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COMBINATORIAL APPROACH TO ENUMERATE THE TOPOLOGIES ON FINITE SET (X_n)

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ABSTRACT. In this article, $Top(X_n)$ is the collection of all topologies on a given set X_n having cardinality n. We introduce the method to find the number of topologies of finite sets of cardinality up to 5 by combinations and counting principles, which enable us to enumerate the classes of homeomorphic disconnected topologies as well as the connected topologies. Moreover we study the graphical aspects of the connected and disconnected topologies.

1. INTRODUCTION

From a Combinatorial point of view, it is interesting to determine how many different topologies there are on a set X_n having n elements, denoted by $Top(X_n)$ and the class of homeomorphic topologies are denoted by τ_n where $n \geq 2$. We refer such kind of classes as homeomorphic classes. In this article τ_1 is fixed for discrete and τ_1' for anti-discrete topology . Moreover τ_0- topology is from Separation axioms of topology. The problem of enumerating the number of topologies of finite sets is considered both theoretically and computationally. Different Mathematicians used different techniques to find the answer of this long

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standing open problem. Stanley [1] used the correspondence between finite τ_0-topologies and partial order sets to find all non-homeomorphic topologies of sets of n elements. He determined that which topologies are τ_0 and which are not. Brikmann [2] has investigated the number $|Top(X_n)|$ of different topologies of set of n-elements by exhaustive enumeration for $n \leq 16$. The general question is still uncertain whether a formula for $|Top(X_n)|$ will ever be obtained, although asymptotic estimates exist. Erne, showed in [7] that $|Top(X_n)|$ is asymptotically equal to $|\tau_0|$, the number of τ_0-topologies (or equivalently partial orders) on set of n-elements.The enumeration of topologies on sets of n elements can be refined by counting $\tau(n,k)$, the number of topologies on n points having k open sets. The most important contributions are due to Erné and Stege, who in [4] computed the values of $\tau(n,k)$, for $n \leq 11$ and arbitrary k, by using Stirling formula of second kind $S(n,k)$, as well as the related numbers of τ_0 and connected topologies, and the corresponding numbers of homeomorphism classes. Further Benoumhani [3] find $\tau(n,k)$ for $k \leq 12$ and Erné and Stege computed the numbers $\tau(n,k)$ for $k \leq 23$ in [8].

In this article we introduced a new dimension to calculate $|Top(X_n)|$, where $n \leq 5$, by combinations and counting principles. Further this methodology is helpful to find out the enumerations of connected topologies as well as number of homeomorphic disconnected topologies. Further we induce the graphical approach in these connected and disconnected topologies which is helpful to elaborate this technique graphically.

For completeness, we recall some preliminaries and notations that are useful for the development of this paper. A topological space (X,τ) is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, (X,τ) is said to be connected. In this paper we use the symbols of $S(orS_i), D(orD_i), T(orT_i), F(orF_i), (i \in N)$ for set of one, two, three and four elements respectively. $< Y >$ is the symbol for describing all topologies by that set Y, and it is interesting observation that the class of topologies where $< Y >$ exists than these are collection of connected topologies, otherwise they are disconnected. Moreover $< Z >_d$ denotes the discrete topology of the set Z. Here τ_A, τ_D are denotation of anti-discrete and discrete topologies where τ_A is connected and τ_D is disconnected topology. For graphical representation we use τ_i, where $i \in N$ and $i \geq 2$, to be the vertices. The vertex τ_i make edge with τ_j for

$i \neq j$ when both are connected topologies A, similarly for disconnected topologies B. They makes complete bipartite graphs. Further, $G = A \times B$ becomes the complete graph of all classes of topologies excluding τ_1 and τ_1'.

We start our enumeration from $Top(X_3)$, since this calculation can be easily checked for $n < 3$.

2. Topologies on X_3 and their graphs

In this section, we make the classes of connected and disconnected topologies of X_3 and compute their numbers by combinations.

(1) $\tau_1 = \tau_A, \tau_1' = \tau_D$
Here τ_A, τ_D are in-discrete and discrete topologies, therefore $|\tau_1 \cup \tau_1'| = 2$.

(2) $\tau_2 = \{\phi, X, < S >\}$
Here S denotes the singleton sets, therefore $|\tau_2| = C_1^3 = 3$.

(3) $\tau_3 = \{\phi, X, < D >\}$
Here $< D >$ denotes the set of two elements and the number of all topologies of D are $| < D > | = |Top(X_2)| = 4$, therefore $|\tau_3| = C_2^3 \times 4 = 12$.

(4) $\tau_4 = \{\phi, X, , S, S^c\}$
Here S^c denotes the compliment of singleton sets, then by calculation we get that $|\tau_4| = C_1^3 = 3$.

(5) $\tau_5 = \{\phi, X, D_1, D_2, S\}$
Here we collect two D_i such that $D_1 \cap D_2 = S$ therefore $|\tau_3|$ depends upon the selection of S since it fix the selection of D_1, D_2 therefore by counting principle we have $|\tau_5| = C_1^3 = 3$.

(6) $\tau_6 = \{\phi, X, < D_1 >_d, D_2\}$
Here we have the same settings as in previous and we discrete each D individually, which gives that $|\tau_6| = C_1^3 \times 2 = 6$.

Here the total number of topologies are $|Top(X_3)|$ =29, and τ_4, τ_5, τ_6 are classes of disconnected homeomorphic topologies. In the graphical representation fig(B) is graph of classes of connected topologies, fig (A) is for disconnected topologies and fig (G) is product of A and B.

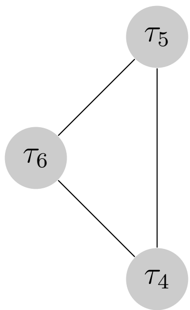


fig (A)

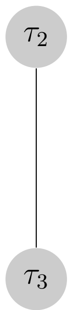
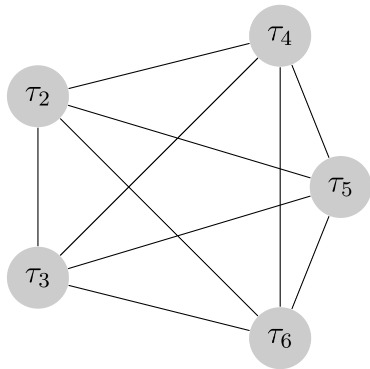


fig (B)



fig(G)

3. Topologies on X_4 and their graphs

In this section, we make the classes of connected and disconnected topologies of $Top(X_4)$ and compute their numbers by combinations.

(1) $\tau_1 = \tau_A, \tau_1' = \tau_D$
Here τ_A, τ_D are in-discrete and discrete topologies and therefore $|\tau_1 \cup \tau_1'| = 2$.

(2) $\tau_2 = \{\phi, X, < S >\}$
Here S denotes the singleton sets, therefore $|\tau_2| = C_1^4 = 4$.

(3) $\tau_3 = \{\phi, X, < D >\}$
Here we collect every topology of each D, whose collection is counted as $|\tau_3| = C_2^4 \times 4 = 24$ since $| < D > | = |\tau(X_2)| = 4$.

(4) $\tau_4 = \{\phi, X, < T >\}$
Here we collect every topology of each triton, which are $|\tau_4| = 4 \times 29 = 116$ since $| < T > | = |Top(X_3)| = 29$.

(5) $\tau_5 = \{\phi, X, S, S^c = T\}$
Here $S \nsubseteq T$ then by combination, we get that $|\tau_5| = C_1^4 = 4$.

(6) $\tau_6 = \{\phi, X, D, D^c\}$
Here we collect the topologies which includes ditons and its complements then $|\tau_6| = \frac{1}{2} \times C_2^3 = 3$.

In the following three collections (7-9), we have the setting where $D \nsubseteq T$ **and** $T \cap D = S$

(7) $\tau_7 = \{\phi, X, T, D, S\}$
Here $T \cap D = S$ implies $D \nsubseteq T$ then we can select S and D in ${}^4C_1$ and ${}^3C_2$ ways respectively therefore $|\tau_7| = C_1^4 \times C_2^3 = 12$.

(8) $\tau_8 = \{\phi, X, T, , D, D^c, S\}$
Here we get the same settings as of previous and include D^c then by same calculation we have $|\tau_8| = 12$.

(9) $\tau_9 = \{\phi, X, T, < D >_d\}$
We take discrete topology of ditons in this collection and get the same calculation to obtain $|\tau_9| = 12$.

From (10-16), we have settings of two tritons such that $T_1 \cap T_2 = D$

(10) $\tau_{10} = \{\phi, X, T_1, T_2, < D >\}$ Here we have two tritons such that $T_1 \cap T_2 = D$ then by combination we get that $|\tau_{10}| = C_2^4 \times 4 = 24$ since $| < D > | = |\tau(X_2)| = 4$.

(11) $\tau_{11} = \{\phi, X, T_1, T_2, D, < D^c >_d\}$ Here we include complement of D, as in $|\tau_8|$ and discrete it, this gives that $|\tau_{11}| = C_2^4 = 6$, as D can be selected in ${}^4C_2$ ways.

(12) $\tau_{12} = \{\phi, X, T_1, T_2, D, S\}$, where $T_1 \cap T_2 = D$ and $S \nsubseteq D$ which gives that $|\tau_{12}| = C_2^4 \times 2 = 12$.

(13) $\tau_{13} = \{\phi, X, T_1, T_2, D, D_1, S\}$, where we include D_1 in previous settings such that $D_1 \cap D = S$ and $D_1 \neq D^c$ then by calculation we get that $|\tau_{13}| = C_2^4 \times C_1^2 \times 2 = 24$.

(14) $\tau_{14} = \{\phi, X, T_1, T_2, < D >_d, D_1, S\}$ It has same setting of open sets as in previous and we discrete D separately here, which gives that $|\tau_{14}| = 24 \times 2$ = 48.

(15) $\tau_{15} = \{\phi, X, < T_1 >_d, T_2\}$ Here we discrete each T individually, and calculate that $|\tau_{15}| = C_3^4 \times 3 = 12$.

In the following classes (16-19), $D_1 \cap D_2 \cap D_3 = S$

(16) $\tau_{16} = \{\phi, X, T_1, T_2, T_3, D_1, D_2, D_3, S\}$ We take T_1, T_2, T_3 and their intersections so we get that$|\tau_{16}| = C_1^4 = 4$, since T_1, T_2, T_3 can be selected in ${}^4C_3$ ways and D_1, D_2, D_3 and S are constructed in natural way.

(17) $\tau_{17} = \{\phi, X, T_1, T_2, T_3, < D_1 >_d, D_2, D_3, S\}$ where $D_1 \cap D_2 \cap D_3 = S$ and discrete each diton individually then by combination we get that $|\tau_{17}| = 4 \times 3 = 12$

(18) $\tau_{18} = \{\phi, X, T_1, T_2, D, < T_3 >_d\}$ where $T_1 \cap T_2 = D$ which gives that $|\tau_{18}| = 4 \times 3 = 12$

(19) $\tau_{19} = \{\phi, X, T_1, T_2, T_3, D_1, D_2, D_3, S, D_4\}$
where $D_4 \neq D_i i \in \{1, 2, 3\}$ $and\ i \neq j$ this gives that S_1 could have three possibilities therefore $|\tau_{19}| = 4 \times 3 = 12$

Here the total number of topologies are 355, and τ_2, τ_3, τ_4 and τ_{10} are connected, while all others are disconnected. In the graphical representation fig(1) is graph of classes of connected topologies, fig (2) is for disconnected topologies and fig (3) is product of A and B.

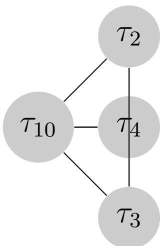


fig (1)

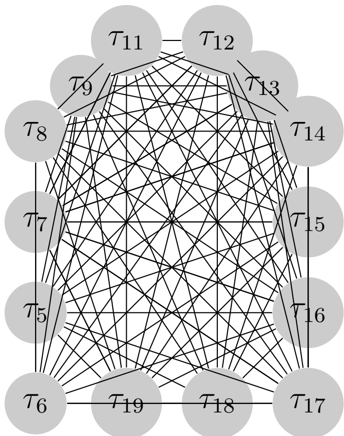
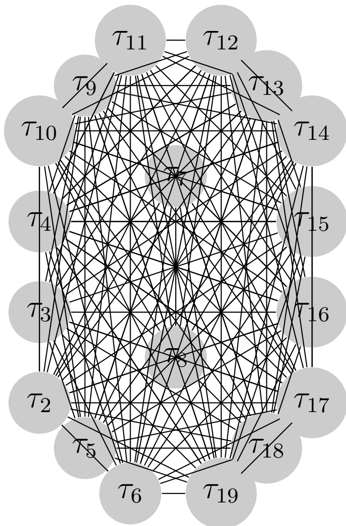


fig (2)



fig(3)

4. Topologies of X_5 $(Top(X_5))$

In this section, we make the classes of connected and disconnected topologies of τ_5 and compute their numbers by combinations.

(1) $\tau_1 = \tau_A, \tau_1' = \tau_D$ where τ_A, τ_D are in-discrete and discrete topologies, therefore $|\tau_1 \cup \tau_1'| = 2$.

(2) $\tau_2 = \{\phi, X, < S >\}$
$< S >$ denotes topologies of singleton open sets, hence by combination $|\tau_2| = C_1^5 = 5$, this implies that $|\tau_2| = 5$.

(3) $\tau_3 = \{\phi, X, < D >\}$
$< D >$ denotes all topologies of D sets of two elements, hence by combination $|\tau_3| = C_2^5 \times$ $|Top(X_2)|$=$C_2^5 \times 4$, this implies that $|\tau_3| = 40$.

(4) $\tau_4 = \{\phi, X, < T >\}$
$< T >$ denotes all topologies of T, sets of three elements, hence by combination $|\tau_4| = C_3^5 \times$number of all topologies of T=$C_3^5 \times 29$, this implies that $|\tau_4| = 290$.

(5) $\tau_5 = \{\phi, X, < F >\}$
$< F >$ denotes all topologies of F, the sets of four elements, hence by combination $|\tau_3| = C_4^5 \times$number of all topologies of F =$C_4^5 \times 355$, this implies that $|Top_5| = 1775$.

(6) $\tau_6 = \{\phi, X, S, S^c = F\}$
S^c denotes the compliment of singletons which becomes unique forton sets for each singleton, hence by combination $|\tau_6| = C_1^5 = 5$.

(7) $\tau_7 = \{\phi, X, D, D^c = [T]\}$
The selection of D is C_2^5 ways then D^c is selected in unique way, hence by combination $|\tau_7| = C_2^5 = 10$.

(8) $\tau_8 = \{\phi, X, T_1, T_2, S\}\ where\ T_1 \cap T_2 = S$
S is selected in C_1^5 ways and T_1, T_2 are selected in $\frac{1}{2}C_2^4$. Then by combination we have $|\tau_8| = C_1^5 \times \frac{1}{2}C_2^4 = 15$.

(9) $\tau_9 = \{\phi, X, T_1, T_2, S, D\}\ where\ T_1 \cap T_2 = S,\ and\ D \subseteq T_1 \cup T_2$
S is selected in C_1^5 ways and D is selected in $\frac{1}{2}C_2^4$ then T_1, T_2 exists in unique way. Then by combination we get that $|\tau_9| = C_1^5 \times C_2^4 = 30 (or C_3^5 \times C_2^3 = 30)$.

(10) $\tau_{10} = \{\phi, X, F, D, S\}\ where\ F \cap D = S,\ D \nsubseteq F$
F can be selected in C_4^5 ways as $F \cap D = S$ therefore D can be selected in C_1^4 ways. Then by combination we have $|\tau_{10}| = C_4^5 \times C_1^4 = 20$.

(11) $\tau_{11} = \{\phi, X, F, < D >_d\}\ where\ F \cap D = S,\ D \nsubseteq F$
Here $< D >_d$ is denotation of discrete topology of diton D. Then by using the previous settings of τ_{10} we have $|\tau_{11}| = C_1^5 \times C_3^4 = 20$.

(12) $\tau_{12} = \{\phi, X, F, D, D^c, S\}\ where\ F \cap D = S,\ D \nsubseteq F$
Then by settings of τ_{10} we get that $|\tau_{12}| = C_1^5 \times C_3^4 = 20$

From (13-20), we have collection of topologies with setting $F \cap T = D$

(13) $\tau_{13} = \{\phi, X, F, T, < D >\}$ *where* $F \cap T = D, i.e. T \nsubseteq F$
Here D can be selected in C_4^5 ways and D can be determined in C_2^4 ways then since $|\langle D \rangle| = 4$, therefore we get that $C_4^5 \times C_2^4 \times 4 = 120)$.

(14) $\tau_{14} = \{\phi, X, F, < T >_d\}$
where $F \cap T = D$ and $< T >_d$ denotes the discrete topology of triton, then as previous calculation we get that $|\tau_{14}| = C_4^5 \times C_2^4 = 30$.

(15) $\tau_{15} = \{\phi, X, F, T, D, D_1, S\}$ where $F \cap T = D$, $T = D \cup D_1$, and where then by calculations we have $|\tau_{15}| = C_1^5 \times C_1^4 \times C_1^3 = 60 (or C_2^5 \times C_2^3 \times C_1^2)$.

(16) $\tau_{16} = \{\phi, X, F, T, < D >_d, D_1\}$ In the settings of (15) $< D >_d$ is discrete topology on D_1 and by same calculations we get that $|\tau_{16}| = C_1^5 \times C_1^4 \times C_1^2 = 60$.

(17) $\tau_{17} = \{\phi, X, F, T, D, < D_1 >_d\}$. Here the same setting as (16) and same calculations give $|\tau_{17}| = 60$.

(18) $\tau_{18} = \{\phi, X, F, T, D, S = F^c\}$
so $|\tau_{18}| = C_4^5 \times C_2^4 = 30 = C_1^5 \times C_2^4$.

(19) $\tau_{19} = \{\phi, X, F, T, D, T^c = D_1\}$ where $F \cap T = D$, $T \nsubseteq F, D_1 \subseteq F$, and $D_1 \cap D = \phi$ then by combination, we have $|\tau_{19}| = C_4^5 \times C_2^4 = 30$.

(20) $\tau_{20} = \{\phi, X, F, T, D, F^c, T^c, D^c\}$ *where* F can be selected in C_4^5 ways, T can be selected in C_1^2 ways, while D is selected uniquely therefore $|\tau_{20}| = C_4^5 \times \frac{1}{2} C_2^4 = 15$.

From (21-24), we collect the topologies with one F and two T_i, where $T_1 \subseteq F,\ T_2 \nsubseteq F,\ T_1 \cap T_2 = S \text{ and } F \cap T_2 = D$

(21) $\tau_{21} = \{\phi, X, F, T_1, T_2, S, D\}$ then by combination we calculate $|\tau_{21}| = C_1^5 \times C_1^4 \times C_2^3 = 60 (or C_4^5 \times C_3^4 \times C_1^3)$.

(22) $\tau_{22} = \{\phi, X, F, T_1, T_2, S, < D >_d\}$ these are calculated as $|\tau_{22}| = C_1^5 \times C_1^4 \times C_2^3 = 60$.

(23) $\tau_{23} = \{\phi, X, F, T_1, T_2, S, < D >_d, T_1^c\}$ as in previous settings we get that $|\tau_{23}| = 60$.

(24) $\tau_{24} = \{\phi, X, F, T_1, T_2, S, D, T_2^c\}$ here we include the open sets T_2^c and by same calculations as previous, we have $|\tau_{24}| = 60$.

(25) $\tau_{25} = \{\phi, X, F, T_1, T_2, D, T_3, D_1, S\}$ Here we have three tritons such that $T_1, T_2 \subseteq F, T_3 \nsubseteq F, T_1 \cap T_3 = S, T_2 \cap T_3 = D_1 = F \cap T_3$,
$T_1 \cap T_2 \cap T_3 = S, T_1 \cap T_2 = D$ then F can be selected in C_4^5 ways, T_1, T_2 can

be selected in C_2^4 ways while T_3 can be selected in $C_2^3 \times C_1^2$ ways therefore $|\tau_{25}| = C_4^5 \times C_2^4 \times C_2^4 = 180$.

From (26-48), we have collections of topologies with two F_i where $i \in \{1, 2\}$ such that $F_1 \cap F_2 = T$.

(26) $\tau_{26} = \{\phi, X, F_1, F_2, < T >\}$ T can be selected in C_3^5 ways, F_1, F_2 can be selected in unique way therefore we get that $|\tau_{26}| = C_3^5 \times |Top(X_3)| = C_3^5 \times 29 = 290$.
(27) $\tau_{27} = \{\phi, X, F_1, F_2, T, S\}$ where $S \nsubseteq T$ then by using combination we have $|\tau_{27}| = C_3^5 \times 2 = 20$.
(28) $\tau_{28} = \{\phi, X, F_1, F_2, T, D, S\}$ where $D \nsubseteq T, T \cap D = S$ then by T can be selected in C_3^5 ways and D can be selected in $C_1^3 \times C_1^2$ ways, hence we get that $|\tau_{28}| = C_1^5 \times C_3^5 \times 2 = 60$ $(or\ C_3^5 \times C_1^3 \times C_1^2)$.
(29) $\tau_{29} = \{\phi, X, F_1, F_2, T, < D >_d\}$then as previous setting of open sets except of $< D >_d$ and same calculations we obtain $|\tau_{29}| = 60$.
(30) $\tau_{30} = \{\phi, X, F_1, F_2, T, < T^c = D >_d\}$. Using previous settings we get that $|\tau_{30}| = C_3^5 = 10$.
(31) $\tau_{31} = \{\phi, X, < F_1 >_d, F_2\}$ then as above, therefore $|\tau_{31}| = C_3^5 \times 2 = 20$, since F_1 can be replaced by F_2.
(32) $\tau_{32} = \{\phi, X, F_1, F_2, T, < T_1 >_d\}$ where$F_1 \cap F_2 = T, T_1 \neq T$ as T_1 can be selected in unique way therefore we have $|\tau_{32}| = C_3^5 \times 9 = 90$.
(33) $\tau_{33} = \{\phi, X, F_1, F_2, T, T_1, D_1, D_2, S\}$ $where T \cap T_1 = S = D_1 \cap D_2, D_1 \cup D_2 = T_1 F_i \cap T_1 = D_i, i \in \{1, 2\}$ then we calculate that $|\tau_{33}| = C_1^5 \times C_2^4 \times 2 = 60 (or C_3^5 \times C_1^3 \times 2) = 60$.
(34) $\tau_{34} = \{\phi, X, F_1, F_2, T, T_1, S, < D_1 >_d, D_2\}$ Here we have same setting as in previous and taking discrete topology of D_1, D_2 separately, hence we have $|\tau_{34}| = 60 \times 2 = 120$.
(35) $\tau_{35} = \{\phi, X, F_1, F_2, T, T_1, S, D_1, D_2, T_1^c\}$. In view of previous setting, we get that $|\tau_{35}| = C_4^5 \times C_3^4 \times C_1^3 = 60$.
(36) $\tau_{36} = \{\phi, X, F_1, F_2, T, T_1, < D >\}$ where $T \cap T_1 = D$. As $|\langle D \rangle| = 4$ then $|\tau_{36}| = C_2^5 \times C_1^3 \times C_1^2 \times 4 = 240 (or C_4^5 \times C_3^4 \times C_2^3 \times 4)$.
(37) $\tau_{37} = \{\phi, X, F_1, F_2, T, T_1, D, S\}$ then include S in previous settings such that $S \nsubseteq D, S \subseteq T_1 or T_2$ then $|\tau_{37}| = C_3^5 \times C_2^3 \times C_1^2 \times 2 = 120 (or C_4^5 \times C_3^4 \times C_2^3 \times 2) = 120$.

(38) $\tau_{38} = \{\phi, X, F_1, F_2, T, T_1, D, < D_1 >_d\}$ keeping in view the previous setting, we include D_1 such that $D_1 = (T_1 \cup T) - D$ then D_1 can be selected in unique way therefore we calculate that $|\tau_{38}| = C_3^5 \times C_2^3 \times C_1^2 = 60 (or C_4^5 \times C_3^4 \times C_2^3) = 60$.

(39) $\tau_{39} = \{\phi, X, F_1, F_2, T, T_1, D, T_1^c, S\}$. Here we include τ_c in previous settings then $T \cap T_1^c = S$ and the number of these topologies are $|\tau_{39}| = C_3^5 \times C_2^3 \times C_1^2 = 60 (or C_4^5 \times C_3^4 \times C_1^3)$.

(40) $\tau_{40} = \{\phi, X, F_1, F_2, T, T_1, D, D_1, S\}$. If $D_1 \cap D = S$ then keeping in view of previous setting then we observe that T can be selected in C_3^5 ways and T_1 can be selected in $C_2^3 \times C_1^2$ ways and D_1 can be selected in $C_1^2 \times C_1^2$ ways therefore $|\tau_{40}| = C_3^5 \times C_2^3 \times C_1^2 \times C_1^2 \times C_1^2 = 240 (or 2 \times C_1^5 \times C_1^4 \times C_1^3 \times C_1^2)$.

(41) $\tau_{41} = \{\phi, X, F_1, F_2, T, T_1, < D >_d, D_1\}$ By keeping in view the previous settings and construct discrete topologies of D, D_1 separately we have $|\tau_{41}| = 240 \times 2 = 480$.

(42) $\tau_{42} = \{\phi, X, F_1, F_2, T, T_1, D, D_1, S_1, D_2, S_2\}$ where $D_1, D_2 \subseteq T \cup T_1$, $D_1 \cap D_2 = \phi, D \cap D_i = S_i \ for \ i \in \{1, 2\}, S_1 \cup S_2 = D$ then we observe that S_1, S_2 can be selected in $C_1^5 \times C_1^4$ ways and D_1 can be selected in $C_1^3 \times C_1^2$ and D_2 can be selected in C_1^2 ways therefore we have $|\tau_{42}| = C_1^5 \times C_1^4 \times C_1^3 \times C_1^2 \times 2 = 240 (or C_4^5 \times C_3^4 \times C_2^3 \times C_1^2 \times 2)$.

(43) $\tau_{43} = \{\phi, X, F_1, F_2, < T >_d, T_1\}$ where $F_1 \cap F_2 = T, T_1 \cap T = D$ then we have $|\tau_{43}| = C_4^5 \times C_3^4 \times C_2^3 = 60 (or C_3^5 \times C_2^3 \times C_1^2)$.

(44) $\tau_{44} = \{\phi, X, F_1, F_2, T, T_1, D_1, T_2, D_2, S\}$ where $T \cap T_i = D_i \ for \ i \in \{1, 2\}$ and $D_1 \cap D_2 = S = T_1 \cap T_2$ then we observe that S can be selected in C_1^5 ways and D, D_1 can be selected in $C_1^4 \times C_1^3$ while T_1, T_2 and F_1, F_2 can be constructed in unique way under given construction therefore we have $|\tau_{44}| = C_1^5 \times C_1^4 \times C_1^3 = 60$.

(45) $\tau_{45} = \{\phi, X, F_1, F_2, T, T_1, D, D^c = T_2, D_1, < D_2 >_d\}$ where $T \cap T_1 = D,\ F_1 \cap T_2 = D_1, F_2 \cap T_2 = D_2$ such that $D_2 \subset T_1 \cup T_2$ then by calculations we get that $|\tau_{45}| = C_3^5 \times C_2^3 \times C_1^2 = 60 (or C_4^5 \times C_3^4 \times C_2^3)$.

(46) $\tau_{46} = \{\phi, X, F_1, F_2, T, T_1, D_1, D_1', S_1, T_2, D_2, S_2\}$. Here we have $T \cap T_2 = D_2, T_1 \cap T_2 = S_2, T_1 \cap T = S_1, D_1 \cap D_1' = S_1$ then under given construction, T can be selected in C_3^5 ways, T_1, T_2 can be selected in $C_1^3 \times C_1^2$ ways such that $D = T \cap T_2$ and $D_2^c = T_1$, then selection of F_1, F_2 are collected in unique way and all the entries of τ_{46} are consequences of unions and intersection processes therefore $|\tau_{46}| = C_3^5 \times C_1^3 \times C_1^2 = 60 (or C_4^5 \times C_3^4 \times C_2^3)$.

(47) $\tau_{47} = \{\phi, X, F_1, F_2, T, T_1, T_2, D_1, D_2, D_3, S\}$ Here we select two tritons T_1 and T_2 which satisfies $T_2 \cap T = D_2, T_1 \cap T_2 = D_3, T_1 \cap T = D_1$ *and* $D_1 \cap D_2 \cap D_3 = S$ then we calculate that $|\tau_{47}| = C^5_1 \times C^4_1 \times C^3_1 \times C^2_1 = 120 (or C^5_3 \times C^3_2 \times C^2_1 \times C^2_1) = 120$.

(48) $\tau_{48} = \{\phi, X, F_1, F_2, T, T_1, T_2, < D_1 >_d, D_2, D_3, S\}$ Here we have same settings as in last one, except the discrete ditons each time seperately, and we get that $|\tau_{48}| = 120 \times 3 = 360$.

From (49 to 61), we select the homeomorphic topologies of F_1, F_2, F_3 such that their intersections are three T_1, T_2, T_3

(49) $\tau_{49} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, < D >\}$ where $T_1 \cap T_2 \cap T_3 = D$ and construction of is C^5_2 ways and the remaining can be constructed in unique ways therefore $|\tau_{49}| = C^5_2 \times 4 = 40$.

(50) $\tau_{50} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, < T_3 >_d\}$ where $T_1 \cap T_2 \cap T_3 = D$ and we discrete each T_i separately, which gives that $|\tau_{50}| = 10 \times 3 = 30$

(51) $\tau_{51} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, < D^c >_d\}$ Here we take same setting as previous and include D^c and discrete it, which gives that $|\tau_{51}| = C^5_2 = 10$

(52) $\tau_{52} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, S\}$ where we include a singleton S such that $S \nsubseteq D$ By combination we have $|\tau_{52}| = C^5_2 \times C^3_1 = 30$

(53) $\tau_{53} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, < D_1 >_d\}$ We include an extra diton D_1 in τ_{51} such that $D \cap D_1 = \phi$ then these type of topologies are $|\tau_{53}| = C^5_2 \times C^3_2 = 30$

(54) $\tau_{54} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, D_1, S\}$ Here we include an extra D_1 in τ_{51} such that $D \cap D_1 = S$ then we calculate that $|\tau_{54}| = C^5_2 \times C^2_1 \times C^3_1 = 60$

(55) $\tau_{55} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, < D_1 >_d\}$ Here, we have same setting as in previous and discrete each D, D_1 separately, which gives us $|\tau_{55}| = 60 \times 2 = 120$

(56) $\tau_{56} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, T_4, D_1, D_2, S\}$ Here we include a T_4 in τ_{51} such that $T_1 \cap T_4 = D_1, T_2 \cap T_4 = D_2, T_3 \cap T_4 = S = T_4 \cap D = S$ then τ_4 can be selected in C^3_2 ways, we get that $|\tau_{56}| = C^5_2 \times C^2_1 \times C^3_2 = 60 (or 5 \times C^4_1 \times C^3_2)$

(57) $\tau_{57} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, < D >_d, T_4, D_1, D_2\}$ Here we discrete each diton from previous settings which gives that $|\tau_{57}| = 60 \times 3 = 180$

(58) $\tau_{58} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, < T_4 >_d\}$ Here we discrete T_4 and this gives $|\tau_{58}| = 60$

(59) $\tau_{59} = \{\phi, X, F_1, F_2, F_3, < T_1 >_d, T_2, T_3, D, T_4, D_1, D_2\}$ This have same setting as of τ_{56} and if we discrete τ_1 and τ_2. We cannot discrete T_3 since it disturbs the construction of τ_{60} by increasing the numbers of $F$and $|\tau_{59}| = 60 \times 2 = 120$

(60) $\tau_{60} = \{\phi, X, F_1, F_2, F_3, < T_1 >_d, < T_2 >_d, T_3, D, T_4, D_1, D_2\}$ Here we discrete both τ_1 and τ_2 at the same time, while combination gives us $|\tau_{60}| = 60$

(61) $\tau_{61} = \{\phi, X, F_1, F_2, F_3, T_1, T_2, T_3, D, D_1, D_2, S_1, S_2\}$ Here $D_1 \cap D_2 = \phi, D \cap D_i = S_i$ which gives that $|\tau_{61}| = C^5_2 \times C^3_2 \times C^2_1 = 60(or C^5_1 \times \frac{1}{2} C^4_1 \times C^3_1 \times C^2_1)$

(62) $\tau_{62} = \{\phi, X, F_1, F_2, F_3, F_4, T_1, T_2, T_3, T_4, T_5, T_6, D_1, D_2, D_3, D_4, S\}$ Here we take four $F's$ and their intersection are six $T's$ whose intersections are four $D's$. By calculations we get that $|\tau_{62}| = C^5_1 = 5$

(63) $\tau_{63} = \{\phi, X, F_1, F_2, F_3, F_4, T_1, T_2, T_3, T_4, T_5, T_6, D_1, D_2, D_3, < D_4 >_d, S\}$ Here we discrete each $D_i,\ i \in \{1, 2, 3, 4\}$ separately and get that $|\tau_{63}| = 5 \times 4 = 20$

(64) $\tau_{64} = \{\phi, X, F_1, F_2, F_3, F_4, T_1, T_2, T_3, T_4, T_5, < T_6 >_d, D_1, D_2, D_3, D_4, S\}$ Here we discrete each $T_i,\ i \in \{1, 2, 3, 4, 5, 6\}$ separately and get that $|\tau_{64}| = 5 \times 6$=30

(65) $\tau_{65} = \{\phi, X, F_1, F_2, F_3, < F_4 >_d, T_1, T_2, T_3, T_4, T_5, T_6, D_1, D_2, D_3, D_4, S\}$ Here we discrete each $F_i, i \in \{1, 2, 3, 4\}$ separately and get that $|\tau_{65}| = 5 \times 4 = 20$

From the above discussion we conclude the following.

Theorem 4.1. *$Top(X_5)$ can be partitioned in to 57 homeomorphic disconnected classes and $|Top(X_5)| = 6942$.*

We conclude this paper with the following interesting open problem.

Open Problem 1: We believe that our method of finding topologies have potential to provide line of direction for development of Algorithm which is indeed responsible for finding topologies on a given set X_n having cardinality n.

Open Problem 2: From group theory it follows that the action of a group G on a given set X partition the set X into equivalence classes (refereed as orbits). Des it possible to identify a group G which decompose $Top(X_n)$ into homeomorphic classes.

5. Conclusion

This article has provided combinatorial approach to enumeration of topologies. An interesting calculations of number of topologies through combinations provides the homeomorphic classes of topologies. The side result of this computational work gives interesting graphical approach to these homeomorphic classes. This method of finding topologies pave a new path for developing the computational algorithm to find topologies of set of cardinality n.

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