

SOBOLEV SPACES ARISING FROM A GENERALIZED SPHERICAL FOURIER TRANSFORM

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This paper is dedicated to Professor Norbert M. Hounkonnou.

ABSTRACT. In this paper, we define Sobolev spaces on a locally compact unimodular group in link with the spherical Fourier transform of type δ . Properties of these spaces are obtained. Analogues of Sobolev embedding theorems are proved.

1. INTRODUCTION

To handle efficiently some problems in functional analysis and particularly in the differential equation theory, Sobolev spaces play a crucial role. That is why they are useful in all areas where differential equations appear : classical/quantum mechanics, physics, engineering, etc. On the other hand, Fourier analysis is present in the study of Sobolev spaces through tempered distributions. For instance, the classical Sobolev space $H^s(\mathbb{R}^n)$ where $s \in \mathbb{R}$, has a characterization via the Fourier transform on \mathbb{R}^n . It is the set of tempered distributions u such that $(1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^n)$ where \hat{u} is the Fourier transform of

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u [6, Definition 3.13]. From then it appears that a generalization of the space $H^s(\mathbb{R}^n)$ can be constructed whenever a generalization of the Fourier transform is available. Also, such study can be conducted by using other operators of Fourier type. See for instance [11] where the q -Dunkel operator plays the role of the classical Fourier operator.

In the context of abstract harmonic analysis, Sobolev spaces on locally compact abelian groups were studied in [4, 5] with the help of the Fourier transform on locally compact abelian groups (Pontrjagin duality). Also the Sobolev spaces on Gelfand pairs were studied in [9] by the use of the spherical Fourier transform on Gelfand pairs.

The aim of this paper is to study the counterpart of the Sobolev space $H^s(\mathbb{R})$ associated with the spherical Fourier transform of type δ on a locally compact unimodular group developed in [7, 8]. The spherical Fourier transform on Gelfand pairs generalizes the Fourier transform on abelian groups while the spherical Fourier transform of type δ generalizes the spherical Fourier transform on Gelfand pairs. Therefore some results obtained here generalize some results in [4, 5, 9].

The paper is organized as follows. Section 2 is devoted to preliminaries on the spherical Fourier transform of type δ . It amounts to the Plancherel formula. Section 3 contains our main results about the Sobolev spaces defined in link with the spherical Fourier transform of type δ .

2. THE SPHERICAL FOURIER TRANSFORM OF TYPE δ

In this section, we give the fundamental ingredients to understand the construction of the spherical Fourier transform of type δ . Our main references are [3, 7, 8, 13]. This theory takes its origin from the article [7]. Informations involving the counterpart of the Bochner theorem and the Plancherel theorem/formula can be found in [3, 13]. The Plancherel formula will play a key role in the sequel (section 3).

Let G be a locally compact unimodular group. We denote by e the neutral element of G . Let K be a compact subgroup of G . We denote by \widehat{K} the unitary dual of K , that is the set of equivalent classes of unitary irreducible representations of K . We choose a class δ in \widehat{K} and still denote an element of this class by δ . Let

ξ_δ be the character of δ , that is

$$(2.1) \quad \xi_\delta(x) = \text{tr}(\delta(x)),$$

where tr denotes the trace. Let us denote by $d(\delta)$ the degree of δ . Set $\chi_\delta = d(\delta)\xi_\delta$. Let $\check{\delta}$ be the contragredient representation of δ . We have

$$(2.2) \quad \overline{\chi_\delta} = \chi_{\check{\delta}} \text{ and } \chi_{\check{\delta}} * \chi_\delta = \chi_{\check{\delta}}.$$

Let E be a finite dimensional complex vector space. Denote by

- $\text{End}(E)$, the set of endomorphisms of E ,
- $\mathcal{C}_c(G, \text{End}(E))$, the set of functions from G into $\text{End}(E)$ that are continuous with compact support.

For $f \in \mathcal{C}_c(G, \text{End}(E))$, define

$$(2.3) \quad {}_\delta f(x) = \overline{\chi_\delta} * f(x) = \int_K \chi_\delta(k) f(kx) dk$$

and

$$(2.4) \quad f_\delta(x) = f * \chi_\delta(x) = \int_K \chi_\delta(k^{-1}) f(xk) dk,$$

where dk is the normalized Haar measure of K . Also define

- $\mathcal{K}_\delta(G, \text{End}(E)) = \{f \in \mathcal{C}_c(G, \text{End}(E)) : {}_\delta f = f_{\check{\delta}} = f\}$,
- $\mathcal{K}^\sharp(G, \text{End}(E)) = \{f \in \mathcal{C}_c(G, \text{End}(E)) : f(kx) = f(xk), \forall x \in G, k \in K\}$,
- $\mathcal{K}_\delta^\sharp(G, \text{End}(E)) = \mathcal{K}_\delta(G, \text{End}(E)) \cap \mathcal{K}^\sharp(G, \text{End}(E))$.

An element of $\mathcal{K}_\delta^\sharp(G, \text{End}(E))$ is called a K - δ invariant function.

A semi-norm on G is a nonnegative lower semi-continuous function ρ which is also bounded on any compact subset of G such that $\forall x, y \in G, \rho(xy) \leq \rho(x)\rho(y)$ [1, 12]. Let \mathcal{B} be a complex Banach space. A function $\phi : G \rightarrow \mathcal{B}$ is said to be *quasi-bounded* if there exists a semi-norm ρ on G such that

$$\|\phi\|_\rho = \sup_{x \in G} \frac{\|\phi(x)\|}{\rho(x)} < \infty.$$

Hereafter is the definition of a spherical function of type δ . Following the papers [3, 7] we set

Definition 2.1. A spherical function of type δ on G is a continuous and quasi-bounded function ω from G into $\text{End}(E)$ such that

$$- \chi_\delta * \omega = \omega = \omega * \chi_\delta.$$

- $\omega(kxk^{-1}) = \omega(x), \forall x \in G, \forall k \in K$.
- the map $f \mapsto \int_G f(x)\omega(x^{-1})dx$ is an irreducible representation of the algebra $\mathcal{K}_\delta^\sharp(G, \text{End}(E))$ on E .

The dimension of the vector space E is called the *height* of the spherical function ω . We denote by \mathcal{S}_δ^m the set of spherical functions of type δ on G and height m ; this set is endowed with the topology of uniform convergence on compact subsets.

Remark 2.1. If $\delta \in \widehat{K}$ is the trivial (class of) representation of dimension 1, that is $\forall x \in G, \delta(x) = Id_{E_\delta}$, where E_δ is the representation space of δ , and if $\dim E_\delta = \dim E = 1$ then we recover the classical notion of spherical function.

From this extended notion of spherical function at our disposal, it is possible to generalize the notion of spherical Fourier transform. This is done in [7].

Definition 2.2. The spherical Fourier transform of type δ of a function $f \in \mathcal{K}_\delta^\sharp(G, \text{End}(E))$ is defined by

$$(2.5) \quad \widehat{f}(\omega) = \int_G f(x)\omega(x^{-1})dx, \omega \in \mathcal{S}_\delta^m.$$

A spherical function of type δ which is positive definite is called a *spherical function of δ -positive type*. Denote by $\mathcal{S}_\delta^{m,+}$ the set of spherical functions of δ -positive type and height m . The set $\mathcal{S}_\delta^{m,+}$ is endowed with the induced topology of \mathcal{S}_δ^m .

The inversion formula is given by [3]:

$$(2.6) \quad f(x) = \int_{\mathcal{S}_\delta^{m,+}} \widehat{f}(\omega)\omega(x)d\sigma(\omega),$$

where σ is a positive Radon measure on $\mathcal{S}_\delta^{m,+}$ the existence of which was proved in [3]. We finish this section by recalling the counterpart of the Plancherel formula. Denote by $L_\delta^{2,\sharp}(G, \text{End}(E))$ the set of K - δ invariant $\text{End}(E)$ -valued functions that are strongly square integrable.

Theorem 2.1. ([3]) If $f \in L_\delta^{2,\sharp}(G, \text{End}(E))$ then $\widehat{f} \in L^2(\mathcal{S}_\delta^{m,+}, \text{End}(E))$ and

$$\int_G \|f(x)\|^2 dx = \int_{\mathcal{S}_\delta^{m,+}} \|\widehat{f}(\omega)\|^2 d\sigma(\omega).$$

3. THE SOBOLEV SPACES $H_{\delta,\gamma}^s(G, \text{End}(E))$

In this section, we present our main results. Mainly, we define the Sobolev spaces $H_{\delta,\gamma}^s(G, \text{End}(E))$ and establish some of their important properties.

Let $\gamma : \mathcal{S}_\delta^{m,+} \rightarrow \mathbb{R}_+$ be a measurable function and let $s \in \mathbb{R}_+$. We define the following Sobolev space

$$H_{\delta,\gamma}^s(G, \text{End}(E)) = \left\{ f \in L_\delta^{2,\sharp}(G, \text{End}(E)) : \int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^s \|\widehat{f}(\omega)\|^2 d\sigma(\omega) < \infty \right\}.$$

This space is provided with the norm

$$\|f\|_{H_{\delta,\gamma}^s} = \left(\int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^s \|\widehat{f}(\omega)\|^2 d\sigma(\omega) \right)^{\frac{1}{2}}.$$

Theorem 3.1. *The space $H_{\delta,\gamma}^s(G, \text{End}(E))$ is a Banach space when it is endowed with the norm $\|\cdot\|_{H_{\delta,\gamma}^s}$.*

Proof. The map $f \mapsto (1 + \gamma(\omega)^2)^{\frac{s}{2}} \widehat{f}$ is an isometric bijection from $H_{\delta,\gamma}^s(G, \text{End}(E))$ onto $L^2(\mathcal{S}_\delta^{m,+}, \sigma, \text{End}(E))$. Since the latter is a Banach space so is $H_{\delta,\gamma}^s(G, \text{End}(E))$. \square

It is possible to construct a pre-Hilbert module structure on $H_{\delta,\gamma}^s(G, \text{End}(E))$. For information about the pre-Hilbert module structure, we refer to [10]. Let us notice that the space $\text{End}(E)$ possesses a C^* -algebra structure (one may identify it with $\mathcal{B}(H)$ the space of operators on some finite dimensional Hilbert space H).

For $f, g \in H_{\delta,\gamma}^s(G, \text{End}(E))$, set

$$(3.1) \quad \langle f, g \rangle_s = \int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^s \widehat{f}(\omega)^* \widehat{g}(\omega) d\sigma(\omega)$$

where $\widehat{f}(\omega)^*$ is the adjoint of the endomorphism $\widehat{f}(\omega)$.

Theorem 3.2. *The map*

$$\langle \cdot, \cdot \rangle_s : H_{\delta,\gamma}^s(G, \text{End}(E)) \times H_{\delta,\gamma}^s(G, \text{End}(E)) \rightarrow \text{End}(E), (f, g) \mapsto \langle f, g \rangle_s$$

turns $H_{\delta,\gamma}^s(G, \text{End}(E))$ into a pre-Hilbert module over the C^ -algebra $\text{End}(E)$.*

Proof. The axioms of the pre-Hilbert module structure are can be easily verified. See [10, page 2]. \square

Theorem 3.3. *The following embedding holds:*

$$H_{\delta,\gamma}^s(G, \text{End}(E)) \hookrightarrow L_{\delta}^{2,\sharp}(G, \text{End}(E)).$$

Moreover, if $f \in H_{\delta,\gamma}^s(G, \text{End}(E))$ then

$$\|f\|_2 \leq \|f\|_{H_{\delta,\gamma}^s}.$$

Proof. Let $f \in H_{\delta,\gamma}^s(G, \text{End}(E))$. Then

$$\begin{aligned} \|f\|_2^2 &= \|\widehat{f}\|_2^2 \quad (\text{Theorem 2.1}) \\ &= \int_{\mathcal{S}_{\delta}^{m,+}} \|\widehat{f}(\omega)\|^2 d\sigma(\omega) \\ &\leq \left(\int_{\mathcal{S}_{\delta}^{m,+}} (1 + \gamma(\omega)^2)^s \|\widehat{f}(\omega)\|^2 d\sigma(\omega) \right) = \|f\|_{H_{\delta,\gamma}^s}^2. \end{aligned}$$

\square

Theorem 3.4. *If $s > \sigma > 0$ then*

$$H_{\delta,\gamma}^s(G, \text{End}(E)) \hookrightarrow H_{\delta,\gamma}^{\sigma}(G, \text{End}(E)).$$

Moreover,

$$\forall f \in H_{\delta,\gamma}^s((G, \text{End}(E))), \quad \|f\|_{H_{\delta,\gamma}^{\sigma}} \leq \|f\|_{H_{\delta,\gamma}^s}.$$

Proof. The results are based on the fact that if $s > \sigma$ then $(1 + \gamma(\omega)^2)^s > (1 + \gamma(\omega)^2)^{\sigma}$ since $1 + \gamma(\omega)^2 > 1$. \square

Let us denote by $\mathcal{C}_{\delta}^{\sharp,b}(G, \text{End}(E))$ the set of $\text{End}(E)$ -valued bounded continuous K - δ invariant functions on G . Hereafter is the counterpart of a Sobolev embedding theorem.

Theorem 3.5. *If $(1 + \gamma^2)^{-\frac{s}{2}} \in L^2(\mathcal{S}_{\delta}^{m,+}, \text{End}(E))$ then $H_{\delta,\gamma}^s(G, \text{End}(E)) \hookrightarrow \mathcal{C}_{\delta}^{\sharp,b}(G, \text{End}(E))$. Moreover,*

$$\forall f \in H_{\delta,\gamma}^s(G, \text{End}(E)), \quad \|f\|_{\infty} \leq 2\|f\|_{H_{\delta,\gamma}^s} \|(1 + \gamma^2)^{-\frac{s}{2}}\|_2.$$

Proof. We mimic a proof from [9] by adapting it to vector valued functions. Let $x_0 \in G$. Let $\varepsilon > 0$. The topology of uniform convergence on compact subsets on $\mathcal{S}_{\delta}^{m,+}$ implies that $\mathcal{S}_{\delta}^{m,+}$ is equicontinuous. Therefore there exists an open

neighbourhood U of x_0 such that $\forall \omega \in \mathcal{S}_\delta^{m,+}$, $\forall x \in U$, $\|\omega(x) - \omega(x_0)\| < \varepsilon$. Now consider $f \in H_{\delta,\gamma}^s(G, \text{End}(E))$.

$$\begin{aligned}
 & \|f(x) - f(x_0)\| \\
 &= \left\| \int_{\mathcal{S}_\delta^{m,+}} \widehat{f}(\omega)(\omega(x) - \omega(x_0)) d\sigma(\omega) \right\| \\
 &\leq \int_{\mathcal{S}_\delta^{m,+}} \|\widehat{f}(\omega)\| \|\omega(x) - \omega(x_0)\| d\sigma(\omega) \leq \varepsilon \int_{\mathcal{S}_\delta^{m,+}} \|\widehat{f}(\omega)\| d\sigma(\omega) \\
 &\leq \varepsilon \int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^{\frac{s}{2}} \|\widehat{f}(\omega)\| (1 + \gamma(\omega)^2)^{-\frac{s}{2}} d\sigma(\omega) \\
 (3.2) \quad &\leq \varepsilon \left(\int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^s \|\widehat{f}(\omega)\|^2 d\sigma(\omega) \right)^{\frac{1}{2}} \\
 &\quad \cdot \left(\int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^{-s} d\sigma(\omega) \right)^{\frac{1}{2}} \\
 &\leq \varepsilon \|f\|_{H_{\delta,\gamma}^s} \|(1 + \gamma^2)^{-\frac{s}{2}}\|_2.
 \end{aligned}$$

Meanwhile we have used the Cauchy-Schwartz inequality (3.2). The continuity of f is proved. Let us prove that f is bounded.

$$\begin{aligned}
 & \|f(x)\| \\
 &= \left\| \int_{\mathcal{S}_\delta^{m,+}} \widehat{f}(\omega)\omega(x) d\sigma(\omega) \right\| \\
 &\leq \sup_{\omega \in \mathcal{S}_\delta^{m,+}} \|\omega(x)\| \int_{\mathcal{S}_\delta^{m,+}} \|\widehat{f}(\omega)\| d\sigma(\omega) \\
 &\leq \sup_{\omega \in \mathcal{S}_\delta^{m,+}} \|\omega(x)\| \int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^{\frac{s}{2}} \|\widehat{f}(\omega)\| (1 + \gamma(\omega)^2)^{-\frac{s}{2}} d\sigma(\omega) \\
 &\leq \sup_{\omega \in \mathcal{S}_\delta^{m,+}} \|\omega(x)\| \left(\int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^s \|\widehat{f}(\omega)\|^2 d\sigma(\omega) \right)^{\frac{1}{2}} \\
 &\quad \cdot \left(\int_{\mathcal{S}_\delta^{m,+}} (1 + \gamma(\omega)^2)^{-s} d\sigma(\omega) \right)^{\frac{1}{2}} \\
 &\leq \sup_{\omega \in \mathcal{S}_\delta^{m,+}} \|\omega(x)\| \|f\|_{H_{\delta,\gamma}^s} \|(1 + \gamma^2)^{-\frac{s}{2}}\|_2.
 \end{aligned}$$

However, a spherical function ω of δ -positive type and height m satisfies the relations $\forall x \in G, \|\omega(x)\| \leq 2\|\omega(e)\|$ and $\omega(e) = I_m$, where I_m is the unit of $\text{End}(E)$. Also $\|I_m\| = 1$. Therefore,

$$\begin{aligned} \|f(x)\| &\leq 2 \sup_{\omega \in \mathcal{S}_\delta^{m,+}} \|\omega(e)\| \|f\|_{H_{\delta,\gamma}^s} (1 + \gamma^2)^{-\frac{s}{2}} \|2\| \\ &\leq 2 \|f\|_{H_{\delta,\gamma}^s} (1 + \gamma^2)^{-\frac{s}{2}} \|2\|. \end{aligned}$$

So f is bounded and $\|f\|_\infty \leq 2 \|f\|_{H_{\delta,\gamma}^s} (1 + \gamma^2)^{-\frac{s}{2}} \|2\|$. □

REFERENCES

- [1] M. AKKOUCI, A. BAKALI, *Fonctions sphériques associées à une mesure bornée, hermitienne et idempotente*, Compositio Mathematica, **90**(3) (1994), 245–266.
- [2] G. CHOQUET, *Topology*, Academic Press Inc, 1966.
- [3] P. COULIBALY, K. KANGNI, *Sur une extension du théorème de Bochner*, Afr. Mat., **25** (2014), 411–416.
- [4] P. GÓRKA, T. KOSTRZEWA, E. G. REYES, *Sobolev spaces on locally compact abelian groups: compact embeddings and local spaces*, Journal of Function Spaces, (2014), Article ID 404738, 6 pages.
- [5] P. GÓRKA, T. KOSTRZEWA, E. G. REYES, *Sobolev spaces on locally compact abelian groups and the bosonic string equation*, J. Aust. Math. Soc., **98** (2015), 39–53.
- [6] D. D. HAROSKE, H. TRIEBEL, *Distributions, Sobolev spaces, elliptic equations*, European Mathematical Society, 2008.
- [7] K. KANGNI, S. TOURE, *Transformation de Fourier sphérique de type δ* , Annales mathématiques Blaise Pascal, **3**(2) (1996), 117–133.
- [8] K. KANGNI, S. TOURE, *Transformation de Fourier sphérique de type δ : Applications aux groupes de Lie semi-simples*, Annales mathématiques Blaise Pascal, **8**(2) (2001), 77–88.
- [9] M. KRUKOWSKI, *Sobolev spaces on Gelfand pairs*, arXiv2003.08519v1.[math.FA], (2020). <https://arxiv.org/abs/2003.08519v1>.
- [10] E. C. LANCE, *Hilbert C^* -modules*, Cambridge University Press, 1995.
- [11] H. MEJJAOLI, N. SRAIEB, *Generalized Sobolev spaces in quantum calculus and applications*, Journal of Inequalities and Special Functions, **3**(2) (2012), 26–45.
- [12] G. WARNER, *Harmonic analysis on semi-simple Lie groups I*, Springer-Verlag Berlin Heidelberg New York, 1972.
- [13] K. D. WODOME, K. KANGNI, K. V. S. ASSIAMOUA, *On an extension of Bochner's theorem*, Far East Journal of Mathematical Sciences, **67**(1) (2012), 1–19.
- [14] J. A. WOLF, *Harmonic Analysis on Commutative Spaces*, American Mathematical Society, Providence, 2007.

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