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ON CONJUGATE OF SUB E-FUNCTIONS

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ABSTRACT. The aim of this article is to introduce the definition of the conjugate of sub E-functions, which plays an important role in linking the concept of duality among sub E-functions. Furthermore, some properties for this class are established.

1. INTRODUCTION

Suppose that $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ along with $a < b$. There are a lot of generalizations of the concept of convex functions see [2, 3, 11]. One way to generalize the concept of convex function is to replace linear functions by another family of functions in the sense of Beckenbach [2]. In 2016, Mohamed S. S. Ali [1] introduced sub E-functions by dealing with a family $\{E(x)\}$ of exponential functions

$$E(x) = A \exp Bx,$$

where A, B are arbitrary constants. More precisely, [1] a positive function $f : I \to (0, \infty)$ is said to be a sub E-function on I, if for all $x \in [a, b] \subset I$,

$$f(x) \leq E(x),$$

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where A and B are chosen such that $E(a) = f(a)$, and $E(b) = f(b)$. In this paper, we deal with this family $\{E(x)\}$ of exponential functions.

In fact, the topic of conjugate convex functions really originates in the paper of Young [14]. This topic has evoked many interests ([10], [5] and [9]), after the work of Fenchel. In [6] and [7] Fenchel greatly generalized the whole idea and applied it to a programming problem. In [12], the optimal solution for the maximum range was interpreted geometrically using the concept of a convex conjugate function. Furthermore, the conjugate convex functions have numerous applications mentioned in [8, 10, 12]. Also, in physical problems [13, 15], the conjugate convex is used to convert functions of one quantity (such as position, pressure, or temperature) into functions of the conjugate quantity (momentum, volume, and entropy, respectively). In this way, it is commonly used in classical mechanics to derive the Hamiltonian formalism out of the Lagrangian formalism and in thermodynamics to derive the thermodynamic potentials, as well as in the solution of differential equations of several variables.

The objective of the present paper is to define a conjugate of sub E-functions on the real line $\mathbb{R}$.

2. Definitions and Preliminary Results

This section is devoted to introduce the main definitions and results about sub E-functions see [1], which will be used in the following.

Definition 2.1. *[1] A positive function $f : I \to (0, \infty)$ is said to be a sub E-function on I, if for all $a, b \in I$ with $a < b$, the graph of $f(x)$ for $a \leq x \leq b$, lies on or under the graph of a function*

$$E(x) = Ae^{Bx},$$

where A and B are taken so that $E(a) = f(a)$, and $E(b) = f(b)$. Equivalently, for all $x \in [a, b]$,

$$\begin{aligned} f(x) &\leq E(x) \\ &= exp\left[\frac{(b-x)\ln f(a) + (x-a)\ln f(b)}{b-a}\right]. \end{aligned} \tag{2.1}$$

There is more than one form for the function $E(x)$ other than that stated in (2.1); for example,

$$E(x) = f(a)e^{B(x-a)};\ B = \frac{\ln f(b) - \ln f(a)}{b-a},$$

or in a multiplicative formula

$$E(x) = [f(a)]^{\frac{b-x}{b-a}} \cdot [f(b)]^{\frac{x-a}{b-a}}.$$

Remark 2.1. *[1] The sub E-functions possess a number of properties analogous to those of convex functions. For example: Let $f : I \to (0,\infty)$ be a sub E-function, then for all $a, b \in I$, the inequality $f(x) \geq E(x)$ holds outside the interval $[a,b]$.*

Definition 2.2. *[4] Assume that $f : I \to (0,\infty)$ is a sub E-function, then a function*

$$T_u(x) = Ae^{Bx}$$

is called a supporting function for $f(x)$ at the point $u \in (a,b)$ if

(1) $T_u(u) = f(u)$;
(2) $T_u(x) \leq f(x)\ \ \forall x \in I$.

That is, if $f(x)$ and $T_u(x)$ agree at $x = u$, then the graph of $f(x)$ lies on or above the support curve.

Proposition 2.1. *[1] Let $f : I \to \mathbb{R}$ be a differentiable sub E-function, then the supporting function for $f(x)$ at the point $u \in I$ has the formula*

$$T_u(x) = f(u)\exp\left[(x-u)\frac{f'(u)}{f(u)}\right]. \tag{2.2}$$

Proposition 2.2. *[1] For a sub E-function $f : I \to (0,\infty)$, the supporting function written at $u \in I$ in the following formula*

$$T_u(x) = f(u)\exp\left[(x-u)\frac{M_{u,f}}{f(u)}\right].$$

The constant $M_{u,f}$ is equal to $f'(u)$ if f is differentiable at the point $u \in I$; otherwise $f'_-(u) \leq M_{u,f} \leq f'_+(u)$.

Theorem 2.1. *[1] If $f : I \to (0,\infty)$ is a two-times continuously differentiable function. The function f is a sub E-function on I if and only if $f(x)f''(x) - (f'(x))^2 \geq 0$ for all x in I.*

Theorem 2.2. *[1] Supposed that a function $f : I \to (0, \infty)$ is a sub E-function on I if and only if there exist a supporting function for $f(x)$ at each point $x \in I$.*

3. Main Results

Lemma 3.1. *Let A be a nonempty, bounded subset of $\mathbb{R}$. If $f : A \to \mathbb{R}$, is an increasing function and continuous on A. Then*

$$f(\sup A) = \sup(f(A)). \tag{3.1}$$

Proof. From the definition of the supremum,

$$x \leq \sup(A), \quad \forall x \in A,$$

since f is increasing, we have

$$f(x) \leq f(\sup(A)), \quad \forall x \in A.$$

Hence,

$$\sup(f(A)) \leq f(\sup(A)). \tag{3.2}$$

Also, let $y = \sup(A)$. Consequently, $\forall \epsilon > 0$, $\exists x_0 \in A$, such that $y \geq x_0 > y - \epsilon$. Implies, $\forall \frac{1}{n} > 0$, $\exists x_n \in A$, such that

$$y - \frac{1}{n} < x_n \leq y.$$

From the squeeze theorem, we obtain that

$$\lim_{n \to \infty} (x_n) = y.$$

Since f is continuous on A, then f is continuous at y, so we get

$$\lim_{n \to \infty} f(x_n) = f(y).$$

This implies, $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that

$$|f(x_n) - f(y)| \leq \epsilon, \text{ when } n \geq n_0,$$

consequently,

$$\begin{aligned} f(y) &= f(y) - f(x_n) + f(x_n) \\ &\leq |f(y) - f(x_n)| + f(x_n) \\ &\leq \epsilon + f(x_n), \quad \forall n \geq n_0 \\ &\leq \epsilon + \sup f(A). \end{aligned}$$

Since $f(y) - \sup f(A) \leq \epsilon$, $\forall \epsilon > 0$. Then

$$f(\sup(D)) = f(y) \leq \sup f(D). \tag{3.3}$$

From inequalities (3.2) and (3.3), then the equation (3.1) is hold. □

Lemma 3.2. *Let $f_\alpha : I \to (0, \infty)$ be an arbitrary family of a sub E-functions and*

$$f(x) = \sup_\alpha (f_\alpha(x)). \tag{3.4}$$

If $J = \{x \in I : f(x) < \infty\}$ is nonempty, then $f : J \to (0, \infty)$ is sub E-function.

Proof. Let $x \in [a, b] \subseteq J \subseteq I$, since $f_\alpha(x)$ is a sub E-function for all α, by using Inequality (2.1) and Equation (3.4). Then,

$$\begin{aligned} f(x) &= \sup_\alpha (f_\alpha(x)) \\ &\leq \sup_\alpha \left(exp\left[\frac{(b-x)\ln f_\alpha(a) + (x-a)\ln f_\alpha(b)}{b-a} \right] \right). \end{aligned}$$

Since the exponential function is continuous and increasing, and by using Lemma 3.1, we get

$$\begin{aligned} f(x) &\leq exp\left(\sup_\alpha \left[\frac{(b-x)\ln f_\alpha(a) + (x-a)\ln f_\alpha(b)}{b-a} \right] \right) \\ &= exp\left[\frac{(b-x)\sup_\alpha \ln f_\alpha(a) + (x-a)\sup_\alpha \ln f_\alpha(b)}{b-a} \right]. \end{aligned}$$

Since the logarithmic function is continuous and increasing, and by using Lemma 3.1, we get

$$\begin{aligned} f(x) &\leq exp\left[\frac{(b-x)\ln \sup_\alpha f_\alpha(a) + (x-a)\ln \sup_\alpha f_\alpha(b)}{b-a} \right], \\ &= exp\left[\frac{(b-x)\ln f(a) + (x-a)\ln f(b)}{b-a} \right]. \end{aligned}$$

Then, from Definition 2.1, we have $f(x) = \sup_\alpha f_\alpha(x)$ is a sub E-function. □

Theorem 3.1. *Let $f : I \to (0, \infty)$ be a sub E-function. Then*

$$f^*(z) = \sup_x \frac{e^{xz}}{f(x)} \tag{3.5}$$

is a sub E-function with domain $I^* = \{z \in \mathbb{R} : f^*(z) < \infty\}$, for all $z \in I^*$.

Proof. We first note that $I^* \neq \emptyset$. If I is single point x_0, then, from Proposition 2.1 and 2.2, we have that

$$T_{x_0}(x) = f(x_0)\exp[(x - x_0)z]$$

is a support function for f for each $z \in \mathbb{R}$. Otherwise, we pick any interior point x_0, choose

$$z \in [\frac{f'_-(x_0)}{f(x_0)}, \frac{f'_+(x_0)}{f(x_0)}],$$

and again $T_{x_0}(x)$ will be a support function for f. In either case,

$$\begin{aligned} T_{x_0}(x) &\leq f(x); \quad \forall x \in I \\ f(x_0)\exp[(x - x_0)z] &\leq f(x); \quad \forall x \in I \\ \frac{e^{xz}}{f(x)} &\leq \frac{e^{x_0 z}}{f(x_0)}; \quad \forall x \in I. \end{aligned}$$

For this choice z, $f^*(z) < \infty$ and $I^* \neq \emptyset$ as claimed. Let $u, v \in I^*$ such that $z = \lambda u + (1 - \lambda)v$, where $\lambda \in [0, 1]$, and let

$$g(z) = \frac{e^{xz}}{f(x)}. \tag{3.6}$$

Consequently,

$$\begin{aligned} g(z) &= g(\lambda u + (1 - \lambda)v) \\ &= \frac{e^{x(\lambda u + (1-\lambda)v)}}{(f(x))^{\lambda + (1-\lambda)}} \\ &= \exp\left[\ln\left(\frac{e^{xu}}{f(x)}\right)^{\lambda}\left(\frac{e^{xv}}{f(x)}\right)^{1-\lambda}\right] \\ &= \exp[\lambda \ln g(u) + (1 - \lambda)\ln g(v)]. \end{aligned}$$

Taking

$$\lambda = \frac{v - z}{v - u}, \quad \text{and} \quad 1 - \lambda = \frac{z - u}{v - u},$$

we get

$$g(z) = \exp\left[\frac{(v - z)\ln g(u) + (z - u)\ln g(v)}{v - u}\right]. \tag{3.7}$$

From Definition 2.1, and (3.7) we get $g(z)$ is sub E-function. By using Lemma 3.2, we obtain that

$$f^*(z) = \sup_x g(z)$$

is a sub E-function. $\square$

Definition 3.1. *If $f : I \to (0, \infty)$ is a sub E-function, then $f^* : I^* \to (0, \infty)$ will denote the conjugate of sub E-function defined by*

$$f^*(z) = \sup_x \frac{e^{xz}}{f(x)}, \tag{3.8}$$

with domain $I^* = \{z \in \mathbb{R} : f^*(z) < \infty\}$.

Example 1. *let $f(x) = 1/\sin x$, for all $x \in (0, \pi)$ be a sub E-function. Consequently, its conjugate*

$$f^*(z) = \begin{cases} e^{xz}, & z \geq 0 \\ 1, & z < 0. \end{cases}$$

is a sub E-function.

By Definition 3.1, we get

$$f^*(z) = \sup_x \frac{e^{xz}}{f(x)} = \sup_x (e^{xz} \sin x).$$

Case 1. If $z \geq 0$, and from $0 < x < \pi$ and $\sin x \leq 1$ we get

$$\begin{aligned} xz &\leq \pi z \\ e^{xz} &\leq e^{\pi z} \\ e^{xz} \sin x &\leq e^{\pi z}; \forall x \in (0, \pi). \end{aligned}$$

Then

$$f^*(z) = \exp[\pi z].$$

Let $z_1, z_2 \in I^*$ such that $z \in [z_1, z_2]$, we have

$$\begin{aligned} f^*(z) &= \exp[\pi(\lambda z_1 + (1 - \lambda) z_2)]; \lambda \in [0, 1] \\ &= \exp[\ln(e^{\pi z_1})^{\lambda} + \ln(e^{\pi z_2})^{1-\lambda}]. \end{aligned}$$

Taking

$$\lambda = \frac{z_2 - z}{z_2 - z_1},$$

we have

$$f^*(z) = \exp\left[\frac{(z_2 - z)\ln(\exp[\pi z_1]) + (z - z_1)\ln(\exp[\pi z_2])}{z_2 - z_1}\right].$$

Then, from Definition 2.1, we get $f^*(z) = \exp[\pi z]$ for all $z \in I^*$.

Case 2. If $z < 0$, and from $0 < x < \pi$ and $\sin x \leq 1$ we have

$$\begin{aligned} xz &< 0 \\ e^{xz} &< e^0 \\ e^{xz}\sin x &< 1. \end{aligned}$$

Consequently,

$$f^*(z) = \sup_x e^{xz}\sin x = 1$$

is a sub E-function by using Theorem 2.1.

Theorem 3.2. *Let $f : I \to (0,\infty)$ be a sub E-function. Then its conjugate $f^* : I^* \to (0,\infty)$ is also a sub E-functions and*

- **(k1):** $\exp(xz) \leq f(x)f^*(z)$.
- **(k2):** *If f is differentiable then $\exp(xz) = f(x)f^*(z)$, if, and only if, $z = f'(x)/f(x)$.*
- **(k3):** $f^{**} = f$.

Proof. We know from Theorem 3.1 that f^* is A sub E-function.

(k1): From definition of f^* and definition of supremum, we get

$$\frac{e^{xz}}{f(x)} \leq f^*(z); \quad \forall x \in I.$$

Since f is sub a E-convex function implies $f(x) > 0,\ \forall x \in I$, we get

$$\exp(xz) \leq f(x)f^*(z). \tag{3.9}$$

(k2): To prove the necessity, by differentiating,

$$\exp(xz) = f(x)f^*(z), \tag{3.10}$$

with respect to x, we have

$$\begin{aligned} z\exp(xz) &= f'(x)f^*(z) \\ z &= f'(x)\frac{f^*(z)}{\exp(xz)}. \end{aligned} \tag{3.11}$$

Since f^*, f are sub E-functions, we have $f^*(z) > 0,\ f(x) > 0$ for all $z \in I^*,\ x \in I$ and from (3.10), one obtains

$$\frac{f^*(z)}{\exp(xz)} = \frac{1}{f(x)}. \tag{3.12}$$

Substituting (3.12) in (3.11), follows that

$$z = \frac{f'(x)}{f(x)}.$$

To show sufficiency, let

$$z = \frac{f'(x)}{f(x)}. \tag{3.13}$$

Integrating both sides of (3.13), gives

$$\int_{x_0}^{x} z dx = \int_{x_0}^{x} \frac{f'(x)}{f(x)} dx$$

for all $x, x_0 \in I$. We observe that

$$\begin{aligned} z(x - x_0) &= \ln f(x) - \ln f(x_0) \\ \exp(xz - x_0 z) &= \frac{f(x)}{f(x_0)} \\ \frac{\exp xz}{f(x)} &= \frac{\exp x_0 z}{f(x_0)}. \end{aligned} \tag{3.14}$$

By Definition 2.1, we have that $T_{x_0}(x)$ is a support function of f at $x_0 \in I$, and from Proposition 2.2, we get

$$T_{x_0}(x) = f(x_0) \exp\left[(x - x_0)\frac{f'(x)}{f(x)}\right] \leq f(x) \quad \forall x \in I.$$

From (3.13) and $f(x) > 0$ for all $x \in I$, we receive

$$\begin{aligned} f(x_0)\exp(xz - x_0 z) &\leq f(x) \\ \frac{\exp xz}{f(x)} &\leq \frac{\exp x_0 z}{f(x_0)} \quad \forall x \in I \\ \sup_x \frac{\exp xz}{f(x)} &\leq \frac{\exp x_0 z}{f(x_0)} \quad \forall x \in I \\ f^*(z) &\leq \frac{\exp x_0 z}{f(x_0)} \quad \forall x \in I. \end{aligned} \tag{3.15}$$

Substituting (3.14) in (3.15), we obtain

$$f^*(z) \leq \frac{\exp xz}{f(x)}$$

$$f^*(z)f(x) \leq \exp(xz). \tag{3.16}$$

From (3.9) and (3.16), hence

$$\exp(xz) = f^*(z)f(x).$$

Remark 3.1. *If f is not differentiable, then $\exp(xz) = f(x)f^*(z)$, if, and only if, $z = K_{x,f}$, where $K_{x,f} \in [\frac{f'_-(x)}{f(x)}, \frac{f'_+(x)}{f(x)}]$.*

(k3): From (3.8) and the definition of supremum, we get

$$\frac{\exp(xz)}{f(x)} \leq f^*(z) \ \ \forall x \in I, \ \ \forall z \in I^*. \tag{3.17}$$

Since f^* is a sub E-function, we have that $f^*(z) \geq 0 \ \ \forall z \in I^*$, consequently from (3.17):

$$\frac{\exp(xz)}{f^*(z)} \leq f(x) \ \ \forall x \in I, \ \ \forall z \in I^*.$$

Then

$$\sup_z \left[\frac{\exp(xz)}{f^*(z)}\right] \leq f(x).$$

Using Definition 3.1 and the above inequality, we obtain that

$$f^{**}(x) \leq f(x), \tag{3.18}$$

for all $x \in I$ and for all $x \in I^{**}$. Since

$$f^{**}(x) = \sup_z \left[\frac{\exp(xz)}{f^*(z)}\right] \ \ \forall x \in I^{**}, \tag{3.19}$$

hence,

$$\frac{\exp(xz)}{f^*(z)} \leq f^{**}(x)$$

$$\exp(xz) \leq f^{**}(x)f^*(z).$$

By $f(x) \geq 0,$ and the above inequality, we have

$$\begin{aligned} \frac{\exp(xz)}{f(x)} &\leq \frac{f^{**}(x)f^{*}(z)}{f(x)} \\ \sup_x \left[\frac{\exp(xz)}{f(x)}\right] &\leq \frac{f^{**}(x)f^{*}(z)}{f(x)} \\ f^{*}(z) &\leq \frac{f^{**}(x)f^{*}(z)}{f(x)}. \end{aligned}$$

Hence,

$$f(x) \leq f^{**}(x). \tag{3.20}$$

From (3.18) and (3.20), we obtain that

$$f^{**}(x) = f(x), \tag{3.21}$$

for all $x \in I$ and for all $x \in I^{**}$. Consequently, $I = I^{**}$ and we have

$$f^{**} = f. \tag{3.22}$$

□

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