Advances in Mathematics: Scientific Journal **10** (2021), no.7, 2933–2946
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.7.2

THE FINITENESS OF THE NUMBER OF EIGENVALUES OF THE FOUR-PARTICLE SCHRÖDINGER OPERATOR WITH THREE-PARTICLE CONTACT INTERACTION

Islom N. Bozorov[1], Usmon R. Shadiev, and Gayerat R. Yodgorov

ABSTRACT. In this paper, we consider the four-particle Schrödinger operator corresponding to the Hamiltonian of a system of four arbitrary quantum particles via a three-particle contact interaction potential on a three-dimensional lattice. The finiteness of the number of eigenvalues of the Schrödinger operator lying to the left of the essential spectrum for zero value of the total quasi-momentum is proved.

1. INTRODUCTION

The spectral properties of many-particle Schrödinger operators in Euclidean space are well studied in the books [9], [15] and [14]. For a system of many particles with arbitrary strongly decreasing interactions, the finiteness of the discrete spectrum was established by Zhislin [4], [5] and Yafaev [17].

In the articles [1] and [8], the finiteness of the three-particle bound states is proved for the three-dimensional discrete Schrödinger operator on the condition that the operators describing the two-particle subsystems have no virtual levels.

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 44A55, 81Q10.

Key words and phrases. Schrödinger operator, essential spectrum, eigenvalue, compact operator, positive operator.

Submitted: 22.06.2021; *Accepted:* 08.07.2021; *Published:* 10.07.2021.

The finiteness of the discrete spectrum of Schrödinger operators corresponding to the systems of the three-particles are investigated in [7], [10] and [11].

In the paper [2], the discrete Schrödinger operator acting in the Hilbert space is considered, which corresponds to the Hamiltonian of a system of four identical particles (bosons) interacting via pair contact attraction potentials on the lattice. It is proved that the number of eigenvalues lying to the left of the essential spectrum is finite for any value of $\mu > 0$ (μ is the interaction energy of two bosons), and their absence is established for sufficiently small $\mu > 0$.

In this paper, we consider the four-particle Schrödinger operator corresponding to the Hamiltonian of a system of four arbitrary quantum particles with a three-particle contact interaction potential on a three-dimensional lattice. The finiteness of the number of eigenvalues of the Schrödinger operator lying to the left of the essential spectrum is proved for any value of $\mu_\alpha > 0$, $\alpha = 1, ..., 4$ (μ_α is the interaction energy of three particles) and zero value of the total quasi-momentum.

2. Formulation of the main results

Let $L_2((\mathbb{T}^3)^3)$ be the Hilbert space of square-integrable functions defined on $(\mathbb{T}^3)^3$, $\mathbb{T} = (-\pi, \pi]$.

We consider the family of operators $H(K)$, $K \in \mathbb{T}^3$ (four-particle discrete Schrödinger operator) corresponding to the Hamiltonian of a system of four arbitrary quantum particles with a three-particle contact interaction potential on a three-dimensional lattice, where $H(K)$ acts in the Hilbert space $L_2((\mathbb{T}^3)^3)$ by the formula (see [13], [16])

$$H(K) = H_0(K) - \sum_{\alpha=1}^{4} \mu_\alpha V_\alpha,$$

μ_α is an interaction energy of particles β, γ and θ, $\beta < \gamma < \theta$, $\{\alpha, \beta, \gamma, \theta\} = \{1, 2, 3, 4\}$, the operators $H_0(K)$ and V_α are defined by the formulas

$$\big(H_0(K)f\big)(\mathbf{p}) = \mathcal{E}_K(\mathbf{p})f(\mathbf{p}), \quad \mathbf{p} = (p_1, p_2, p_3), \quad p_i \in \mathbb{T}^3,$$

$$\mathcal{E}_K(\mathbf{p}) = \sum_{i=1}^{3} \varepsilon_i(p_i) + \varepsilon_4(K - p_1 - p_2 - p_3), \quad \varepsilon_\alpha(p) = \frac{1}{m_\alpha} \sum_{i=1}^{3} (1 - \cos p_i), \quad \alpha = 1, ..., 4,$$

m_i mass i-th particle,

$$(V_1 f)(\mathbf{p}) = (2\pi)^{-6} \int\limits_{(\mathbb{T}^3)^2} f(p_1, q_2, q_3) dq_2 dq_3,$$

$$(V_2 f)(\mathbf{p}) = (2\pi)^{-6} \int\limits_{(\mathbb{T}^3)^2} f(q_1, p_2, q_3) dq_1 dq_3,$$

$$(V_3 f)(\mathbf{p}) = (2\pi)^{-6} \int\limits_{(\mathbb{T}^3)^2} f(q_1, q_2, p_3) dq_1 dq_2,$$

$$(V_4 f)(\mathbf{p}) = (2\pi)^{-6} \int\limits_{(\mathbb{T}^3)^2} f(q_1, q_2, p_1 + p_2 + p_3 - q_1 - q_2) dq_1 dq_2.$$

We denote by $A^{\frac{1}{2}}$ the positive square root of the operator $A \geq 0$, and by $\sigma(A)$ and $\sigma_{ess}(A)$, respectively, the spectrum and the essential spectrum of the operator A.

Note that the operator V_α is a positive operator and the equality $V_\alpha^{\frac{1}{2}} = V_\alpha$, $\alpha = 1, ..., 4$ holds, i.e. V_α is projector. Therefore, the positive square root of the operator $\mu_\alpha V_\alpha$ is equal to the operator $\sqrt{\mu_\alpha} V_\alpha$, $\alpha = 1, ..., 4$.

Let $H_\alpha(K) = H_0(K) - \mu_\alpha V_\alpha$. The operator $H_\alpha(K)$ is called the *channel operator* corresponding to the Hamiltonian of the system $\{\alpha\}$, $\{\beta, \gamma, \theta\}$, $\{\alpha, \beta, \gamma, \theta\} = \{1, 2, 3, 4\}$.

We set

$$W_\alpha := W_\alpha(K, z) = (I - \mu_\alpha V_\alpha^{\frac{1}{2}} R_0 V_\alpha^{\frac{1}{2}})^{-1}, \quad R_0 := R_0(K, z) = (H_0(K) - zI)^{-1},$$

where $z \notin \sigma(H_\alpha(K))$, $\alpha = 1, ..., 4$, I is identity operator in $L_2((\mathbb{T}^3)^3)$.

By virtue of the lemma 3.1 (see below) the operator $W_\alpha(K, z)$ exists if and only if $z \notin \sigma(H_\alpha(K))$, in addition $W_\alpha(K, z)$ is a positive operator for all $z < \tau_\alpha(K)$, $\tau_\alpha(K) = \inf \sigma_{ess}(H_\alpha(K))$.

Let $\tau_K = \inf \bigcup\limits_{\alpha=1}^{4} \sigma\big(H_\alpha(K)\big)$. We define the matrix operators acting in the Hilbert space $L_2^{(4)}((\mathbb{T}^3)^3) = L_2((\mathbb{T}^3)^3) \otimes L_2((\mathbb{T}^3)^3) \otimes L_2((\mathbb{T}^3)^3) \otimes L_2((\mathbb{T}^3)^3)$ as

$$\mathbf{A}(K, z) = \mathbf{W}(K, z) \mathbf{L}(K, z), \qquad z \in \mathbb{C} \setminus \bigcup_{\alpha=1}^{4} \sigma\big(H_\alpha(K)\big),$$

$$\mathbf{B}(K, z) = \mathbf{W}^{\frac{1}{2}}(K, z) \mathbf{L}(K, z) \mathbf{W}^{\frac{1}{2}}(K, z), \quad z < \tau_K,$$

where

$$\mathbf{L}(K,z)=\begin{pmatrix} 0 & \sqrt{\mu_1\mu_2}V_1R_0V_2 & \sqrt{\mu_1\mu_3}V_1R_0V_3 & \sqrt{\mu_1\mu_4}V_1R_0V_4 \\ \sqrt{\mu_2\mu_1}V_2R_0V_1 & 0 & \sqrt{\mu_2\mu_3}V_2R_0V_3 & \sqrt{\mu_2\mu_4}V_2R_0V_4 \\ \sqrt{\mu_3\mu_1}V_3R_0V_1 & \sqrt{\mu_3\mu_2}V_3R_0V_2 & 0 & \sqrt{\mu_3\mu_4}V_3R_0V_4 \\ \sqrt{\mu_4\mu_1}V_4R_0V_1 & \sqrt{\mu_4\mu_2}V_4R_0V_2 & \sqrt{\mu_4\mu_3}V_4R_0V_3 & 0 \end{pmatrix},$$

$$\mathbf{W}(K,z)=\begin{pmatrix} W_1 & 0 & 0 & 0 \\ 0 & W_2 & 0 & 0 \\ 0 & 0 & W_3 & 0 \\ 0 & 0 & 0 & W_4 \end{pmatrix}.$$

Note that each nonzero element $V_iR_0(K,z)V_j,\ i\neq j$ of the matrix operator $\mathbf{L}(K,z)$ is an integral operator and its kernel is a continuous function on the $(\mathbb{T}^3)^3\times(\mathbb{T}^3)^3$ for each $z\notin\mathbb{C}\backslash\bigcup_{i=1}^4\sigma\big(H_i(K)\big)$. Therefore $V_iR_0(K,z)V_j$ belongs to the Hilbert-Schmidt class. Hence, the operator $\mathbf{L}(K,z)$ is compact. It follows from the boundedness of $\mathbf{W}(K,z)$ that the operator $\mathbf{A}(K,z)$ is compact [16].

Remark 2.1. *The equation $\mathbf{A}(K;z)\varphi=\varphi$ is an analogue of the Faddeev equation, which is obtained for the three-particle continuous Schrödinger operator (see [9]).*

Note that for any $K\in\mathbb{T}^3$ the essential spectrum $\sigma_{ess}\big(H(K)\big)$ of the operator $H(K)$ coincides with the union of the spectra channel operators (see [12], [16]), i.e.

$$\sigma_{ess}\big(H(K)\big)=\bigcup_{\alpha=1}^{4}\sigma\big(H_\alpha(K)\big),$$

$$\sigma(H_\alpha(K))=\sigma_{ess}(H_\alpha(K))=\bigcup_{p\in\mathbb{T}^3}\Big\{\sigma\big(h_\alpha(K-p)\big)+\varepsilon_\alpha(p)\Big\}, \tag{2.1}$$

where the notation $A+b$ means that $A+b=\{\lambda+b:\ \lambda\in A\}$, the operator $h_\alpha(k)$ acts in the space $L_2((\mathbb{T}^3)^2)$ by

$$\big(h_\alpha(k)f\big)(p_\beta,p_\gamma)=\big[\varepsilon_\beta(p_\beta)+\varepsilon_\gamma(p_\gamma)+\varepsilon_\theta(k-p_\beta-p_\gamma)\big]f(p_\beta,p_\gamma)-$$

$$-\frac{\mu_\alpha}{(2\pi)^6}\int\limits_{(\mathbb{T}^3)^2}f(q_1,q_2)dq_1dq_2,\quad \{\alpha,\beta,\gamma,\theta\}=\{1,2,3,4\}.$$

The main results of this work are the following statements.

Lemma 2.1. *Suppose that the family of operators* $\mathbf{B}(K,z)$, $z<\tau_K$ *uniformly converges to some operator* $\mathbf{B}(K,\tau_K)$ *as* $z\to\tau_K-0$. *Then the operator* $H(K)$ *on the interval* $(-\infty,\tau_K)$ *can have only a finite number of eigenvalues.*

Using the Lemma 2.1, we obtain

Theorem 2.1. *For all* $\mu=(\mu_1,\mu_2,\mu_3,\mu_4)$, *where* $\mu_\alpha\geq 0$, $\alpha=1,...,4$ *the operator* $H(\mathbf{0})$, $\mathbf{0}=(0,0,0)$ *can have only a finite number of eigenvalues lying to the left of the essential spectrum.*

3. Proof of the main results

The rest of the work is devoted to the proof of the main results.

Lemma 3.1. *The operator* $W_\alpha(K,z)$ *exists if and only if* $z\notin\sigma\big(H_\alpha(K)\big)$, *in addition,* $W_\alpha(K,z)$ *is a positive operator for all* $z<\tau_\alpha(K)$.

Proof. Let $z\notin\sigma_{ess}\big(H_\alpha(K)\big)$. Since

$$H_0(K)-V_\alpha-zI=\big(H_0(K)-zI\big)\big[I-R_0(K,z)V_\alpha\big],$$

the operator $H_\alpha(K)-zI$ is invertible if and only if $I-R_0(K,z)V_\alpha$ is invertible. The equality $\sigma\big(R_0(K,z)V_\alpha\big)\setminus\{0\}=\sigma\big(V_\alpha^{\frac{1}{2}}R_0(K,z)V_\alpha^{\frac{1}{2}}\big)\setminus\{0\}$ gives that $1\in\sigma\big(R_0(K,z)V_\alpha\big)$ if and only if $1\in\sigma(V_\alpha^{\frac{1}{2}}R_0(K,z)V_\alpha^{\frac{1}{2}})$. Therefore, the operator $I-R_0(K,z)V_\alpha$ is invertible if and only if the operator $I-V_\alpha^{\frac{1}{2}}R_0(K,z)V_\alpha^{\frac{1}{2}}$ is invertible. Since $V_\alpha^{\frac{1}{2}}R_0(K,z_1)V_\alpha^{\frac{1}{2}}<V_\alpha^{\frac{1}{2}}R_0(K,z_2)V_\alpha^{\frac{1}{2}}$ for $z_1<z_2<m_K$, we have $I-V_\alpha^{\frac{1}{2}}R_0(K,z_1)V_\alpha^{\frac{1}{2}}>I-V_\alpha^{\frac{1}{2}}R_0(K,z_2)V_\alpha^{\frac{1}{2}}$.

Hence we obtain the assertion of the Lemma 3.1. □

We denote by $N(K;z)$, $z<\tau_K$ the number of eigenvalues of the operator $H(K)$ lying to the left of z. Let A be a self-adjoint operator acting in the Hilbert space $\mathcal{H}$, and let $\mathcal{H}_A(\lambda)$, $\lambda>\sup\sigma_{ess}(A)$ be the subspace consisting of the elements of $f\in\mathcal{H}$ satisfying the inequality $(Af,f)>\lambda(f,f)$.

We set

$$n(\lambda,A)=\sup_{\mathcal{H}_A(\lambda)}\dim\mathcal{H}_A(\lambda).$$

The number $n(\lambda,A)$ coincides with the number of eigenvalues of the operator A lying to the right of λ.

We set

$$N(K,z) = n\big(-z, -H(K)\big), \quad -z > -\tau_K.$$

Lemma 3.2. *For each $z < \tau_K$, the equality*

$$N(K,z) = n\big(1, \mathbf{B}(K,z)\big) \tag{3.1}$$

holds.

Proof. We prove lemma by the method used in [7]. Assume $f \in \mathcal{H}_H(z)$, $z < \tau$, i.e.,

$$\Big(\big(H_0(K) - zI\big)f, f\Big) < (Vf, f), \quad V = \mu_1 V_1 + \mu_2 V_2 + \mu_3 V_3 + \mu_4 V_4.$$

Therefore,

$$(y, y) < \Big(R_0^{\frac{1}{2}}(K,z) V R_0^{\frac{1}{2}}(K,z) y, y\Big), \quad y = \big(H_0(K) - zI\big)^{\frac{1}{2}} f.$$

Thus $N(K,z) \le n\big(1, R_0^{\frac{1}{2}}(K,z) V R_0^{\frac{1}{2}}(K,z)\big)$.

Arguing similarly, we obtain the converse statement

$$N(K,z) \ge n\big(1, R_0^{\frac{1}{2}}(K,z) V R_0^{\frac{1}{2}}(K,z)\big).$$

This implies the equality

$$N(K,z) = n\big(1, R_0^{\frac{1}{2}}(K,z) V R_0^{\frac{1}{2}}(K,z)\big).$$

We consider the equation for the eigenfunctions $f \in L_2((\mathbb{T}^3)^2)$ of the operator $R_0^{\frac{1}{2}} V R_0^{\frac{1}{2}}$, $R_0^{\frac{1}{2}} := R_0^{\frac{1}{2}}(K,z)$

$$\lambda f = R_0^{\frac{1}{2}} \Big[\mu_1 V_1 + \mu_2 V_2 + \mu_3 V_3 + \mu_4 V_4\Big] R_0^{\frac{1}{2}} f, \quad \lambda > 1. \tag{3.2}$$

Let

$$\varphi_i = \mu_i^{\frac{1}{2}} V_i^{\frac{1}{2}} R_0^{\frac{1}{2}} f. \tag{3.3}$$

Taking into account the equality $V_i^{\frac{1}{2}} = V_i$, $i = 1, \ldots, 4$ and (3.3), we write equality (3.2) in the form

$$\lambda f = R_0^{\frac{1}{2}} \Big[\mu_1^{\frac{1}{2}} V_1 \varphi_1 + \mu_2^{\frac{1}{2}} V_2 \varphi_2 + \mu_3^{\frac{1}{2}} V_3 \varphi_3 + \mu_4^{\frac{1}{2}} V_4 \varphi_4\Big].$$

Substituting this expression in (3.3), we obtain the system of integral equations

$$\begin{cases} \lambda\varphi_1 = \mu_1^{\frac{1}{2}} V_1 R_0 \Big(\mu_1^{\frac{1}{2}} V_1\varphi_1 + \mu_2^{\frac{1}{2}} V_2\varphi_2 + \mu_3^{\frac{1}{2}} V_3\varphi_3 + \mu_4^{\frac{1}{2}} V_4\varphi_4 \Big), \\ \lambda\varphi_2 = \mu_2^{\frac{1}{2}} V_2 R_0 \Big(\mu_1^{\frac{1}{2}} V_1\varphi_1 + \mu_2^{\frac{1}{2}} V_2\varphi_2 + \mu_3^{\frac{1}{2}} V_3\varphi_3 + \mu_4^{\frac{1}{2}} V_4\varphi_4 \Big), \\ \lambda\varphi_3 = \mu_3^{\frac{1}{2}} V_3 R_0 \Big(\mu_1^{\frac{1}{2}} V_1\varphi_1 + \mu_2^{\frac{1}{2}} V_2\varphi_2 + \mu_3^{\frac{1}{2}} V_3\varphi_3 + \mu_4^{\frac{1}{2}} V_4\varphi_4 \Big), \\ \lambda\varphi_4 = \mu_4^{\frac{1}{2}} V_4 R_0 \Big(\mu_1^{\frac{1}{2}} V_1\varphi_1 + \mu_2^{\frac{1}{2}} V_2\varphi_2 + \mu_3^{\frac{1}{2}} V_3\varphi_3 + \mu_4^{\frac{1}{2}} V_4\varphi_4 \Big), \end{cases},$$

i.e., the equation

$$\lambda\Phi = \mathbf{T}(K,z)\Phi, \tag{3.4}$$

has a nontrivial solution $\Phi = (\varphi_1, \varphi_3, \varphi_3, \varphi_4)$, where the operator $\mathbf{T}(K,z)$ acts in $L_2^{(4)}((\mathbb{T}^3)^3)$ as

$$\mathbf{T}(K,z) = \begin{pmatrix} \mu_1 V_1 R_0 V_1 & \sqrt{\mu_1\mu_2} V_1 R_0 V_2 & \sqrt{\mu_1\mu_3} V_1 R_0 V_3 & \sqrt{\mu_1\mu_4} V_1 R_0 V_4 \\ \sqrt{\mu_2\mu_1} V_2 R_0 V_1 & \mu_2 V_2 R_0 V_2 & \sqrt{\mu_2\mu_3} V_2 R_0 V_3 & \sqrt{\mu_2\mu_4} V_2 R_0 V_4 \\ \sqrt{\mu_3\mu_1} V_3 R_0 V_1 & \sqrt{\mu_3\mu_2} V_3 R_0 V_2 & \mu_3 V_3 R_0 V_3 & \sqrt{\mu_3\mu_4} V_3 R_0 V_4 \\ \sqrt{\mu_4\mu_1} V_4 R_0 V_1 & \sqrt{\mu_4\mu_2} V_4 R_0 V_2 & \sqrt{\mu_4\mu_3} V_4 R_0 V_3 & \mu_4 V_4 R_0 V_4 \end{pmatrix}.$$

It is easy to check that the equations (3.4) and (3.2) are equivalent. From this we obtain the equality

$$N(K,z) = n\big(1, \mathbf{T}(K,z)\big).$$

If we show that

$$n\big(1, \mathbf{T}(K,z)\big) = n\big(1, \mathbf{W}^{\frac{1}{2}}(K,z)\mathbf{L}(K,z)\mathbf{W}^{\frac{1}{2}}(K,z)\big),$$

then Lemma 3.2 will be proved.

Suppose $f \in \mathcal{H}_{\mathbf{T}(K,z)}(1)$, that is $\big(\mathbf{T}(K,z)f, f\big) > (f,f)$ or

$$\Big(\big[\mathbf{E} - V^{\frac{1}{2}} R_0 V^{\frac{1}{2}}\big] f, f\Big) < (\mathbf{L}(K,z)f, f),$$

where $\mathbf{E}$ is the identity operator in $L_2^{(4)}((\mathbb{T}^3)^3)$ and

$$V^{\frac{1}{2}} R_0 V^{\frac{1}{2}} = \begin{pmatrix} \mu_1 V_1 R_0 V_1 & 0 & 0 & 0 \\ 0 & \mu_2 V_2 R_0 V_2 & 0 & 0 \\ 0 & 0 & \mu_3 V_3 R_0 V_3 & 0 \\ 0 & 0 & 0 & \mu_4 V_4 R_0 V_4 \end{pmatrix}.$$

Hence,

$$(\varphi,\varphi) < \big(\mathbf{W}^{\frac{1}{2}}(K,z)\mathbf{L}(K,z)\mathbf{W}^{\frac{1}{2}}(K,z)\varphi, \varphi\big), \quad f = \mathbf{W}^{\frac{1}{2}}(K,z)\varphi.$$

Thus, $n\big(1,\mathbf{T}(K,z)\big)\le n\big(1,\mathbf{W}^{\frac12}(K,z)\mathbf{L}(K,z)\mathbf{W}^{\frac12}(K,z)\big)$.

Arguing similarly, we obtain the converse statement

$$n\big(1,\mathbf{T}(K,z)\big)\ge n\big(1,\mathbf{W}^{\frac12}(K,z)\mathbf{L}(K,z)\mathbf{W}^{\frac12}(K,z)\big).$$

This implies the equality (3.1). □

Proof of the Lemma 2.1. Note that for any $z\le\tau_K$ the operator $\mathbf{B}(K;z)$ is a compact and continuous operator-valued function for $z\le\tau_K$. Therefore, using the Weyl inequality [3]

$$n(a+b,A+B)\le n(a,A)+n(b,B)$$

for compact operators A,B and according the Lemma 3.2 we obtain the proof of the Lemma 2.1 [7].

In what follows we assume that $K=\mathbf{0}$, $\mathbf{0}=(0,0,0)$. We set

$$\Delta_{\mu_\alpha}(k;z)=1-\mu_\alpha F_\alpha(k;z),\quad F_\alpha(k;z)=\frac{1}{(2\pi)^6}\int\limits_{(\mathbb{T}^3)^2}\frac{dq_1dq_2}{\varepsilon_k^\alpha(q_1,q_2)-z},$$

$z\in\mathbb{C}\setminus[m_\alpha(k),M_\alpha(k)]$, where $m_\alpha(k)=\min\limits_{p,q\in\mathbb{T}^3}\varepsilon_k^\alpha(p,q),\quad M_\alpha(k)=\max\limits_{p,q\in\mathbb{T}^3}\varepsilon_k^\alpha(p,q)$.

Lemma 3.3. *The number $z\in\mathbb{C}\setminus\sigma_{ess}(h_\alpha(k))$ is an eigenvalue of the operator $h_\alpha(k)$ if and only if $\Delta_{\mu_\alpha}(k;z)=0$.*

A similar lemma is proved in [6].

The Lemma 3.3 and the equality (2.1) imply

Lemma 3.4. *For the spectrum of the operator $H_\alpha(\mathbf{0})=H_0(\mathbf{0})-\mu_\alpha V_\alpha$, $\alpha\in\{1,2,3,4\}$ the equality*

$$\sigma\big(H_\alpha(\mathbf{0})\big)=[0,M_0]\cup\sigma_\alpha$$

holds, where $\sigma_\alpha=\Big\{z\in\mathbf{C}\backslash[0,M_\mathbf{0}]:\ \Delta_\alpha\big(p;z-\varepsilon_\alpha(p)\big)=0$ *for at least one* $p\in\mathbb{T}^3\Big\}$.

Let

$$f_\alpha(x,z)=\int\limits_{\mathbb{T}^2}\frac{dsdt}{\frac{1}{m_\beta}(1-\cos s)+\frac{1}{m_\gamma}(1-\cos t)+\frac{1}{m_\theta}(1-\cos(s+t+x)-z},$$

$x\in\mathbb{T}$, $z<0$.

Then the function $f_\alpha(x,z)$ is analytic on $\mathbb{T}$ for each $z<0$.

Proposition 3.1. *Let $z<0$. Then $\max\limits_x f_\alpha(x,z)=f_\alpha(0,z)$.*

Proof. A simple calculation shows that from the monotonicity of the function $f_\alpha(x,\cdot)$ on $(-\infty,0)$. Therefore, if the function $f_\alpha(\cdot,z)$ for each $z<0$ reaches its maximum value at the point x', then it reaches its maximum value for all $z'\neq z$, $z'<0$ at the same point x'.

Suppose that $f_\alpha(x',z)=\max\limits_x f_\alpha(x,z)>f_\alpha(0,z)$ for $z<0$. Then $\lim\limits_{z\to-0} f_\alpha(x',z)$ exists and it is finite, and $\lim\limits_{z\to-0} f_\alpha(0,z)=+\infty$. Therefore, there exists $z^*<0$ such that $f_\alpha(x',z^*)<f_\alpha(0,z^*)$. This contradiction proves Proposition 3.1. □

From the Proposition 3.1, we obtain the following

Corollary 3.1. *Let $z\leq 0$ and $\max\limits_p F_\alpha(p;z)=F_\alpha(p';z)$. Then $p'=\mathbf{0}$.*

It follows from the Corollary 3.1 that if $\sigma_\alpha\neq\emptyset$, then $\inf\sigma_\alpha=z$, $\Delta_\alpha(\mathbf{0};z)=0$. From here and the Lemma 3.4 we have

Proposition 3.2. *For all $\mu_\alpha\geq 0$ the equality*

$$\tau_\alpha=:\tau_\alpha(\mathbf{0})=\inf\sigma\big(H_\alpha(\mathbf{0})\big)=\min\{0,\ \lambda_\alpha\},$$

holds, where λ_α such that $\Delta_{\mu_\alpha}(\mathbf{0};\lambda_\alpha)=0$, herewith $\tau_\alpha=\tau_\alpha(\mu_\alpha)\leq 0$.

The expression for the essential spectrum of the operator $H(K)$ implies

$$\tau_{\mathbf{0}}=\inf\sigma_{ess}\big(H(\mathbf{0})\big)=\min\{\tau_1,\tau_2,\tau_3,\tau_4\}.$$

We denote

$$\mu_\alpha(z)=\Big[\frac{1}{(2\pi)^3}F_\alpha(\mathbf{0};z)\Big]^{-1},\quad z\leq 0.$$

A simple calculation shows that

$$\Delta_{\mu_\alpha}(k;z)=\begin{cases}>0 & \text{for}\quad \mu_\alpha<\mu_\alpha(z),\\ 0 & \text{for}\quad \mu_\alpha=\mu_\alpha(z),\\ <0 & \text{for}\quad \mu_\alpha>\mu_\alpha(z).\end{cases}$$

Lemma 3.5. *There exists a number $C>0$ such that, for all $\mu_\alpha\geq 0$, $\alpha=1,...,4$ and $z\leq\tau_\alpha$ the inequality*

$$\Delta_{\mu_\alpha}\big(p;z-\varepsilon_\alpha(p)\big)\geq Cp^2 \tag{3.5}$$

holds.

Proof. The function $\Delta_{\mu_\alpha}\big(p; z-\varepsilon_\alpha(p)\big)$ is continuous in z and p, and strictly monotonically decreases in $z<\tau_\alpha$ for any fixed $p\in\mathbb{T}^3$. Hence, if $\mu_\alpha<\mu_\alpha(0)$, then

$$\Delta_{\mu_\alpha}\big(p; z-\varepsilon_\alpha(p)\big) > \Delta_{\mu_\alpha(0)}\big(p; 0-\varepsilon_\alpha(p)\big) \ge \Delta_{\mu_\alpha(0)}\big(\mathbf{0}; 0-\varepsilon_\alpha(\mathbf{0})\big)=0.$$

Therefore, inequality (3.5) for $\mu_\alpha<\mu_\alpha(0)$ holds.

Let $\mu_\alpha\ge\mu_\alpha(0)$. Then $\mu_\alpha(\tau_\alpha)F_\alpha\big(\mathbf{0};\tau_\alpha\big)=1$ and $\Delta_{\mu_\alpha}(p; z-\varepsilon_\alpha(p))\ge\Delta_{\mu_\alpha(\tau_\alpha)}\big(p;\tau_\alpha-\varepsilon_\alpha(p)\big)$. By the Corollary 3.1, we have

$$\begin{aligned}
&\Delta_{\mu_\alpha(\tau_\alpha)}\big(p;\tau_\alpha-\varepsilon_\alpha(p)\big)\\
&=\mu_\alpha(\tau_\alpha)F_\alpha(\mathbf{0};\tau_\alpha)-\frac{\mu_\alpha(\tau_\alpha)}{(2\pi)^6}\int\limits_{(\mathbb{T}^3)^2}\frac{dq_1dq_2}{\varepsilon_\alpha(p)+\varepsilon_p^\alpha(q_1,q_2)-\tau_\alpha}\\
&\ge\frac{\mu_\alpha(\tau_\alpha)}{(2\pi)^6}\int\limits_{(\mathbb{T}^3)^2}\frac{dq_1dq_2}{\varepsilon_{\mathbf{0}}^\alpha(q_1,q_2)-\tau_\alpha}-\frac{\mu_\alpha(\tau_\alpha)}{(2\pi)^6}\int\limits_{(\mathbb{T}^3)^2}\frac{dq_1dq_2}{\varepsilon_\alpha(p)+\varepsilon_{\mathbf{0}}^\alpha(q_1,q_2)-\tau_\alpha}\\
&=\varepsilon_\alpha(p)\,\frac{\mu_\alpha(\tau_\alpha)}{(2\pi)^6}\int\limits_{(\mathbb{T}^3)^2}\frac{dq_1dq_2}{(\varepsilon_\alpha(p)+\varepsilon_{\mathbf{0}}^\alpha(q_1,q_2)-\tau_\alpha)(\varepsilon_{\mathbf{0}}^\alpha(q_1,q_2)-\tau_\alpha)}.
\end{aligned}\tag{3.6}$$

Since the point $p=\mathbf{0}$ is the unique nondegenerate minimum point for the function $\varepsilon_\alpha(\cdot)$, there exists $C>0$ such that for all $p\in\mathbb{T}^3$ the inequality

$$\varepsilon_\alpha(p)\ge Cp^2\tag{3.7}$$

holds. Therefore, the integral

$$\int\limits_{(\mathbb{T}^3)^2}\frac{dq_1dq_2}{(\varepsilon_\alpha(p)+\varepsilon_{\mathbf{0}}^\alpha(q_1,q_2)-\tau_\alpha)(\varepsilon_{\mathbf{0}}^\alpha(q_1,q_2)-\tau_\alpha)}$$

converges for all $\tau_\alpha\le 0$ and $p\in\mathbb{T}^3$. From here and according to (3.7) from the inequality (3.6) we obtain the inequality (3.5) for $\mu_\alpha\ge\mu_\alpha(0)$. □

Using $V_\alpha=V_\alpha^2$ and performing elementary calculations on integral equations, it is easy to check that the operator $W_\alpha(\mathbf{0},z)$ acts as

$$W_\alpha(\mathbf{0},z)=I+\mu_\alpha\Delta_\alpha^{-1}V_\alpha R_0(\mathbf{0},z)V_\alpha,$$

where Δ_α^{-1} is the operator of multiplication by $[\Delta_{\mu_\alpha}(p;z-\varepsilon_\alpha(p))]^{-1}$. Moreover, the operator $W_\alpha(\mathbf{0},z)$ is continuous in $z<\tau_\alpha\le 0$.

Lemma 3.6. *For all $z<\tau_\alpha$, the operator $W_\alpha^{\frac12}(z)=:W_\alpha^{\frac12}(\mathbf{0},z)$ represents as*

$$W_\alpha^{\frac12}(z)=[\mu_\alpha\Delta_\alpha^{-1}]^{\frac12}M_\alpha(z)^{\frac12}V_\alpha+\widetilde{W}_\alpha(z),$$

where $M_\alpha(z)$ *is operator of multiplication by the function* $V_\alpha R_0(\mathbf{0},z)\varphi_0$, $\varphi_0 \equiv 1$, $\widetilde{W}_\alpha(z)$ *is a bounded operator for each* $z \in (-\infty, \tau_\alpha]$, *and the operator function* $\widetilde{W}_\alpha(\cdot)$ *is continuous on the interval* $(-\infty, \tau_\alpha]$.

Proof. Note that the operator $[\mu_\alpha \Delta_\alpha^{-1}]^{\frac{1}{2}} M_\alpha(z)^{\frac{1}{2}} V_\alpha$ is a positive root of the positive operator $\mu_\alpha \Delta_\alpha^{-1} V_\alpha R_0(\mathbf{0},z) V_\alpha$. From here and according to the well-known inequality [3]

$$\|B^{\frac{1}{2}} - A^{\frac{1}{2}}\| \le \|B - A\|^{\frac{1}{2}},$$

for all positive operators A and B, we have

$$\|W_\alpha^{\frac{1}{2}}(z) - [\mu_\alpha \Delta_\alpha^{-1}]^{\frac{1}{2}} M_\alpha(z)^{\frac{1}{2}} V_\alpha\| \le I.$$

This gives the representation

$$W_\alpha^{\frac{1}{2}}(z) = [\mu_\alpha \Delta_\alpha^{-1}]^{\frac{1}{2}} M_\alpha(z)^{\frac{1}{2}} V_\alpha + \widetilde{W}_\alpha(z).$$

The nonzero elements $W_\alpha^{\frac{1}{2}} V_\alpha^{\frac{1}{2}} R_0 V_\beta^{\frac{1}{2}} W_\beta^{\frac{1}{2}}$, $\alpha \ne \beta$ of the matrix operator $\mathbf{B}(\mathbf{0},z)$ have the form

$$W_\alpha^{\frac{1}{2}} V_\alpha^{\frac{1}{2}} R_0 V_\beta^{\frac{1}{2}} W_\beta^{\frac{1}{2}} = \mu_\alpha^{\frac{1}{2}} \mu_\beta^{\frac{1}{2}} W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z), \quad K_{\alpha\beta}(z) = V_\alpha R_0(\mathbf{0},z) V_\beta,$$

$\alpha \ne \beta$. □

Lemma 3.7. *Let* $\mu_\alpha \ge 0$, $\alpha = 1,\ldots,4$. *Then* $W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z)$, $\alpha \ne \beta$ *is a compact operator for* $z \in (-\infty, \tau_\mathbf{0}]$ *and the operator function* $W_\alpha^{\frac{1}{2}}(\cdot) K_{\alpha\beta}(\cdot) W_\beta^{\frac{1}{2}}(\cdot)$, $\alpha \ne \beta$ *is continuous on the interval* $(-\infty, \tau_\mathbf{0}]$.

Proof. According to the Lemma 3.6, we write

$$\begin{aligned} &W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z) \\ &= \left[\mu_\alpha \Delta_\alpha^{-1}\right]^{\frac{1}{2}} M_\alpha(z)^{\frac{1}{2}} K_{\alpha\beta}(z) M_\beta(z)^{\frac{1}{2}} \left[\mu_\beta \Delta_\beta^{-1}\right]^{\frac{1}{2}} + \widetilde{W}_\alpha(z) K_{\alpha\beta}(z) M_\beta(z)^{\frac{1}{2}} \left[\mu_\beta \Delta_\beta^{-1}\right]^{\frac{1}{2}} \\ &+ \left[\mu_\alpha \Delta_\alpha^{-1}\right]^{\frac{1}{2}} M_\alpha(z)^{\frac{1}{2}} K_{\alpha\beta}(z) \widetilde{W}_\beta(z) + \widetilde{W}_\alpha(z) K_{\alpha\beta}(z) \widetilde{W}_\beta(z). \end{aligned}$$

The operators $M_\alpha(z)$ and $\widetilde{W}_\alpha(z)$ are bounded for all $z \le \tau_\mathbf{0}$ and $\alpha = 1,\ldots 4$. Therefore, taking into account the commutativity of the operators $M_\alpha(z)$ and $\mu_\alpha \Delta_\alpha^{-1}$, according to the last equality, for the compactness of the operator $W_\alpha^{\frac{1}{2}}(z)$ $K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z)$, $z \le \tau_\mathbf{0}$ it suffices to show that the operators $K_{\alpha\beta}(z)$, $\left[\mu_\alpha \Delta_\alpha^{-1}\right]^{\frac{1}{2}}$ $K_{\alpha\beta}(z)$, $K_{\alpha\beta}(z)\left[\mu_\beta \Delta_\beta^{-1}\right]^{\frac{1}{2}}$ and $\left[\mu_\alpha \Delta_\alpha^{-1}\right]^{\frac{1}{2}} K_{\alpha\beta}(z) \left[\mu_\beta \Delta_\beta^{-1}\right]^{\frac{1}{2}}$ are compact for all $z \le \tau_\mathbf{0}$. These operators belong to the Hilbert-Schmidt class for all $z \le \tau_\mathbf{0}$. Indeed, for convenience, we show this for $\alpha = 1$ and $\beta = 2$. The operator $K_{12}(z)$ acts in $L_2((\mathbb{T}^3)^3)$ as

$$(K_{12}(z)f)(p_1, p_2, p_3) = \int\limits_{(\mathbb{T}^3)^2} \int\limits_{(\mathbb{T}^3)^2} \frac{(2\pi)^{-12} f(q_1, q_2', q_3) dq_1 dq_2 dq_2' dq_3'}{\varepsilon_1(p_1) + \varepsilon_2(q_2') + \varepsilon_3(q_3') + \varepsilon_4(p_1 + q_2 + q_3') - z}.$$

According to (3.7), the kernel

$$K_{12}(z; p_1, q_2', q_3) = \int\limits_{\mathbb{T}^3} \frac{(2\pi)^{-12} dq_3'}{\varepsilon_1(p_1) + \varepsilon_2(q_2') + \varepsilon_3(q_3') + \varepsilon_4(p_1 + q_2 + q_3') - z}$$

of the integral operator $K_{12}(z)$ is bounded for all $p_1, q_2', q_3 \in \mathbb{T}^3$ and $z \leq \tau_{\mathbf{0}}$. Therefore, the operator $K_{12}(z)$ belongs to the Hilbert-Schmidt class for all $z \leq \tau_{\mathbf{0}}$.

Since $[\mu_\alpha \Delta_\alpha^{-1}]^{\frac{1}{2}}$ is a multiplication operator, according to the Lemma 3.5, from the boundedness of the kernel of the operator $K_{12}(z)$, we obtain boundedness of the kernels of the operators $[\mu_\alpha \Delta_1^{-1}]^{\frac{1}{2}} K_{12}(z)$, $K_{12}(z)[\mu_2 \Delta_2^{-1}]^{\frac{1}{2}}$ and $[\mu_1 \Delta_1^{-1}]^{\frac{1}{2}} K_{12}(z)$ $\frac{1}{2}$, respectively, by functions $\frac{C}{\sqrt{p_1^2}}$, $\frac{C}{\sqrt{p_2^2}}$ and $\frac{C}{\sqrt{p_1^2}\sqrt{p_2^2}}$ for all $p_1, p_2 \in \mathbb{T}^3$ and $z \leq \tau_{\mathbf{0}}$. Therefore, the operators $[\mu_\alpha \Delta_1^{-1}]^{\frac{1}{2}} K_{12}(z)$, $K_{12}(z)[\mu_2 \Delta_2^{-1}]^{\frac{1}{2}}$ and $[\mu_1 \Delta_1^{-1}]^{\frac{1}{2}} K_{12}(z)$ $\frac{1}{2}$ belong to the Hilbert-Schmidt class for all $z \leq \tau_{\mathbf{0}}$.

Thus, we have proved that $W_1^{\frac{1}{2}}(z) K_{12}(z) W_2^{\frac{1}{2}}(z)$ is a compact operator for all $z \leq \tau_{\mathbf{0}}$.

By similar way are proved the compactness of the operators $W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z)$ $W_\beta^{\frac{1}{2}}(z)$, $\alpha \neq \beta$ for all $z \leq \tau_{\mathbf{0}}$.

The kernel of the integral operator $W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z)$, $\alpha \neq \beta$ is continuous in $\mathbb{T}^3 \times \mathbb{T}^3$ for all $z < \tau_{\mathbf{0}}$. Therefore, the function $W_\alpha^{\frac{1}{2}}(\cdot) K_{\alpha\beta}(\cdot) W_\beta^{\frac{1}{2}}(\cdot)$, $\alpha \neq \beta$ is continuous on the interval $(-\infty, \tau_{\mathbf{0}})$. The continuity of the function $W_\alpha^{\frac{1}{2}}(\cdot) K_{\alpha\beta}(\cdot) W_\beta^{\frac{1}{2}}(\cdot)$, $\alpha \neq \beta$, at the point $z = \tau_{\mathbf{0}}$ follows from the dominated convergence theorem. $\square$

From the Lemma 3.7 we obtain

Lemma 3.8. *For each* $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ *with* $\mu_\alpha \geq 0$, $\alpha = 1, ..., 4$, *the family of operators* $\mathbf{B}(\mathbf{0}, z)$, $z < \tau_{\mathbf{0}}$ *uniformly converges to some operator* $\mathbf{B}(\mathbf{0}, \tau_{\mathbf{0}})$ *for* $z \to \tau_{\mathbf{0}} - 0$.

The proof of the Theorem 2.1 follows from the Lemmas 3.8 and 2.1.

ACKNOWLEDGMENT

The authors express their deep gratitude to M.I. Muminov for the valuable critical remarks.

REFERENCES

[1] ZH.I. ABDULLAEV, S.N. LAKAEV: *Finiteness of discrete spectrum of three particle Schrödinger operator on the lattice*, Theoret. and Math. Phys., **111**(1) (1997), 467–479.

[2] S.A. ALBEVERIO, S.N. LAKAEV, ZH.I. ABDULLAEV: *On the Finiteness of the Discrete Spectrum of a Four-Particle Lattice Schrödinger Operator*, Funct. Anal. Appl., **36**(3) (2002), 212–216.

[3] M.SH. BIRMAN, M.Z. SOLOMYAK: *pectral Theory of Self-Adjoint Operators in Hilbert Space*, Textbook, Leningrad. Univ. Press, Leningrad, 1980 [in Russian]. English transl. (Math. and Its Appl. Sov. Series, Vol. 5), Kluwer, Dordrecht (1987).

[4] *G.M. Zhislin On the finiteness of the discrete spectrum of energy operators of multiparticle quantum systems*, (English. Russian original), Sov. Math., Dokl. **13**, 1445-1449; translation from Dokl. Akad. Nauk SSSR, **207** (1972), 25–28.

[5] G.M. ZHISLIN: *Finiteness of the discrete spectrum in the quantum n-particle problem*, Theoret. and Math. Phys., **21**(1) (1974), 971–980.

[6] S.N. LAKAEV: *On Efimov's Effect in a System of Three Identical Quantum Particles*, Funct. Anal. Appl., **27**(3) (1993), 166–175.

[7] S.N. LAKAEV, M.I. MUMINOV: *Essential and Discrete Spectra of the Three-Particle Schrödinger Operator on a Lattice*, Theoret. and Math. Phys., **135**(3) (2003), 849–871.

[8] S.N. LAKAEV, S.M. SAMATOV: *Finiteness of the Discrete Spectrum of the Hamiltonian of a System of Three Arbitrary Particles on a Lattice*, Theoret. and Math. Phys., **129**(3) (2001), 1655–1668.

[9] S.P. MERKUR'EV, L.D. FADDEEV: *Quantum Scattering Theory for Several Particle Systems*, Nauka, Moscow, 1985.

[10] M.I. MUMINOV: *The infiniteness of the number of eigenvalues in the gap in the essential spectrum for the three-particle Schrödinger operator on a lattice*, Theoret. and Math. Phys., **159**(2) (2009), 667–683.

[11] M.E. MUMINOV, E.M. SHERMATOVA: *On finiteness of discrete spectrum of three-particle Schrödinger operator on a lattice*, Russian Math. (Iz. VUZ), **60**(1) (2016), 22–29.

[12] M.E. MUMINOV, U.R. SHODIEV: *Spectral properties of a Hamiltonian of a four-particle system on a lattice*, Russian Math. (Iz. VUZ), **54**(12) (2010), 27–37.

[13] M.E. MUMINOV, U.R. SHODIEV: *On the essential spectrum of a four-particle Schrödinger operator on a lattice*, Siberian Adv. Math., **21**(4) (2011), 292–303.

[14] M. REED, B. SIMON: *Methods of Modern Mathematical Physics* Vol. 4: Analysis of Operators, New York, Academic Press, 1978.

[15] H. CYCON, R. FROESE, W. KIRSCH, AND B. SIMON: *Schrödinger Operators*, Springer, Berlin, 1987.

[16] U.R. SHODIEV: *On the essential spectrum of the four-particle Schrödinger operator with three-particle contact interaction on the lattice*, *Uzbek Math. Jour.* **3** (2008), 110–120 [in Russian].

[17] D.R. YAFAEV: *On the point spectrum in the quantum-mechanical many-body problem*, Math. USSR-Izv., **10**(4) (1976), 861–896.

DEPARTMENT OF DIGITAL TECHNOLOGIES
SAMARKAND STATE UNIVERSITY
UNIVERSITY BOULEVARD 15, SAMARKAND,
UZBEKISTAN.
Email address: islomnb@mail.ru

DEPARTMENT OF DIGITAL TECHNOLOGIES
SAMARKAND STATE UNIVERSITY
UNIVERSITY BOULEVARD 15, SAMARKAND,
UZBEKISTAN.
Email address: usmon555@mail.ru

DEPARTMENT OF PHYSICS AND MATHEMATICS
NAVOI STATE PEDAGOGICAL INSTITUTE
IBN SINO 45, NAVOI,
UZBEKISTAN.
Email address: g.yodgorov@mail.ru