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I_2-LOCALIZED DOUBLE SEQUENCES IN METRIC SPACES

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ABSTRACT. In this article, the notions of I_2-localized and I_2^*-localized sequences in metric spaces are defined. Besides, we study some properties associated to I_2-localized and I_2-Cauchy sequences. On the other hand, we define the notion of uniformly I_2-localized sequences in metric spaces.

1. INTRODUCTION AND PRELIMINARIES

The notion of localized sequence was defined in [10] as a generalization of a Cauchy sequence in metric spaces. Besides, by using the properties of localized sequence and the locator of a sequence, many results take into account closure operators in metric spaces were obtained in [10]. If X is a metric space with a metric $d(\cdot,\cdot)$ and (x_n) be a sequence of points in X, we can call the sequence (x_n) to be localized in some subset $M \subset X$ if the number sequence $\alpha_n = d(x_n, x)$ converges for all $x \in X$. The maximal subset on which (x_n) is a localized sequence is said to be the locator of (x_n). Besides, if (x_n) is localized on X, then it becomes localized everywhere. If the locator of a sequence (x_n) contains all elements of this sequence, except of a finite number of elements, then (x_n) is

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said to be localized in itself. It is important to recall that, every Cauchy sequence in X is localized everywhere. In addition, if $B : X \to X$ is a function with the condition $d(B_x, B_y) \leq d(x, y)$ for all $x, y \in X$, thus for every $x \in X$ the sequence $(B^n x)$ is localized at every fixed point of the function B. This means that fixed points of the function B is contained in the locator of the sequence $(B^n x)$.

Otherwise, The concept of I-convergence of real sequences was defined by Kostyrko et al. [8] as a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers.

On the other hand, the notion of statistical convergence was introduced for double sequences by Muresaleen [12], as well as, Móricz [11] who defined it for multiple sequences. Recently, some works on I-convergence of double sequences have been studied (see [15], [7], [2]). Motivated by the notions mentioned above, Nabiev et al. [13] introduced the notion of I-localized sequences in metric spaces and they obtained some interesting results.

In this paper, the purpose is to generalize the concept of I-localized sequence by using the notions of ideal I of subset of the set $\mathbb{N}$ ($\mathbb{N}$ denotes the set of natural number) of positive integers and double sequences I_2-convergent.

Definition 1.1. *([8, 9]) Let X be a non-empty set, the family $I \subset 2^X$ is said to be and ideal if satisfies that if $A \subset I$ and $B \subset A$, then $B \in I$, besides if $A, B \in I$, then $A \cup B \in I$. Additionally, a non-empty family of subsets of $F \subset 2^X$ is a filter on X if satisfies that $\emptyset \in F$, if $A, B \in F$, then $A \cap B \in F$, moreover if $A \in F$ and $A \subset B$, then $B \in F$. In addition, an ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. Thus, $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X. Finally, a non-trivial ideal I is said to be admissible if $\{\{x\} : x \in X\} \subset I$.*

Remark 1.1. *Throughout this paper, I_2 is an admissible ideal on $\mathbb{N} \times \mathbb{N}$.*

Definition 1.2. *([1]) A double sequence $x = (x_{nm})$ of elements of X is said to be I_2-convergent to $L \in X$ if for every $\epsilon > 0$, $\{(n, m) \in \mathbb{N} \times \mathbb{N} : d(x_{nm}, L) \geq \epsilon\} \in I_2$, we write $I_2\text{-}\lim_{n,m\to\infty} x_{nm} = L$.*

Definition 1.3. *([14]) A double sequence $x = (x_{nm})$ of elements of X is said to be I_2-Cauchy sequence if for every $\epsilon > 0$ there exits $n_0 = n_0(\epsilon), m_0 = m_0(\epsilon) \in \mathbb{N}$ such that $\{(n, m) \in \mathbb{N} \times \mathbb{N} : d(x_{nm}, x_{n_0 m_0}) \geq \epsilon\} \in I_2$.*

Definition 1.4. *([1]) A double sequence $x = (x_{nm})$ of elements of X is said to be I_2^*-convergent to $L \in X$ if there exits $M \in F(I)$, i.e. $R = \mathbb{N} \times \mathbb{N} - M \in I_2$ such that $\lim_{k,i\to\infty} d(x_{n_k m_i}, L) = 0$ and $M = \{n_1 < ... < n_k; m_1 < ... < m_i\} \subset \mathbb{N} \times \mathbb{N}$.*

Definition 1.5. *([3]) A double sequence $x = (x_{nm})$ of elements of X is said to be I_2^*-Cauchy sequence if there exits a set $M = \{n_1 < ... < n_k; n_1 < ... < n_p; m_1 < ... < m_i; m_1 < ... < m_j\}$ such that $\lim_{k,i,p,j\to\infty} d(x_{n_k m_i}, x_{n_p m_j}) = 0$.*

We can see that I_2^*-convergent and I_2^*-Cauchy sequences imply I_2-convergent and I_2-Cauchy sequences, respectively. Moreover, if I is an ideal with property (AP2) (see [1]), then I and I^*-convergence coincide, as well as, I_2-Cauchy and I_2^*-Cauchy sequences coincide.

2. I_2 AND I_2^*-LOCALIZED SEQUENCES

In this section, (X, d) denotes a metric space and I_2 is a non-trivial ideal on $\mathbb{N} \times \mathbb{N}$.

Definition 2.1. *A double sequence (x_{nm}) of elements of X is said to be I_2-localized in the subset $M \subset X$ if for each $L \in M$, I_2-$\lim_{n,m\to\infty} d(x_{nm}, L)$ exits, i.e. the real number $\alpha_{nm} = d(x_{nm}, L)$ is I_2-convergent.*

Remark 2.1. *The maximal set on which a sequence (x_{nm}) is I_2-localized we will call the I_2-locator of (x_{nm}) and we will denote this set as $loc_{I_2}(x_{nm})$.*

Definition 2.2. *A double sequence (x_{nm}) is said to be I_2-localized everywhere if I_2-locator of (x_{nm}) coincides with X, i.e. $loc_{I_2}(x_{nm}) = X$*

Definition 2.3. *A double sequence (x_{nm}) is said to be I_2-localized in itself if $\{(n, m) \in \mathbb{N} \times \mathbb{N} : x_{nm} \notin loc_{I_2}(x_{nm})\} \subset I_2$.*

Remark 2.2. *By the Definitions mentioned above, we can imply that if (x_{nm}) is an I_2-Cauchy sequence, then it is I_2-localized elsewhere. In fact, since $|d(x_{nm}, L) - d(x_{n_0 m_0}, L)| \leq d(x_{nm}, x_{n_0 m_0})$. Then, we have that $\{(n, m) \in \mathbb{N} \times \mathbb{N} : |d(x_{nm}, L) - d(x_{n_0 m_0}, L)| \geq \epsilon\} \subset \{(n, m) \in \mathbb{N} \times \mathbb{N} : d(x_{nm}, x_{n_0 m_0}) \geq \epsilon\}$. This indicates that the sequence is I_2-localized, indeed it is I_2-Cauchy sequence.*

Remark 2.3. *By the Definitions mentioned above, we can also imply that every I_2-convergent sequence is I_2-localized.*

Remark 2.4. *If I_2 is an admissible ideal, then it is an open problem if every double localized sequence in X is I_2-localized sequence in X.*

Remark 2.5. *If X is a vector space, and (x_{nm}), (y_{nm}) are two I_2-localized sequences, then $(x_{nm}y_{nm})$, $(\dfrac{x_{nm}}{y_{nm}})$, where $y_{nm} \neq \emptyset$ and $(x_{nm} + y_{nm})$ are I_2-localized sequences.*

Definition 2.4. *A double sequence (x_{nm}) is said to be I_2^*-localized in a metric space X if the sequence $d(x_{nm}, L)$ is I_2^*-convergent for each $L \in X$.*

Remark 2.6. *By the above Definition, every I_2^*-convergent or I_2^*-Cauchy sequence in a metric space X is I_2^*-localized.*

Now, we show some results which were obtained taking into account the previously notions.

Lemma 2.1. *Let I_2 be an admissible ideal on $\mathbb{N} \times \mathbb{N}$ and X be a metric space. If a double sequence $(x_{nm}) \subset X$ is I_2^*-localized in the set $M \subset X$. Then, (x_{nm}) is I_2-localized in the set M and $loc_{I_2^*}(x_{nm}) \subset loc_{I_2}(x_{nm})$.*

Proof. Let (x_{nm}) be I_2^*-localized in M. Then, there exits a set $R \in I_2$ such that for $R^c = \mathbb{N} \times \mathbb{N} - R = \{k_1 < ...k_i; w_1 < ... < w_j\}$, we have that $\lim_{i,j\to\infty} d(x_{ij}, L)$, for each $L \in M$. Then, the sequence $d(x_{nm}, L)$ is an I_2^*-Cauchy sequence, this implies that $d(x_{nm}, L)$ is an I_2-Cauchy sequence. Therefore, the double number sequence $d(x_{nm}, L)$ is I_2-convergent. This means that (x_{nm}) is I_2-localized in the set M. □

Remark 2.7. *By the Lemma 2.1, we proved that $loc_{I_2^*}(x_{nm}) \subset loc_{I_2}(x_{nm})$, but under which conditions the equality is satisfied. This is an open problem.*

Proposition 2.1. *Every I_2-localized sequence is I_2-bounded.*

Proof. Let (x_{nm}) be I_2-localized. Then, the double number sequence $d(x_{nm}, L)$ is I_2-convergent for some $L \in X$. This implies that $\{(n,m) \in \mathbb{N} \times \mathbb{N} : d(x_{nm}, L) > U\} \in I_2$ for some $U > 0$. In consequence, the double sequence (x_{nm}) is I_2-bounded. □

Theorem 2.1. *Let I_2 be an admissible ideal with the (AP2) property and $P = loc_{I_2}(x_{nm})$. Besides, a point $L_1 \in X$ be such that for any $\epsilon > 0$ there exits $L \in P$ which satisfies*

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : |d(L, x_{nm}) - d(L_1, x_{nm})| \geq \epsilon\} \in I_2 \tag{2.1}$$

Then, $L_1 \in P$.

Proof. It will be sufficient if we show that the double number sequence $\alpha_{nm} = d(x_{nm}, L_1)$ is an I_2-Cauchy sequence. Now, let $\epsilon > 0$ and $L \in P = loc_{I_2}(x_{nm})$ is a point with the property (2.1). By the (AP2) property of I_2, we have that

$$|d(L, x_{k_n k_m}) - d(L_1, x_{k_n k_m})| \to 0 \text{ as } n, m \to \infty,$$

and

$$|d(x_{k_n k_m}, L) - d(x_{k_s k_t}, L) \to 0 \text{ as } n, m, s, t \to \infty,$$

where $R = \{k_1 < \cdots < k_n < \cdots ; k_1 < \cdots < k_m < \cdots\} \in F(I_2)$. Thus, for any $\epsilon > 0$ there is $n_0, m_0 \in \mathbb{N}$ such that

$$|d(L, x_{k_n k_m}) - d(L_1, x_{k_n k_m})| < \frac{\epsilon}{3} \tag{2.2}$$

and

$$|d(L, x_{k_n k_m}) - d(L, x_{k_s k_t})| < \frac{\epsilon}{3} \tag{2.3}$$

for all $n \geq n_0$, $m \geq m_0$, $s \geq n_0$ and $t \geq m_0$. Now, combing (2.2) and (2.3) with the following estimation

$$\begin{aligned} |d(L_1, x_{k_n k_m}) - d(L_1, x_{k_s k_t})| &\leq |d(L_1, x_{k_n k_m}) - d(L, x_{k_n k_m})| \\ &+ |d(L, x_{k_n k_m}) - d(L, x_{k_s k_t})| + |d(L, x_{k_s k_t}) - d(L_1, x_{k_s k_t})| \end{aligned}$$

We have that $|d(L_1, x_{k_n k_m}) - d(L_1, x_{k_s k_t})| < \epsilon$, for all $n \geq n_0$, $m \geq m_0$, $s \geq n_0$ and $t \geq m_0$, which give

$$|d(L_1, x_{k_n k_m}) - d(L_1, x_{k_s k_t})| \to 0 \text{ as } n, m, s, t \to \infty$$

for $R = (k_{nm}) \subset \mathbb{N} \times \mathbb{N}$ and $R \in F(I_2)$. This implies that $d(x_{nm}, L_1)$ is an I_2-Cauchy. □

Definition 2.5. *If X is a metric space. Then,*

(1) *The point L_1 is an I_2-limit point of the double sequence $(x_{nm}) \in X$ if there is a set $R = \{k_1 < ...k_i; w_1 < ...w_i\} \subset \mathbb{N} \times \mathbb{N}$ such that $R \notin I_2$ and $\lim_{k,w\to\infty} x_{n_k m_w} = L_1$.*

(2) *([14]) A point L_1 is said to be an I_2-cluster point of the double sequence (x_{nm}) if for each $\epsilon > 0$, $\{(n,m) \in \mathbb{N} \times \mathbb{N} : d(x_{nm}, L_1) < \epsilon\} \notin I_2$. Additionally, if $R = \{k_1 < ...; w_1 < ...\} \in I_2$, the double subsequence $(x_{k_n w_m})$ of the sequence (x_{nm}) is called I_2-thin subsequence of the double sequence (x_{nm}). Besides, if $M = \{s_1 < ...; t_1 < ...\} \notin I_2$, the double sequence $x_M = (x_{st})$ is called I_2-nonthin double subsequence of (x_{nm}).*

Proposition 2.2. *All I_2-limit points (I_2-cluster points) of the I_2-localized double sequence (x_{nm}) have the same distance from each point L of the locator $loc_{I_2}(x_{nm})$.*

Proof. If L_1 and L_2 are two I_2-limit points of the double sequence (x_{nm}). Then, the double numbers $d(L_1, L)$ and $d(L_2, L)$ are I_2-limit points of the I_2-convergent sequence $d(L, x_{nm})$. Therefore, $d(L_1, L) = d(L_2, L)$.

With I_2-cluster point is proved similarly. □

Proposition 2.3. *$loc_{I_2}(x_{nm})$ does not contain more than I_2-limit (I_2-cluster) point of the double sequence (x_{nm}).*

Proof. If $L, L_1 \in loc_{I_2}(x_{nm})$ are two I_2-limit points of the double sequence (x_{nm}), then by the Proposition 2.2, $d(L, L) = d(L, L_1)$. But, $d(L, L) = 0$. This implies that $d(L, L_1) = 0$ for $L \neq L_1$ and this is a contradiction.

With I_2-cluster point is proved similarly by using Proposition 2.2. □

Proposition 2.4. *If the double sequence (x_{nm}) has an I_2-limit point $L_1 \in loc_{I_2}(x_{nm})$. Then, $I_2\text{-}\lim_{n,m\to\infty} x_{nm} = L_1$.*

Proof. The double sequence $(d(x_{nm}, L_1))$ is I_2-convergent and some I_2-nonthin subsequence of this double sequence converges to zero. Then, (x_{nm}) is I_2-convergent to L_1. □

Definition 2.6. *For the given I_2-localized double sequence (x_{nm}), with the I_2-locator $P = loc_{I_2}(x_{nm})$, the number*

$$\sigma_2 = \inf_{L\in P}(I_2\text{-}\lim_{n,m\to\infty} d(L, x_{nm}))$$

is called the I_2-barrier of (x_{nm}).

Theorem 2.2. *Let $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ is an ideal with the property (AP2). Then, and I_2-localized double sequence is I_2-Cauchy if and only if $\sigma_2 = 0$.*

Proof. Let (x_{nm}) be an I_2-Cauchy double sequence in a metric space X. Then, there is a set $R = \{k_1 < .. < k_n; k_1 < ...k_m\} \subset \mathbb{N} \times \mathbb{N}$ such that $R \in F(I)$

and $\lim_{n,m,s,t\to\infty} d(x_{k_nk_m}, x_{k_sk_t}) = 0$. In consequence, for each $\epsilon > 0$ there exits $n_0, m_0 \in \mathbb{N}$ such that

$$d(x_{k_nk_m}, x_{k_{n_0}k_{m_0}}) < \epsilon \text{ for all } n \geq n_0 \text{ and } m \geq m_0.$$

Since (x_{nm}) is an I_2-localized double sequence, I_2-$\lim_{n,m\to\infty} d(x_{k_nk_m}, x_{k_{n_0}k_{m_0}})$ exist and we have that I_2-$\lim_{n,m\to\infty} d(x_{k_nk_m}, x_{k_{n_0}k_{m_0}}) \leq \epsilon$. Therefore, $\sigma_2 \leq \epsilon$, this is due to $\epsilon > 0$, we have that $\sigma_2 = 0$.

Now, let's prove the converse by taking $\sigma_2 = 0$. Then, for each $\epsilon > 0$ there is a $L \in loc_{I_2}(x_{nm})$ such that $d(L) = I_2$-$\lim_{n,m\to\infty} d(L, x_{nm}) < \frac{\epsilon}{2}$. In this case

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : |d(L) - d(L, x_{nm})| \geq \frac{\epsilon}{2} - d(L)\} \in I_2.$$

This implies that $\{(n,m) \in \mathbb{N} \times \mathbb{N} : d(L, x_{nm}) \geq \frac{\epsilon}{2}\} \in I_2$. Therefore, I_2-$\lim_{n,m\to\infty} d(L, x_{nm}) = 0$, this means that (x_{nm}) is an I_2-Cauchy double sequence. ☐

Theorem 2.3. *If the double sequence (x_{nm}) is I_2-localized in itself and (x_{nm}) contains an I_2-nonthin Cauchy subsequence, then (x_{nm}) will be an I_2-Cauchy double sequence itself.*

Proof. Let (y_{nm}) be an I_2-nonthin Cauchy subsequence. It might be assumed that all members of (y_{nm}) belong to the $loc_{I_2}(x_{nm})$. Since (y_{nm}) is a double Cauchy sequence, by the Theorem 2.2, $\inf_{y_{nm}} \lim_{s,t\to\infty} d(y_{st}, y_{nm}) = 0$. Otherwise, since (x_{nm}) is I_2-localized in itself, then

$$I_2\text{-}\lim_{s,t\to\infty} d(x_{st}, y_{nm}) = I_2\text{-}\lim_{s,t\to\infty} d(y_{st}, y_{nm}) = 0.$$

Therefore, the I_2-barrier of (x_{nm}) is equal to zero. Then, we have that (x_{nm}) is and I_2-Cauchy double sequence. ☐

Definition 2.7. *A double sequence (x_{nm}) in a metric space X is said to be uniformly I_2-localized on a subset $M \subset X$ if the double sequence $(d(L, x_{nm}))$ is uniformly I_2-convergent for all $L \in M$.*

Lemma 2.2. *Let (x_{nm}) be a double sequence uniformly I_2-localized on the set $M \subset X$ and $L_1 \in Y$ is such that for every $\epsilon > 0$ there is $L_2 \in M$ for which*

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : |d(L_1, x_{nm}) - d(L_2, x_{nm})| \geq \epsilon\} \in I_2$$

is satisfied. Then, $L_1 \in loc_{I_2}(x_{nm})$ and (x_{nm}) are uniformly I_2-localized on the set of such points L_1.

Proof. The prove of this Lemma is analogously to the Theorem 2.1. □

3. Conclusion

The purpose of this paper was to extend the notion of I-localized sequences in double sequences. As we could see, we obtained some interesting properties and results. Nevertheless, there are some open problems (see Remarks 2.4 and 2.7) which would be interesting whether we study them for future work. Furthermore, for future work, it would also be interesting whether we make a deeper study of the uniformly I_2-localized double sequence. On the other hand, it would also be interesting to see if these notions might be extended in triple sequences.

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