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ANALYTICAL APPROACH TO THE PENNE'S BIOHEAT EQUATION FOR THE EVALUATION OF TEMPERATURE FOR DEEP SEATED TISSUES

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ABSTRACT. In clinical practice, body temperature is an acclaimed indicator for diagnosis and treatments amongst other follow - ups. This work studied the temperature distribution in deep-seated tissues using Penne's bio-heat equation (PBE). The analytical solution of the PBE was formulated using the separation of variable technique. The resulting solution gives an expression for tissue temperature in organs that are located inside the body. This revealed that the tissue temperature is subject to changes in tissue thickness, the distance of tissue from the skin surface and other thermal properties of tissues under consideration.

1. INTRODUCTION

The existing relationship between human body temperature and disease is almost as old as medicine itself hence its importance as a good health indicator cannot be over-emphasized [1,2]. The usage of temperature as a clinical diagnosis criterion for humans began around 400BC. Disease conditions are accompanied by inflammation and increased vascularity which elevates temperature and could be exploited in distinguishing normal physiological processes within the body [3]. Temperature study and its applications in the fields of science

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has gained relevance all over the world. The discovery by William Herschel in 1800 and recording of the first thermal images by his son John Herschel ushered in new dimensions in the field of temperature measurement [4]. Infrared (thermal) radiation is an electromagnetic radiation with wavelengths longer than those of visible light but shorter than radio waves. Its wavelength ranges from 0.75 to 1000 μm and is classified into three parts; Near-infrared (0.75 to 1.5μm), Medium infrared (1.5 to 5.6μm) and Far-infrared (5.6 to 1000μm) [3]. All electromagnetic radiations including infrared are emitted by every object with a temperature above absolute zero and the amount of energy radiated, energy spectrum and intensity are related to temperature as expressed by the Stefan – Boltzmann Law:

$$E = \sigma T^4, \tag{1.1}$$

where is the amount of energy radiated in W/m^2, is the Stefan-Boltzmann constant (5.68 x10$^{-8} Wm^{-2}K^{-4}$) and T is temperature in Kelvin (K). The Planck's equation is given as:

$$I(\lambda, T) = \frac{2h\pi c^2}{\lambda^4}(\exp^{\frac{hc}{\pi kT}} - 1)^{-1}, \tag{1.2}$$

where h is the Planck constant, is the intensity of radiation, c is the speed of light in vacuum, λ is the wavelength, T is temperature and k is Boltzmann constant [5]. The human body radiates heat energy from the surface of the skin and its emissivity value for human skin is 0.98, which is approximately that of a perfect black body. In order words, the thermal radiation emitted by the skin is closely related to the actual skin temperature.

IR thermography (IRT) is most suitable for skin surface temperature measurement. Therefore, accurate temperature values can be created from measurements of the infrared radiation from the skin. The recording of the temperature distribution of a body using the infrared radiation emitted by the surface of that body at wavelengths between 0.8 μm and 1.0 μm is referred to as infrared thermography and this can be achieved using an infrared camera in a process known as thermal imaging. Thermography provides a great advantage of real-time two-dimensional temperature measurement. With innovative technology, a single image may contain thousands of temperature points, recorded in a short period. It has been used for research over the last 50 years to study some diseases where skin temperature could reflect the presence of inflammation in the

underlying tissues, or where there is an increase or decrease in blood flow resulting from a clinical abnormality. In principle, thermal imaging can be applied in medicine either as a diagnostic test or as an outcome measure for clinical trials [6]. This non- contact and non-invasive imaging modality was reported by Lawson in 1956 [7] as a new tool in the investigation of breast lesions and was widely researched between the 1970s and 1980s [8]. It was approved by the US Food and Drug Administration (FDA) as an adjunctive tool for the diagnosis of breast cancer, but not as a stand-alone diagnostic tool for breast malignancy, due to its low sensitivity and specificity [9,10].

Presently, mammography is being used as the main standard for breast cancer screening and diagnosis; this modality can reveal masses or micro-calcifications in the earliest stages of the disease. Nevertheless, it was shown to have a low sensitivity in younger women and those with dense breasts [11]. Besides, this modality employs the use of high quality and expensive equipment, uses ionizing radiation and has high costs [12]. Based on this scenario, thermography has been widely accepted for its safety, non-invasive, non-ionizing and low-cost potential [13]. Literature has shown more favourable results for thermography and some of its applications such as: treatment of diabetes [14], quick detection of seasonal influenza [15], study of eye diseases [2], analysis of chronic pain [16] and mainly in the diagnosis of one of the most dreaded diseases of our time, "cancer" [3].

Cancer diagnosis using IR thermography is now possible based on the existing relationship between temperature rise in the skin and the presence of a tumour, as established by Lawson [7]. Owing to this fact, the detection of multiple types of cancer, such as melanoma [17], is possible. Specifically, for breast cancer detection, it is said that asymmetrical breast temperature can be an indication of either vascular problems or cancer in the region of interest [18]. In recent times, the sensitivity of infrared cameras has improved considerably and is now close to 0.025°C, so that it is now possible to measure small changes in surface temperature. An important aspect of this technology is that it is most appropriate for tissues situated at the skin surface, hence for deep-seated tissues some parameters need to be considered to account for energy lost in the body as it travels from one tissue layer to another before arrival at the skin surface. Tissues are anisotropic materials characterized by the presence of a vascular structure and local heat generation due to metabolism. Some mathematical models

have been designed to understand the concept of thermoregulatory mechanisms and heat transfer in living organisms. These models are geared towards developing temperature control during hypothermia and hyperthermia; estimating the temperature distributions in core organs where direct temperature measurement is highly invasive and determining changes in organ temperature following changes in metabolic and hormonal activities in the body [19].

Of the models, the Penne's bio-heat approach is conventionally used to study heat transfer in tissues due to its simplicity. It involves thermal conduction, convection, perfusion of blood, and metabolic heat generation in tissue [20].When there is a variance between the temperature of the blood and that of the tissue through which it flows, convective heat transport will occur, altering the temperatures of both the blood and the tissue. Perfusion based heat transfer interaction is critical to many physiological processes such as thermoregulation and inflammation. The blood/tissue thermal interaction is a function of several parameters including the rate of perfusion and the vascular anatomy, which vary widely among the different tissues, organs of the body, and pathology. The main objective of this work is to derive an analytical solution from the existing bio-heat equation for temperatures of tissues situated deep under the skin taking into account the thermal properties of the tissues such as density, conductivity, thermal diffusivity etc.

2. Mathematical conceptualization

One of the basic roles of blood flow in the body is its ability to heat or cool tissues of the body based on the tissue temperature. Upon this assumption, [20] proposed the heat transfer model popularly known as the Pennes bio-heat equation which involves convection, thermal conduction, perfusion of blood and metabolic heat generation [21]. This equation is a combination of the Fourier heat conduction law and two additional terms which account for metabolic effect and thermal change between blood and tissue [20]. It is defined by the expression;

$$\rho c \frac{\delta T}{\delta t} = \nabla(k \nabla T) + Q + w_b C_b (T_b - T). \tag{2.1}$$

This equation is subjected to the following initial and boundary conditions:

Initial Conditions

$$T(x,t) = T_c = T_o \tag{2.2}$$

This is because the temperature of tissues is assumed to be 37^oC as that of the core body due to metabolic regulatory mechanism.

Boundary Conditions

(i) $T(x,t)|_{t=0} = f(x)$, where f(x) is a function of distance.
(ii) $T(x,t)|_{x=0.001} = T_b + Q_b$
(iii) $\theta(x,t)|_{x=0.001} = 0$
(iv) $\theta(x,t)|_{t=0} = f(x)$

These parameter are described in table 1 below.

TABLE 1. Definition of parameters

Parameters	Description
ρ	Density of tissue (kg/m^3)
c	Specific heat capacity of tissue (J/kg.K)
$\frac{\delta T}{\delta t}$	rate of temperature rise (^{o}C/s)
k	thermal conductivity of the tissue (W/m.K)
Q	metabolic heat generation per unit volume from the tissue (W/m^3)
w_b	mass flow rate of blood per unit volume of tissue (kg/(s.m^3)
c_b	Specific heat capacity of blood (J/kg.K)
T_b	Arterial blood temperature (^{o}C)
T	Tissue temperature (^{o}C)
x	Distance from tissue to skin surface (m)

For a small volume of human tissue, assuming a one-dimensional situation, the equation above can be expressed as:

$$\rho c\frac{\delta T}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta T}{\delta x}) + Q + w_b c_b(T_b - T). \tag{2.3}$$

This model assumes that:

(i) $Q = B_b Q_c$
(ii) $B_b = \omega_b C_b$

(iii) $T - T_b - Q_o = \theta$

(iv) Q_c is a constant

$$\rho c \frac{\delta T}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta T}{\delta x}) + B_b Q_o + B_b(T_b - T) \tag{2.4}$$

$$\rho c \frac{\delta T}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta T}{\delta x}) + B_b Q_o + B_b T_b - B_b T \tag{2.5}$$

$$\rho c \frac{\delta T}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta T}{\delta x}) - B_b T + B_b Q_o + B_b T_b \tag{2.6}$$

$$\rho c \frac{\delta T}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta T}{\delta x}) - B_b(T - T_b - Q_o) \tag{2.7}$$

Let

$$T - T_b - Q_o = \theta \tag{2.8}$$

$$\rho c \frac{\delta \theta}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta \theta}{\delta x}) - B_b \theta \tag{2.9}$$

Let

$$x = \xi \exp^{ax} \tag{2.10}$$

$$k(x) = \xi k_o \alpha^2 X^2, \tag{2.11}$$

where ξ is a scaling parameter k_o = constant.

$$\frac{\delta x}{\delta X} = \alpha \xi \exp^{ax} = \alpha x \tag{2.12}$$

$$\delta x = \alpha x \delta X \tag{2.13}$$

$$\delta x^2 = \alpha^2 x^2 \delta X^2 \tag{2.14}$$

Considering $\frac{\delta}{\delta x}(k(x)\frac{\delta \theta}{\delta x})$ alone

$$k(x)\frac{\delta^2 \theta}{\delta X^2} + \frac{\delta k(x)}{\delta x}\frac{\delta \theta}{\delta x} \tag{2.15}$$

Substituting (2.15) into (2.9), it is obtained that:

$$\rho c \frac{\delta \theta}{\delta t} = k(x)\frac{\delta^2 \theta}{\delta x^2} + \frac{\delta k(x) \delta \theta}{\delta x \delta x} - B_b \theta. \tag{2.16}$$

Putting (2.11), (2.13) and (2.14) into (2.16), we obtain:

$$\rho c\frac{\delta\theta}{\delta t} = \xi k_o\alpha^2X^2\frac{\delta^2\theta}{\alpha^2x^2\delta X^2} + \frac{\delta(\xi K_o\alpha^2)}{\delta x}\frac{\delta\theta}{\alpha x\delta X} - B_b\theta. \tag{2.17}$$

Using the method of separable variables:

$$\theta(x,t) = \Sigma(x)\Omega(t), \tag{2.18}$$

$$\rho c\frac{\Sigma\delta\Omega}{\delta t} = \xi K_o\frac{\Omega\delta^2\Sigma}{\delta X^2} + 2\xi K_o\alpha\frac{\Omega\delta\xi}{\delta X} - B_b\xi\Omega. \tag{2.19}$$

Multiply through by $\frac{1}{\Sigma\Omega}$ we have:

$$\frac{\rho c}{\Omega}\frac{\delta\Omega}{\delta t} = \frac{\xi K_o}{\Sigma}\frac{\delta^2\Sigma}{\delta X^2} + \frac{2\xi K_o\alpha}{\Sigma}\frac{\delta\Sigma}{\delta X} - B_b. \tag{2.20}$$

Let the above equation be equal to a constant. The properties of blood are fairly the same in time and different locations so then assume this constant to be B_b so that:

$$\frac{\rho c}{\Omega}\frac{\delta\Omega}{\delta t} = \xi K_o\{\frac{1}{\Sigma}\frac{\delta^2\Sigma}{\delta X^2} + \frac{2\alpha}{\Sigma}\frac{\delta\Sigma}{\delta X}\} - B_b = B_b. \tag{2.21}$$

The (2.21) implies that:

$$\frac{\rho c}{\Omega}\frac{\delta\Omega}{\delta t} = B_b \tag{2.22}$$

and

$$\xi K_o\{\frac{1}{\Sigma}\frac{\delta^2\Sigma}{\delta X^2} + \frac{2\alpha}{\Sigma}\frac{\delta\Sigma}{\delta X}\} - B_b = B_b, \tag{2.23}$$

$$\frac{\xi K_o}{\Sigma}\frac{\delta^2\Sigma}{\delta X^2} + \frac{2\xi K_o\alpha}{\Sigma}\frac{\delta\Sigma}{\delta X}\} - 2B_b = 0. \tag{2.24}$$

Multiplying through by Σ,

$$\xi K_o\frac{\delta^2\Sigma}{\delta X^2} + 2\xi K_o\alpha\frac{\delta\Sigma}{\delta X}\} - 2B_b\Sigma = 0. \tag{2.25}$$

Rewriting (2.22), we obtain:

$$\frac{\delta\Omega}{\Omega} = \frac{B_b}{\rho c}\delta t. \tag{2.26}$$

Integrating through yields:

$$\int\frac{\delta\Omega}{\Omega} = \int\frac{B_b}{\rho c}\delta t$$

$$\ln\Omega = \frac{B_b}{\rho c}t + C$$

$$\Omega = \exp^{\frac{B_b}{\rho c}t+C}$$

$$\Omega = A_o \exp^{\frac{B_b}{\rho c}t} \tag{2.27}$$

To solve (2.25), we assume the first term to be zero

$$2\xi K_o \alpha \frac{\delta \Sigma}{\delta X}\} - 2B_b\Sigma = 0 \tag{2.28}$$

$$\frac{\delta \Sigma}{\delta X}\} - \frac{B_b\Sigma}{\xi K_o \alpha} = 0 \tag{2.29}$$

Let

$$m = \frac{B_b}{\xi K_o \alpha}$$

$$\frac{\delta \Sigma}{\delta X}\} - m\Sigma = 0 \tag{2.30}$$

$$\frac{\delta \Sigma}{\Sigma} = m\delta X$$

$$\int \frac{\delta \Sigma}{\Sigma} = \int m\delta X$$

$$\ln \Sigma = mx + C$$

$$\Sigma = \exp^{mx+C}$$

$$\Sigma = P_o \exp^{mx} \tag{2.31}$$

Here P_o is an arbitrary constant.

$$\frac{\delta \Sigma}{\delta x} = mP_o \exp^{mx} \tag{2.32}$$

$$\frac{\delta^2 \Sigma}{\delta x^2} = m^2 P_o \exp^{mx} \tag{2.33}$$

Substituting (2.33) and (2.32) into (2.25)

$$\xi K_o m^2 P_o \exp^{mx} + 2\xi K_o \alpha m P_o \exp^{mx} - 2B_b P_o \exp^{mx} = 0, \tag{2.34}$$

$$P_o \exp^{mx}[\xi K_o m^2 + 2\xi K_o \alpha m - 2B_b] = 0 \tag{2.35}$$

i.e

$$P_o \exp^{mx} = 0 or \xi K_o m^2 + 2\xi K_o \alpha m - 2B_b = 0.$$

Assuming ξ is unity, then:

$$K_o m^2 + 2K_o \alpha m - 2B_b = 0. \tag{2.36}$$

Recalling the quadratic equation form $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.37}$$

By comparing a = K_o; b = $2K_o\alpha$; c = -2B_b; x = m.Then the solution of the (2.36) will be:

$$m = \frac{-2K_o\alpha \pm \sqrt{(2K_o\alpha)^2 - 4K_o(-2B_b)}}{2K_o}$$

$$m = \frac{-2K_o\alpha \pm \sqrt{4K_o^2\alpha^2 + 8K_oB_b}}{2K_o}$$

$$m = -\alpha \pm \frac{\sqrt{K_o^2\alpha^2 + 2K_oB_b}}{K_o}$$

we can write that

$$m_1 = -\alpha + \frac{\sqrt{K_o^2\alpha^2 + 2K_oB_b}}{K_o} \tag{2.38}$$

$$m_2 = -\alpha - \frac{\sqrt{K_o^2\alpha^2 + 2K_oB_b}}{K_o} \tag{2.39}$$

Therefore,

$$\Sigma = P_o \exp^{m_1 x}$$

$$\Sigma = P_o \exp^{m_2 x}.$$

If $\Sigma = U$ and $\Sigma = V$ are two solutions, so also is $\Sigma = U + V$.
If $\Sigma A_1 \exp^{m_1 x}$ and $\Sigma A_2 \exp^{m_2 x}$, then

$$\Sigma = A_1 \exp^{m_1 x} + A_2 \exp^{m_2 x} \tag{2.40}$$

Recalling (2.18) and substituting the values of Σ and Ω in the (2.27) and (2.40) respectively to obtain:

$$\theta(x,t) = A_o \exp^{\frac{B_b}{\rho c}t}(A_1 \exp^{m_1 x} + A_2 \exp^{m_2 x}) \tag{2.41}$$

Recall that $x = \xi \exp^{ax}$, $\frac{x}{\xi} = \exp^{ax}$, $ln\frac{x}{\xi} = aX$, and

$$X = \frac{1}{\alpha} ln \frac{x}{\xi}. \tag{2.42}$$

Substitute (2.42) into (2.41)

$$\theta(x,t) = A_o \exp^{\frac{B_b}{\rho c}t}(A_1 \exp^{\frac{m_1}{\alpha} ln\frac{x}{\xi}} + A_2 \exp^{\frac{m_2}{\alpha} ln\frac{x}{\xi}}). \tag{2.43}$$

Considering $m^2 - n^2 = 0$, $m^2 = n^2$, $m = \pm n$, $y = A\exp^{mx}$

$$y = Acoshmx + Bsinhnx, \tag{2.44}$$

writing equation (2.43) in the form of (2.44):

$$\theta(x,t) = (A_1A_o \exp^{\frac{m_1}{\alpha}ln\frac{x}{\xi}} + A_2A_o \exp^{\frac{m_2}{\alpha}ln\frac{x}{\xi}}) \exp^{\frac{B_b}{\rho c}t}. \tag{2.45}$$

Let

$$A_oA_1 = P_1 and A_2A_o = P_2$$

$$\theta(x,t) = (P_1 \exp^{\frac{m_1}{\alpha}ln\frac{x}{\xi}} + P_2 \exp^{\frac{m_2}{\alpha}ln\frac{x}{\xi}}) \exp^{\frac{B_b}{\rho c}t}$$

$$\theta(x,t) = (P_1 cosh\frac{m_1}{\alpha}ln\frac{x}{\xi} + P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{\xi}) \exp^{\frac{B_p}{\rho c}t}. \tag{2.46}$$

We cannot evaluate the temperature at x = 0 so we choose a reference point x as 0.001 m and consequently $\xi = X$, then the contribution to the measured temperature T(x,t) is only from T_b and Q_o. Then,

$$T - T_b - Q_o = 0. \tag{2.47}$$

But from (2.8),

$$T - T_b - Q_o = \theta.$$

At x = 0.001 and $\xi = x$,

$$\theta(x,t) = (P_1 cosh\frac{m_1}{\alpha}ln\frac{0.001}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{0.001}{0.001}) \exp^{\frac{B_b}{\rho c}t}$$

$$\theta(x,t) = (P_1 cosh\frac{m_1}{\alpha}ln(1) + P_2 sinh\frac{m_2}{\alpha}ln(1)) \exp^{\frac{B_b}{\rho c}t}$$

$$\theta(x,t) = (P_1 cosh\frac{m_1}{\alpha}(0) + P_2 sinh\frac{m_2}{\alpha}(0)) \exp^{\frac{B_b}{\rho c}t}$$

$$\theta(x,t) = 0$$

$$\text{At} \quad x = 0.001, \theta(x,t) = 0 \tag{2.48}$$

From (2.9)

$$\rho c\frac{\delta\theta}{\delta t} = \frac{\delta}{\delta x}(k(x)\frac{\delta\theta}{\delta x}) - B_b\theta$$

$$\frac{\delta T}{\delta t} = \frac{1}{\rho c}\{\frac{\delta}{\delta x}(k(x)\frac{\delta\theta}{\delta x}) - B_b\theta\}$$

$$\delta T = [\frac{1}{\rho c}\{\frac{\delta}{\delta x}(k(x)\frac{\delta\theta}{\delta x}) - B_b\theta\}]\delta t \tag{2.49}$$

$$\int \delta T = \int [\frac{1}{\rho c}\{\frac{\delta}{\delta x}(k(x)\frac{\delta\theta}{\delta x}) - B_b\theta\}]\delta t$$

$$T(x,t)\frac{1}{\rho c}\{\frac{\delta}{\delta x}(k(x)\frac{\delta\theta}{\delta x}) - B_b\theta\}t + f(x).$$

At t = 0

$$T(x,0)\frac{1}{\rho c}\{\frac{\delta}{\delta x}(k(x)\frac{\delta\theta}{\delta x}) - B_b\theta\}(0) + f(x)$$

$$T(x,t) = f(x) \tag{2.50}$$

Similarly, at x = 0.001

$$T(0.001,t) = T_b + Q_o. \tag{2.51}$$

The general solution will be

$$T(x,t) = f(x) + T_b + Q_o \tag{2.52}$$

But from (2.5),

$$T(x,t) = T_b + Q_o + \theta. \tag{2.53}$$

Let (2.53) be equal to (2.52)

$$T_b + Q_o + \theta = f(x) + T_b + Q_o$$

$$\theta(x,t)|_{t=0} = f(x) = ig(x). \tag{2.54}$$

At t = 0:

$$\theta(x,0) = (P_1 cosh\frac{m_1}{\alpha}ln\frac{x}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{0.001})\exp^{\frac{B_p}{\rho c}(0)}$$

$$\theta(x,0) = (P_1 cosh\frac{m_1}{\alpha}ln\frac{x}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{0.001})\exp^{0}$$

$$\theta(x,0) = (P_1 cosh\frac{m_1}{\alpha}ln\frac{x}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{0.001}).1$$

$$\theta(x,0) = P_1 cosh\frac{m_1}{\alpha}ln\frac{x}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{0.001}. \tag{2.55}$$

At x = 0.001 and t = 0, equation () will be:

$$\theta(x,t) = (P_1 cosh\frac{m_1}{\alpha}ln\frac{0.001}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{0.001}{0.001})\exp^{\frac{B_p}{\rho c}(0)} = 0$$

$$P_1 cosh(0) + P_2 sinh(0) = 0$$

$$P_1 = 0. \tag{2.56}$$

If (2.56) is true, then (2.46) will be:

$$\theta(x,t) = (0.cosh\frac{m_1}{\alpha}ln\frac{x}{0.001} + P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{0.001})\exp^{\frac{B_p}{\rho c}t} = 0$$

$$\theta(x,t) = P_2 sinh\frac{m_2}{\alpha}ln\frac{x}{0.001})\exp^{\frac{B_p}{\rho c}t} = 0 \tag{2.57}$$

To evaluate (2.57) we make the following assumptions: $m_2 = \lambda$, but $m_2 = -\alpha - \frac{\sqrt{K_o^2\alpha^2+2K_oB_b}}{K_o}$; $\lambda = (-\alpha - \frac{\sqrt{K_o^2\alpha^2+2K_oB_b}}{K_o})$; $\lambda^2 = (-\alpha - \frac{\sqrt{K_o^2\alpha^2+2K_oB_b}}{K_o})^2$. Multiply both sides by χ,

$$\chi(\lambda)^2 = \chi(-\alpha - \frac{sqrtK_o^2\alpha^2 + 2K_oB_b}{K_o})^2. \tag{2.58}$$

By comparing (2.27) and (2.31), we have:

$$m = \frac{B_b}{\rho c}.$$

Following (2.40), (2.58) will be

$$\chi(\lambda)^2 = \chi(-\lambda - \frac{sqrtK_o^2 + \alpha^2 + 2K_oB_b}{K_o})^2 = \chi(\frac{B_b}{\rho c})$$

$$\chi(\lambda)^2 = \chi(\frac{B_b}{\rho c}), \tag{2.59}$$

where ξ and λ are new tissue-dependent parameters that must be determined. Using (2.57):

$$\theta(x,t) = (P_2 sinh\frac{\lambda}{\alpha}ln\frac{x}{0.001})\exp^{\chi\lambda^2.t}. \tag{2.60}$$

Recall that ξ = 0.001 and (2.43), then substituting into (2.60), we have that

$$\theta(x,t) = P_2 sinh(\lambda x)\exp^{\chi\lambda^2.t}. \tag{2.61}$$

Since there is no significant restriction on λ, we can replace P_2 with$P_2\lambda$ with and still, have a solution. Furthermore, we can integrate over λ from 0 to ∞ and still have a solution. Hence, a possible solution is given as

$$\theta(x,t) = \int_{\infty}^{0} P_2 sinh(\lambda x)\exp^{\chi\lambda^2.t}\,\partial\lambda. \tag{2.62}$$

From (2.54), (2.62) becomes

$$f(x) = \int_{\infty}^{0} P_2(\lambda) sinh(\lambda x)\partial\lambda. \tag{2.63}$$

Since f(x) contains only a sine term, it is an odd function. Following from Fourirer transform

$$\frac{2}{\pi}\int_{\infty}^{o} f(v)sinh(\lambda v)\partial v. \tag{2.64}$$

Substituting (2.64) into (2.62) we have:

$$\theta(x,t) = \frac{2}{\pi}\int_{\infty}^{o}\int_{\infty}^{o} f(v)\exp^{\chi\lambda^2.t} sinh(\lambda v)sinh(\lambda x)\partial\lambda\partial v. \tag{2.65}$$

From trigonometry identities [23]:

$$sinhAsinhB = \frac{1}{2}[cosh(A+B) - cosh(A-B)]. \tag{2.66}$$

Substituting into (2.65)

$$\theta(x,t) = \frac{2}{\pi}\int_{\infty}^{o}\int_{\infty}^{o} f(v)\exp^{\chi\lambda^2.t}[cosh(\lambda v+\lambda x) - cosh(\lambda v-\lambda x)]\partial\lambda\partial v \tag{2.67}$$

$$\theta(x,t) = \frac{1}{\pi}\int_{\infty}^{0} f(v). \tag{2.68}$$

The integral in (2.68) will take the form below as given by [24]

$$\int_{\infty}^{0}\exp^{\alpha\lambda^2} coshb\lambda\partial\lambda = \frac{1}{2i}\sqrt{\frac{\pi}{a}}\exp^{\frac{-b^2}{4a}}. \tag{2.69}$$

Comparing (2.69) and (2.68), a = χt; b= (v+x), (v-x), and substituting,

$$\theta(x,t) = \frac{1}{\pi}\int_{\alpha}^{0} f(v)\{\frac{1}{2i}\sqrt{\frac{\pi}{\chi t}}\exp^{\frac{-(v+x)^2}{4\chi}t} - \frac{1}{2i}\sqrt{\frac{\pi}{\chi t}}\exp^{\frac{-(v-x)^2}{4\chi t}}\}\partial v$$

$$\theta(x,t) = \frac{1}{2i\sqrt{\pi\chi t}}\{\int_{\infty}^{0} f(v)\exp^{\frac{-(v+x)^2}{4\chi t}}\partial v - \int_{\infty}^{0} f(v)\exp^{\frac{-(v-x)^2}{4\chi t}}\partial v\} \tag{2.70}$$

$$\frac{(v+x)^2}{4\chi t} = \frac{v+x}{2\sqrt{\chi t}}. \tag{2.71}$$

Let $\frac{(v+x)}{2\sqrt{\chi t}} = \epsilon$ in the first integral and $\frac{v-x}{2\sqrt{\chi t}} = \epsilon$ in the second integral. From $\frac{(v+x)}{2\sqrt{\chi t}} = \epsilon$, v+x = 2 $\epsilon\sqrt{\chi t}$. Let x= 0, v = 2$\epsilon\sqrt{\chi t}$,

$$\partial v = 2\sqrt{\chi t}\partial\epsilon. \tag{2.72}$$

Considering $\frac{v+x}{2\sqrt{\chi t}} = \epsilon$, v+x = 2$\epsilon\sqrt{\chi t}$. Let v =0, x = 2$\epsilon\sqrt{\chi t}$,

$$\frac{x}{2\sqrt{\chi t}} = \epsilon. \tag{2.73}$$

Similarly for $\frac{v-x}{2\sqrt{\chi t}} = \epsilon$,

$$\frac{-x}{2\sqrt{\chi t}} = \epsilon. \tag{2.74}$$

Changing the limit from (0 to ∞) to ($\frac{X}{2\sqrt{\chi t}\infty}$) as v = 0:

$$\theta(x,t) = \frac{1}{2i\sqrt{\pi\chi t}}\{\int_{\frac{x}{2\sqrt{\chi t}}}^{\infty} f(v)\exp^{-\epsilon^2} 2\sqrt{\chi t}\partial\epsilon - \int_{\frac{-x}{2\sqrt{\chi t}}}^{\infty} f(v)\exp^{-\epsilon^2} 2\sqrt{\chi t}\partial\epsilon\}$$

$$\theta(x,t) = \frac{1}{i\sqrt{\pi}\{\int_{\frac{x}{2\sqrt{\chi t}}}^{\infty} f(2\epsilon\sqrt{\chi t}-x)\exp^{-\epsilon^2}\partial\epsilon - \int_{\frac{-x}{2\sqrt{\chi t}}}^{\infty} f(2\epsilon\sqrt{\chi t}+x)\exp^{-\epsilon^2}\partial\epsilon\}} \tag{2.75}$$

However if the initial temperature T_o is a constant and considering our boundary conditions: $T(x,t)|_{t=0} = T_b + Q_o + \theta\ \theta(x,t)|_{t=0} = f(x) = ig(x) = iT_o$. Then: $f(x) = f(2\epsilon\sqrt{\chi t} - x) = iT_o$, $f(x) = f(2\epsilon\sqrt{\chi t} + x) = iT_o$,

$$\theta(x,t) = \frac{1}{i\sqrt{\pi}}\{\int_{x}^{2\sqrt{\chi t}} iT_o\exp^{-\epsilon^2}\partial\epsilon - \int_{-x}^{2\sqrt{\chi t}} iT_o\exp^{-\epsilon^2}\partial\epsilon\},$$

$$\theta(x,t) = \frac{T_o}{\sqrt{\pi}}\{\int_{x}^{2\sqrt{\chi t}} \exp^{-\epsilon^2}\partial\epsilon - \int_{-x}^{2\sqrt{\chi t}} \exp^{-\epsilon^2}\partial\epsilon\}. \tag{2.76}$$

The contemporary error function is given as (2.21):

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty}\exp^{-t^2}\partial t = 1 - erf(x), \tag{2.77}$$

$$erfc(x) = \int_{x}^{\infty}\exp^{-t^2}\partial t = \frac{\sqrt{\pi}}{2}(1 - erf(x)). \tag{2.78}$$

Extracting the terms in equation (2.75) separately and comparing with equation (2.77):

$$\int_{\frac{x}{2\sqrt{\chi t}}}^{\infty}\exp^{-\epsilon^2}\partial\epsilon = \frac{\sqrt{\pi}}{2}(1 - erf(\frac{x}{2\sqrt{\chi t}})) \tag{2.79}$$

$$\int_{\frac{-x}{2\sqrt{\chi t}}}^{\infty}\exp^{-\epsilon^2}\partial\epsilon = \frac{\sqrt{\pi}}{2}(1 - erf(\frac{x}{2\sqrt{\chi t}})) = \frac{\sqrt{\pi}}{2}(1 + erf(\frac{x}{2\sqrt{\chi t}})). \tag{2.80}$$

Substituting equations (2.78) and (2.79) into (2.75):

$$\theta(x,t) = \frac{T_o}{\sqrt{\pi}}\{\frac{\sqrt{\pi}}{2}(1 - erf(\frac{x}{2\sqrt{\chi t}}))\frac{\sqrt{\pi}}{2}(1 + erf(\frac{x}{2\sqrt{\chi t}}))\},$$

$$\theta(x,t) = \frac{T_o}{2}\{1 - erf(\frac{x}{2\sqrt{\chi t}}) - 1 - erf(\frac{x}{2\sqrt{\chi t}}))\},$$

$$\theta(x,t) = \frac{T_o}{2}\{-2erf(\frac{x}{2\sqrt{\chi t}})\},$$

$$\theta(x,t) = -T_o erf(\frac{x}{2\sqrt{\chi t}}), \tag{2.81}$$

Substituting the value of x in (2.43) into (2.80):

$$\theta(x,t) = -T_o erf(\frac{x}{2\sqrt{\chi t}}).\frac{1}{\alpha} ln\frac{x}{\xi},$$

$$\theta(x,t) = -T_o erf(\frac{1}{2\sqrt{\chi t}}).ln\frac{x}{0.001}. \tag{2.82}$$

Putting (2.81) into (2.53),

$$T(x,t) = T_b - T_o erf(\frac{1}{2\alpha\sqrt{\chi t}}) ln\frac{x}{0.001} + Q_o.$$

But $Q = B_b Q_o$, $Q_o = \frac{Q}{B_b}$ and $B_b = \omega_b C_b$, $Q_o = \frac{Q}{\omega_b C_b}$,

$$T(x,t) T_b - T_o erf(\frac{1}{2\alpha\sqrt{\chi t}}) ln\frac{x}{0.001} + \frac{Q}{\omega_b C_b}, \tag{2.83}$$

where T = tissue temperature (^oK), T_0 = core body temperature (^oK), α = thermal diffusivity (m^2/s), χ = tissue thickness (m), t = time (s).

Equation (2.82) above was used to obtain the temperature value for some deep-seated human tissues. For the calculations, table 2 and the following parameters (specific heat capacity of blood = 3800 J/kg/^oK, blood perfusion rate = 0.5 $m^3/kg/s$, Core body temperature, $T_o = 36.8^oC$ and the arterial blood temperature = 37^oC) was employed.

TABLE 2. Thermal parameters of some tissues

Tissue	K (W/m.K)	P (kg/m3)	C (J/kg/∘K)	α (W/J.m2)	χ (m)	x (m)	Q (W/m2)
Liver	0.57	1050	3600	15000000	0.12	0.02	700
Kidney	0.54	1050	3900	13000000	0.03	0.07	900
Brain	0.37	1045	4200	15000000	0.0025	0.17	1200

3. Results and discussion

The tissue temperature obtained using the derived model for varying thermal parameters available in literature are presented in table 3.

Table 3. Obtained tissue temperature of some tissues

Tissue	T(^oK)
Liver	310.52
Kidney	310.62
Brain	310.78

Figures 1a-1c is a graphical presentation of the variation of temperature with tissue depth. It is observed that temperature is proportional to tissue depth. The simulation result in figures 2a-2c gives the relationship between metabolic heat production (Q) and temperature (T) such that as Q changes, there is a corresponding effect on T. In cases of pathology such as cancer, Q increases to provide enough energy as demanded by the proliferating cancer cells leading to an increase in the amount of heat produced thereby elevating T [25]. The relationship between tissue thickness and temperature is an exponential one as illustrated in figures 3a-3c below. From above, it is observed that tissue temperature is affected by metabolic heat produced in the tissue, tissue thickness and distance of tissue from skin surface or tissue depth which suggests that the skin temperature measured using the thermometer placed under the armpit cannot be assumed to be equal to the temperature of the tissue of interest especially in describing pathologies like cancer. Upon comparison with observations from existing works of literature [26,27], the temperature profile seems to be in agreement as the same model was employed under different conditions.

4. Conflict of interest

The authors declare no conflict of interest.

5. Acknowledgment

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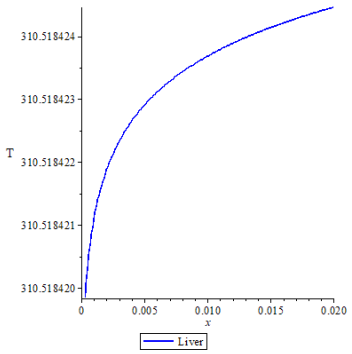

FIGURE 1. Fig.1a: A plot of T against x for liver tissue

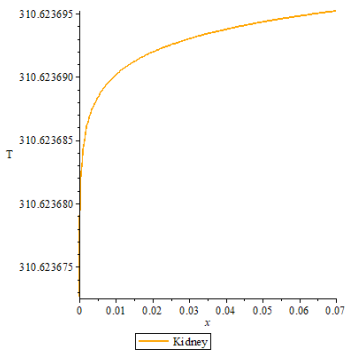

FIGURE 2. Fig. 1b: A plot of T against x for kidney tissue

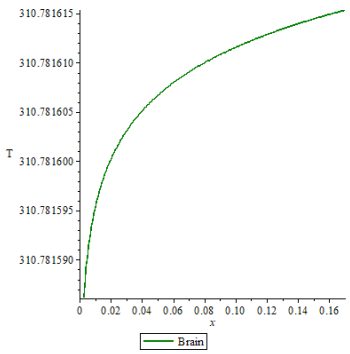

FIGURE 3. Fig.1c: A plot of T against x for brain tissue

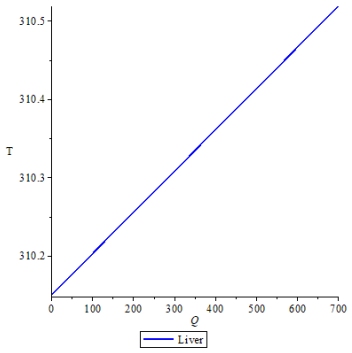

FIGURE 4. Fig.1a: A plot of T against Q for liver tissue

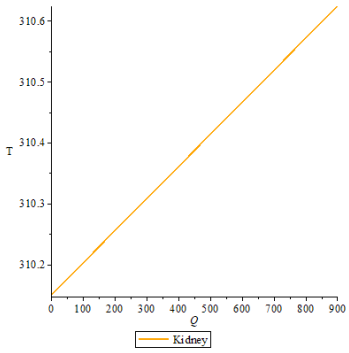

FIGURE 5. Fig. 1b: A plot of T against Q for kidney tissue

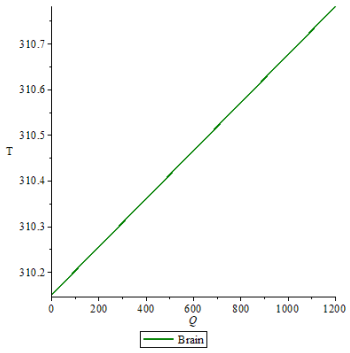

FIGURE 6. Fig.1c: A plot of T against Q for brain tissue

6. CONCLUSION

In this study, Penne's bio-heat equation has been employed in the determination of the temperature value of deep-seated tissues in the body. This analytical model provides an approximate temperature value for the tissue, which from

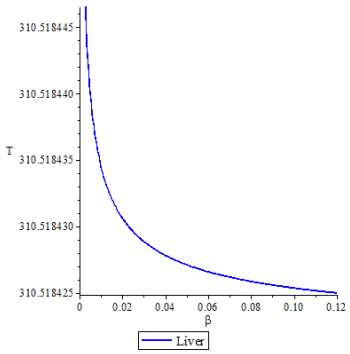

FIGURE 7. Fig.1a: A plot of T against β for liver tissue

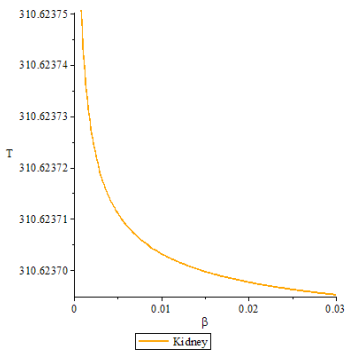

FIGURE 8. Fig. 1b: A plot of T against β for kidney tissue

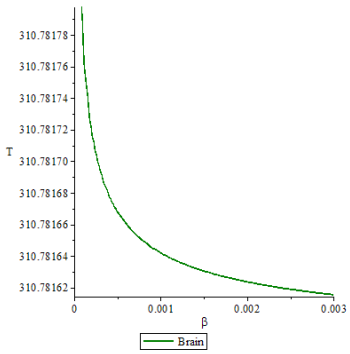

FIGURE 9. Fig.1c: A plot of T against β for brain tissue

theoretical analysis was found to be slightly different from skin temperature, especially in cases of pathology. The results were seen to be in agreement with other investigations in related areas for medical applications such as thermographic imaging of lesions as no mathematical derivation has been given with regards to this area.

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