta_{n+1},\eta_{n+p}) \leq R^2\Psi^2 Ka\zeta_{a,b,c}\{RM(\eta_a,\eta_b,\eta_c)\},$$

for all a,b,c such that $n-2 \leq a \leq n, n-1 \leq b \leq n+1, c = n+p-1$.

Following same procedure, we get

$$M(\eta_n,\eta_{n+1},\eta_{n+p}) \leq R^n\Psi^n Ka\zeta_{a,b,c}\{RM(\eta_a,\eta_b,\eta_c)\}, \tag{2.7}$$

for all a,b, c such that $0 \leq a \leq n$, $1 \leq b \leq n+1, c = p$. Let K be the bound of $O(UV^{-1}, U\zeta_0)$. Then it follows from (2.7) that

$$M(\eta_n,\eta_{n+1},\eta_{n+p}) \leq R^n\Psi^n(RK).$$

Thus, by lemma 2.2 , $\{\eta_n\}$ is an $MR-$Cauchy sequence in $O(UV^{-1}, U\zeta_0)$. □

3. Main Results

Theorem 3.1. *Let U and V be two self-mappings of an $MR-$metric space $(\mathbb{X}, M)$ such that:*

- (i) $U(\mathbb{X}) \subseteq V(\mathbb{X})$;
- (ii) *The pair (U, V) is an MR-semi-compatible and V is continuous;*
- (iii) *For some $\zeta_0 \in \mathbb{X}$, some orbit $\{\eta_n\} = O(UV^{-1}, U\zeta_0)$ is bounded and complete;*
- (iv) *For all $\zeta, \eta \in O(V^{-1}U, \zeta_0) \cup O(UV^{-1}, U\zeta_0)$, for some Ψ, where Ψ is a $(c)-$ $comparison$ function with base R and $\forall \xi \in \mathbb{X}$,*

$$\begin{aligned}
&M(U\zeta, U\eta, U\xi) \leq \frac{1}{R}\Psi \max\{RM(V\zeta, V\eta, V\xi),\\
&RM(U\zeta, V\zeta, V\xi), RM(U\eta, V\eta, V\xi), RM(U\zeta, V\eta, V\xi), RM(U\eta, V\zeta, V\xi)\}.
\end{aligned}$$

Then U and V have a unique common fixed point in $\mathbb{X}$.

Proof. Let $\zeta_0 \in \mathbb{X}$, construct two sequences $\{\zeta_n\}$ and $\{\eta_n\} \in X$ as $U\zeta_{n-1} = V\zeta_n = \eta_n$, for all $n \in \mathbb{N}$. By Lemma 2.2, $\{\eta_n\}$ is an $MR-$Cauchy sequences in $O(UV^{-1}, U\zeta_0)$, thus $\{\eta_n\}$ is complete. Consequently

$$\eta_n = V\zeta_n = U\zeta_{n-1} \longrightarrow s \in \mathbb{X} \tag{3.1}$$

As V is continuous and (U, V) is $MR-$semi-compatible, we have

$$V^2\zeta_n \to Vs, UV\zeta_n \to Vs. \tag{3.2}$$

We will divide the proof into three steps :

Step 1: Setting $\zeta = V\zeta_n, \eta = V\zeta_n$ and $\xi = s$ in (iv) we have

$$\begin{aligned} M(UV\zeta_n, UV\zeta_n, Us) \leq \frac{1}{R}\Psi \max \{&RM(VV\zeta_n, VV\zeta_n, Vs),\\ RM(UV\zeta_n, VV\zeta_n, Vs), &RM(UV\zeta_n, VV\zeta_n, Vs), RM(UV\zeta_n, VV\zeta_n, Vs),\\ RM(UV\zeta_n, VV\zeta_n, Vs)\}&. \end{aligned}$$

Let $n \to \infty$, by (3.2) we have,

$$M(Vs, Vs, Us) = 0,$$

implies

$$Vs = Us. \tag{3.3}$$

Step 2: Set $\zeta = \zeta_n, \eta = \zeta_n$ and $\xi = s$ in (iv) we obtain,

$$M(U\zeta_n, U\zeta_n, Us) \leq \frac{1}{R}\Psi \max \left\{ \begin{array}{c} RM(V\zeta_n, V\zeta_n, Vs), RM(U\zeta_n, V\zeta_n, Vs), \\ RM(U\zeta_n, V\zeta_n, Vs), RM(U\zeta_n, V\zeta_n, Vs), \\ RM(U\zeta_n, V\zeta_n, Vs) \end{array} \right\}.$$

Let $n \to \infty$, by (3.1), (3.3) and Ψ is a $(c) - comparison$ function with base R we have,

$$M(s, s, Us) \leq \frac{1}{R}\Psi\{RM(s, s, Us)\} < M(s, s, Us), \text{ if } M(s, s, Us) > 0,$$

which is a contradiction. Thus $M(s, s, Us) = 0$, implies $s = Us$. Hence $s = Us = Vs$; that is, s is a common fixed point of U and V.

Step 3: To prove the uniqueness. Let u be a common fixed point of U and V, then $u = Uu = Vu$. Set $\zeta = \zeta_n, \eta = \zeta_n$ and $\xi = u$ in (iv) we obtain,

$$M(U\zeta_n, U\zeta_n, Uu) \leq \frac{1}{R}\Psi \max \left\{ \begin{array}{c} RM(V\zeta_n, V\zeta_n, Vu), RM(U\zeta_n, V\zeta_n, Vu), \\ RM(U\zeta_n, V\zeta_n, Vu), RM(U\zeta_n, V\zeta_n, Vu), \\ RM(U\zeta_n, V\zeta_n, Vu) \end{array} \right\}.$$

Let $n \to \infty$, by Ψ is a $(c) - comparison$ function with base R, we have,

$$M(s,s,u) \leq \frac{1}{R}\Psi\{RM(s,s,u)\} < M(s,s,u), \text{ if } M(s,s,u) > 0,$$

which is a contradiction. Thus $M(s,s,u) = 0$, implies $s = u$. Hence $s = u$; That is, s is a unique common fixed point of U and V. □

Remark 3.1. *From (i) of remark (2.1) it gives there are $MR-$semi-compatible maps (U,V) which are not $MR-$compatible even if V is continuous. The above theorem examines the common fixed points of such $MR-$semi-compatible maps (U,V) in $MR-$metric spaces.*

Lemma 3.1. *Let $\mathbb{X}$ be an $MR-$metric space and U, V be two self -mappings of $\mathbb{X}$ satisfying there exists a sequence $\{\zeta_n\} \in \mathbb{X}$ such that*

(1) $U\zeta_n = V\eta_{n+1}$ *for* $n \in \mathbb{N}$;

(2) $\delta_M(O_U(V\zeta_0)) < \infty$.

Then the following are equivalent:

$(b1)$ *Let* $\epsilon > 0$, *there exist* $\epsilon^{'}, \epsilon^{''}$ *such that* $0 < \epsilon^{'} < \epsilon < \epsilon^{''}$ *and let*

$$\begin{aligned} &M(V\zeta,V\eta,V\xi) \\ &= \frac{1}{R}\max\left\{\begin{array}{c} RM(V\zeta,V\eta,V\xi), RM(V\zeta,U\zeta,V\xi), RM(V\eta,U\zeta,V\xi), \\ RM(V\zeta,U\zeta,V\xi), RM(V\eta,U\eta,V\xi) \end{array}\right\}. \end{aligned} \tag{3.4}$$

If $M(V\zeta,V\eta,V\xi) < \epsilon^{''}$ *gives* $M(U\zeta,U\eta,U\xi) < \epsilon^{'}$

$(b2)$ *There exists an increasing upper seicontinuous* $\phi : R^{+} \to R^{+}$ *and* $\phi(t) < t$ $\forall > 0$ *such that*

$$M(U\zeta,U\eta,\xi) \leq \frac{1}{R}\phi(RM(V\zeta,V\eta,V\xi)) \text{ for all } \zeta,\eta,\xi \in \mathbb{X}. \tag{3.5}$$

Proof. It is obvious that (3.5) implies (3.4). Suppose that (3.4) is satisfied. Define $\phi : R^{+} \to R^{+}$as

$$\phi(\zeta) = \left\{\begin{array}{cc} \frac{1}{9}\zeta, & 0 \leq \zeta < 1, \\ \frac{1}{9}(\zeta + [\zeta]), & [\zeta] \leq \zeta < [\zeta] + 1, 1 \leq \zeta, \\ \frac{3}{8}(\zeta + [\zeta]), & \zeta = 1 + [\zeta], 1 \leq \zeta, \end{array}\right\}.$$

Here $[\zeta]$ is the greatest integer not exceeding ζ. Then (3.5) comes from (3.4) and definition of ϕ. □

Lemma 3.2. *Let $\mathbb{X}$ be an $MR-$metric space and U, V be two self -mappings of $\mathbb{X}$ satisfying*

(I) $U\zeta_n = V\eta_{n+1}$ *for* $n \in \mathbb{N}$;
(II) $\delta_M(O_U(V\zeta_0)) < \infty$, *and (3.4).*
Then $\{\zeta_n\}$ *is an* $MR-$*Cauchy sequence.*

Proof. Let $\lambda_n = \delta_M(O_U(V\zeta_n)), \forall n \in \mathbb{N}$. Then by the definition 2.9 and (II), we attain $\lambda_1 < \infty$. Since $\{\lambda_n\}$ is a decreasing sequence of nonnegative real numbers, there exists $\epsilon \geq 0$ such that

$$\lim_{n\to\infty} \lambda_n = \epsilon.$$

By Lemma 3.1, $\lambda_{n+1} \leq \phi(\lambda_n)$. Now, it is enough to show that $\epsilon = 0$. If not, then from Lemma 3.1, we get that $\epsilon \leq \phi(\epsilon) < \epsilon$, which is a contradiction. Therefore $\epsilon = 0$.Thus, $\{V\zeta_n\}$ is an $MR-$Cauchy sequence. □

Theorem 3.2. *Let* $\mathbb{X}$ *be an* $MR-$*complete metric space and* U, V *be two self - mappings of* $\mathbb{X}$ *satisfying*
$(s1)$ $\delta_M(O_U(V\zeta_0)) < \infty$ *and (3.4);*
$(s2)$ V *is continuous;*
$(s3)$ $U\mathbb{X} \subseteq V\mathbb{X}$;
$(s4)$ (U, V) *is a pair of an* $MR-$*compatible;*
$(s5)$ *The* $MR-$*metric is a continuous function on* $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$.
Then U *and* V *have a unique common fixed point in* $\mathbb{X}$.

Proof. By $(s3)$, we get a sequence $\{\zeta_n\} \in \mathbb{X}$ such that

$$U\zeta_n = V\eta_{n+1} \text{ for } n \in \mathbb{N}.$$

Thus, by Lemma 3.2, $\{V\zeta_n\}$ is an $MR-$Cauchy sequence. Since $\mathbb{X}$ be an MR-complete metric space, $\{V\zeta_n\}$ converges to $s \in \mathbb{X}$. We have to show that s is a unique fixed point of U and V. Now V is continuous implies that

$$\lim_{n\to\infty} VU\zeta_n = Vs \text{ for } n \in \mathbb{N}.$$

Since $\forall n, m, r \in \mathbb{N}$ such that $n < m < r$,

$$M(UV\zeta_n, UV\zeta_m, VU\zeta_r) \leq \beta M(V\zeta_n, V\zeta_m, U\zeta_r),$$

by $s2$, $s4$ and $s5$, we get

$$\lim_{n\to\infty} UV\zeta_n = Vs.$$

Take $Vs \neq s$. Let $\epsilon = M(s, s, Vs)$. Then $0 < \epsilon$. Now

$$\begin{aligned}
&\lim_{n\to\infty} M(V\zeta_n, V\zeta_n, VU\zeta_n)\\
= &\lim_{n\to\infty} \frac{1}{R} \max \left\{ \begin{array}{c} RM(V\zeta_n, V\zeta_n, VU\zeta_n), RM(V\zeta_n, U\zeta_n, VU\zeta_n), \\ RM(V\zeta_n, U\zeta_n, VU\zeta_n) RM(V\zeta_n, U\zeta_n, VU\zeta_n), \\ RM(V\zeta_n, U\zeta_n, VU\zeta_n) \end{array} \right\}\\
= &M(s, s, Vs).
\end{aligned}$$

Since

$$\lim_{n\to\infty} M(V\zeta_n, V\zeta_n, VU\zeta_n) = M(s, s, Vs) = \epsilon > 0,$$

From Lemma 3.1, $\epsilon \leq \phi(\epsilon) < \epsilon$, which is a contradiction. Thus $Vs = s$.Take $Us \neq s$. Since

$$\begin{aligned}
&\lim_{n\to\infty} M(Vs, V\zeta_n, VU\zeta_n)\\
= &\lim_{n\to\infty} \frac{1}{R} \max \left\{ \begin{array}{c} RM(Vs, V\zeta_n, VU\zeta_n), RM(Vs, U\zeta_n, VU\zeta_n), \\ RM(V\zeta_n, Us, VU\zeta_n), RM(V\zeta_n, Us, VU\zeta_n), \\ RM(V\zeta_n, U\zeta_n, VU\zeta_n) \end{array} \right\}\\
= &M(s, s, Us) = \epsilon > 0,
\end{aligned}$$

From Lemma 3.1, $\epsilon \in\leq \phi(\epsilon) < \epsilon$, which is a contradiction. Thus $Us = s$.

To prove the uniqueness, let $s_1 \neq s_2$ be two common fixed points of U and V. Then

$$M(Vs_1, Vs_1, Vs_2) = M(s_1, s_1, s_2) := \epsilon > 0.$$

Then from Lemma 3.1,

$$\epsilon = M(Us_1, Us_1, Us_2) = M(s_1, s_1, s_2) \leq \phi(\epsilon) < \epsilon.$$

Which is a contradiction. Hence U and V have a unique common fixed in $\mathbb{X}$ □

We obtain the following corollary.

Corollary 3.1. *Let $\mathbb{X}$ be a complete $MR-$metric space and U, V be two self-mappings of an $MR-$metric space $(\mathbb{X}, M)$ such that:*

(1) $U(\mathbb{X}) \subseteq V(\mathbb{X})$;

(2) *The pair (U, V) is an MR-compatible and V is continuous;*

(3) $\delta_d(O_U(V\zeta_0)) < \infty$*;*

(4) *For some* $q \in [0,1)$ *and for all* $\zeta, \eta\ \xi \in \mathbb{X}$,

$$M(U\zeta, U\eta, U\xi) \leq \frac{q}{R}\Psi \max\left\{ \begin{array}{c} RM(V\zeta, V\eta, V\xi), RM(U\zeta, V\zeta, V\xi), RM(U\eta, V\eta, V\xi), \\ RM(U\zeta, V\eta, V\xi), RM(U\eta, V\zeta, V\xi) \end{array} \right\}.$$

Then U *and* V *have a unique common fixed point in* $\mathbb{X}$*.*

Proof. Since the 4^{th} condition from the above corollary gives (3.4), the result comes from Theorem (3.2). □

We generalize the above corollary.

Corollary 3.2. *Let* U *and* V *be two self-mappings of an* $MR-$*metric space* $(\mathbb{X}, M)$ *satisfying:* $(i), (iii), (iv)$ *of Theorem (3.1) and the pair* (U,V) *is* $MR-$*compatible and* V *is continuous; Then* U *and* V *have a unique common fixed point in* $\mathbb{X}$*.*

Proof. From Theorem (3.1) and proposition 2.1. □

Note that: Corollary 3.2 is a particular case of Corollary 3.1.

Theorem 3.3. *Let* U *and* V *be two self-mappings of an* $MR-$*metric space* $(\mathbb{X}, M)$ *satisfying:*

1. $(i), (iii)$ *of Theorem (3.1) and the pair* (U,V) *is an MR-semi-compatible and* U *is continuous;*
2. *For some* $q \in [0,1)$ *and for all* $\zeta, \eta\ \xi \in \mathbb{X}$

$$M(U\zeta, U\eta, U\xi) \leq \frac{1}{R}\Psi \max\left\{ \begin{array}{c} RM(V\zeta, V\eta, V\xi), RM(U\zeta, V\zeta, V\xi), RM(U\eta, V\eta, V\xi), \\ RM(U\zeta, V\eta, V\xi), RM(U\eta, V\zeta, V\xi) \end{array} \right\}.$$

Then U *and* V *have a unique common fixed point in* $\mathbb{X}$*.*

Proof. Let $\zeta_0 \in \mathbb{X}$, construct two sequences $\{\zeta_n\}$, $\{\eta_n\} \in \mathbb{X}$ as in proof of theorem 3.1. Thus (i) of Theorem 3.1 is satisfied. Because U is continuous, we obtain $UV\zeta_n \to Us$, and (U,V) is an MR-semi-compatible, we have

$$UV\zeta_n \to Us.$$

Now, the limit of a sequence is unique, we attain $Us = Vs$ and the rest of the proof comes from steps 2 and 3 of Theorems 3.1.

□

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