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THE EXISTENCE OF A SOLUTION OF THE NONLINEAR INTEGRAL EQUATION VIA THE FIXED POINT APPROACH

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ABSTRACT. In this paper, we present some fixed point results for a generalized class of nonexpansive mappings in the framework of uniformly convex hyperbolic space and also propose a new iterative scheme for approximating the fixed point of this class of mappings in the framework of uniformly convex hyperbolic spaces. Furthermore, we establish some basic properties and some strong and $\triangle$-convergence theorems for these mappings in uniformly convex hyperbolic spaces. Finally, we present an application to the nonlinear integral equation and also, a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature with different choices of parameters and initial guesses. The results obtained in this paper extends and generalizes corresponding results in uniformly convex Banach spaces, CAT(0) spaces and other related results in literature.

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1. Introduction

Many real life problems in mathematics, engineering, physics, economics, game theory, and other fields can be turned into fixed point problems, making fixed point theory a useful field of study. In general, it is nearly impossible to solve fixed point problems analytically, necessitating the use of an iterative solution for fixed point problems. Researchers have devised multiple iterative approaches for addressing fixed point problems for various operators over the years. However, work to build quicker and more efficient iterative algorithms is currently ongoing. At the very least, a good and dependable fixed point iteration must have the following characteristics:

(1) it should converge to a fixed point of an operator;
(2) it should be T-stable;
(3) it should be fast compared to other existing iterations in literature;
(4) it should show data dependence result.

The Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{1.1}$$

is one of the earliest iterative processes used to approximate Equation (1.1), where T is a contraction mapping. When T is a nonexpansive mapping, the Picard iterative method fails to approach Equation (1.1), even when the presence of the fixed point is guaranteed. Browder [8] shown that a fixed point exists for the class of nonexpansive self mappings on a closed and bounded subset of a uniformly convex Banach space. Following Browder result, researchers in this field devised many iterative procedures to approximate the fixed point of a nonexpansive mappings and a variety of other nonlinear mappings. Developing quicker and more efficient iterative techniques for approximating fixed points of nonlinear mappings is still a developing area of research. There are several studies on the approximation of fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, and total asymptotically nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces (for example, see [1, 7, 9, 10, 14, 15, 17, 24–28, 31, 33, 35, 36] and the references therein).

In 2017 Karakaya et al. in [16] introduced a new iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $r_0 \in C$, the sequence $\{r_n\}$ in C is defined by

$$(1.2)\qquad \begin{cases} p_n = Tr_n, \\ q_n = (1-\alpha_n)p_n + \alpha_n T p_n \\ r_{n+1} = Tq_n, \ n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that their iterative process converges faster than all of Picard [13], Mann [23], Ishikawa [14], Noor [26], Abass et al. [1] processes and some other existing ones in the literature. Hereafter, for brevity we will call this the *Karakaya Algorithm.*

In 2018 Ullah et al. in [34] introduced a new iteration process called M iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $u_0 \in C$, the sequence $\{u_n\}$ in C is defined by

$$(1.3)\qquad \begin{cases} w_n = (1-\alpha_n)u_n + \alpha_n T u_n, \\ v_n = Tw_n \\ u_{n+1} = Tv_n, \ n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process, the above listed iterative process and some existing ones.

Motivated by the iterative processes (1.3) and (1.2), Abass et. al. [2] introduced the following iterative process. Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $u_0 \in C$, the sequence $\{u_n\}$ in C is defined by

$$(1.4)\qquad \begin{cases} w_n = Tu_n, \\ v_n = Tw_n \\ u_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They established that the rate of convergence of iterative process (1.2), (1.3) and (1.4) are the same, which in turn is faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process, the above listed iterative process and some existing ones in literature.

The role played by the spaces involved in the study of fixed point theory is also quite essential, in addition to the nonlinear mappings involved. In the literature, there are several fixed point results and iterative techniques for estimating the fixed points of nonlinear mappings in Hilbert and Banach spaces. In particular, the reader should see [1, 3–6, 26, 33]. Due to its convex structures, dealing with Banach space is easy. Metric space, on the other hand, does not naturally have this structure. As a result, it becomes necessary to add convex structures to it. Takahashi [32] was the first to establish the notion of convex metric space, studying the fixed points for nonexpansive mappings in convex metric spaces. Several attempts have been made since then to introduce various convex structures on metric spaces. The hyperbolic space is an example of a metric space having a convex structure. Different convex structures have been applied to hyperbolic spaces, resulting in several hyperbolic space definitions, (see [12, 20, 29]). Kohlenbach's [20] class of hyperbolic spaces is slightly more restricted than the class of hyperbolic spaces introduced in [12], but it is more general than the class of hyperbolic spaces introduced in [20]. Moreover, Banach spaces and CAT(0) spaces are well-known examples of hyperbolic spaces described in [29]. Hadamard manifords, Hilbert balls with the hyperbolic metric, Catesian products of Hilbert balls, and R-trees are some further examples of hyperbolic spaces.

In 2020 Chuadchawna et al. in [9] introduced the concept of generalized M-iteration process for approximating the fixed points from Banach spaces to hyperbolic spaces. Using their new iterative process, the authors were able to establish $\triangle$-convergence and strong convergence theorems for the class of mappings satisfying the condition (C_λ) and the condition (E) which is the generalization of Suzuki generalized nonexpansive mappings in the setting of hyperbolic spaces. Let C be a convex subset of a hyperbolic space X and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = W(x_n, Tx_n, \beta_n) \\ y_n = W(Tz_n, z_n, \gamma_n) \\ x_{n+1} = W(Ty_n, y_n, \alpha_n), \ n \geq 0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. They established that the above iterative algorithm converges faster than the M-iteration, as such, converges faster than a whole lot of existing iterative processes in the literature.

Question 1: Now a natural question that arises is, can one construct an iterative process that approximate better, converges faster than iterative process (1.5) and a host of others in the framework of hyperbolic spaces?

We recall the following. Let C be a nonempty subset of a metric space X and $T : C \to C$ a self mapping. A point $x \in X$ is said to be a fixed point of T if $Tx = x$. A mapping $T : C \to C$ is said to be

(1) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;
(2) mean nonexpansive, if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that $\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$, for all $x, y \in C$;
(3) satisfy condition (C), if $\frac{1}{2}\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$; for all $x, y \in C$;
(4) satisfy condition (C_λ), if $\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$, where $\lambda \in [0, 1)$ for all $x, y \in C$;
(5) Suzuki mean nonexpansive mapping if there exist $\alpha, \beta, \in [0, 1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$, $\frac{1}{2}\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$;
(6) generalized mean nonexpansive mapping if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$, $\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$;
(7) α-nonexpansive mapping if there exists $\alpha < 1$ such that for all $x, y \in C$, $\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$;
(8) quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T.

It is worth mentioning that nonexpansive mappings are continuous on their domains but mean nonexpansive, generalized mean nonexpansive, mappings satisfying condition (C), condition (C_λ) need not be continuous. Due to this fact, these mappings are more fascinating and applicable compare to the nonexpansive mappings.

Question 2: Is it possible to introduce a class of mapping, that contains mean nonexpansive, Suzuki mean nonexpansive mapping, generalized mean nonexpansive, mappings satisfying condition (C), condition (C_λ), α-nonexpansive mappings and other nonexpansive type of mappings that is in existence in the literature?

Motivated by all these facts, we provide and affirmative answer to the question raised by introducing a new class of generalized nonexpansive mappings, we study some fixed points properties and demiclosedness principle for this class of mappings in uniformly convex hyperbolic space introduced in [20], and establish both strong and $\triangle$-convergence theorems for approximating the fixed point of this class of generalized nonexpansive mappings using our newly introduced iterative scheme.

2. Preliminaries

We carry out part of our study in the framework of the hyperbolic space introduced by Kohlenbach [20].

Definition 2.1. *A hyperbolic space (X, d, W) is a metric space (X, d) together with a convex mapping $W : X^2 \times [0, 1] \to X$ satisfying:*

(1) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$;
(2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
(3) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$;
(4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$;
for all $w, x, y, z \in X$ and $\alpha, \beta \in [0, 1]$.

Example 1 ([30]). *Let X be a real Banach space which is equipped with norm $||.||$. Define the function $d : X^2 \to [0, \infty)$ by*

$$d(x, y) = ||x - y||$$

as a metric on X. Then, we have that (X, d, W) is a hyperbolic space with mapping $W : X^2 \times [0, 1] \to X$ defined by $W(x, y, \alpha) = (1 - \alpha)x + \alpha y$.

Definition 2.2 ([30]). *Let X be a hyperbolic space with a mapping $W : X^2 \times [0, 1] \to X$.*

(i) *A nonempty subset C of X is said to be convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$.*
(ii) *X is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$*

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided that $d(x, z) \leq r, d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$.

(iii) *A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for a given $r > 0$ and $\epsilon \in (0,2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r (for a fixed ϵ).*

Definition 2.3. *Let C be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in C. For $x \in X$, let $r(\cdot, \{x_n\}) : X \to [0,\infty)$ be a continuous functional defined by*

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(C, \{x_n\})$ of $\{x_n\}$ with respect to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x, \{x_n\}) = \inf\{r(y, \{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C, \{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then we simply denote them by $r(\{x_n\})$ and $A(\{x_n\})$ respectively. It is well-known that in uniformly convex Banach spaces and CAT(0) spaces, bounded sequences have unique asymptotic center with respect to closed convex subsets.

Definition 2.4 ([18])**.** *A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{nk}\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$.*

Remark 2.1 ([21])**.** *We note that $\triangle$-convergence coincides with the usual weak convergence known in Banach spaces with the usual Opial property.*

Lemma 2.1 ([22])**.** *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Lemma 2.2 ([9])**.** *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{n_k}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

Lemma 2.3 ([17]). *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x^*) \leq c$, $\limsup_{n\to\infty} d(y_n, x^*) \leq c$ and $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x^*) = c$, for some $c > 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Definition 2.5. *The mapping $T : C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ satisfying $f(0) = 0$ and $f(t) > 0$ for all $t > (0,\infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$.*

3. Generalized Nonexpansive Mappings

In this section we introduce the notion of generalized nonexpansive mappings and establish some basic properties for this class of mapping.

Definition 3.1. *Let C be a nonempty subset of a hyperbolic pace X. A mapping $T : C \to C$ will be called generalized nonexpansive mapping if there exist $\beta, \gamma.\alpha \in [0,1)$, with $\gamma + \beta < 1$ such that for all $x, y \in C$,*

$$\begin{aligned}(1-\alpha)d(Tx,x) \leq d(x,y) \Rightarrow d(Tx,Ty) &\leq \beta d(y,Tx) + \gamma d(x,Ty) \\ &+ (1-(\gamma+\beta))d(x,y).\end{aligned} \tag{3.1}$$

Proposition 3.1.

(1) Every nonexpansive mapping is a generalized nonexpansive mapping.
(2) Every mean nonexpansive mapping is a generalized nonexpansive mapping.
(3) Every Suzuki mean nonexpansive mapping is a generalized nonexpansive mapping.
(4) All mappings satisfying condition (C) is a nonexpansive mapping.
(5) All mappings satisfying condition (C_λ) is a nonexpansive mapping.

Proposition 3.2. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Then T is quasi-nonexapansive.*

Proof. Let $x \in F(T)$ and $y \in C$,

$$(1-\alpha)d(Tx,x) = (1-\alpha)d(x,x) = 0 < d(x,y).$$

So, we have

$$\begin{aligned} d(x,Ty) = d(Tx,Ty) &\le \beta d(y,Tx) + \gamma d(x,Ty) + (1-(\beta+\gamma))d(x,y) \\ &= \beta d(y,x) + \gamma d(x,Ty) + (1-(\beta+\gamma))d(x,y) \\ \Rightarrow (1-\gamma)d(x,Ty) &\le (1-\gamma)d(x,y) \\ \Rightarrow d(x,Ty) &\le d(x,y). \end{aligned}$$

Hence T is quasi-nonexpanisve. □

Theorem 3.1. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a generalized nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$. Since

$$(1-\alpha)d(Tx_n, x_n) = (1-\alpha)d(x_n, x_n) = 0 < d(x_n, y).$$

so, we have

$$\begin{aligned} d(x_n, Ty) &= d(Tx_n, Ty) \\ &\le \beta d(y, Tx_n) + \gamma d(x_n, Ty) + (1-(\beta+\gamma))d(x_n, y) \\ \Rightarrow d(x_n, Ty) &\le d(x_n, y). \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, y) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n, Ty) = 0.$$

As such, we have that

$$Ty = y.$$

Hence $F(T)$ is closed.

Now suppose that X is strictly convex and C is convex. We show that $F(T)$ is convex. Let $x, y \in F(T), z = W(x, y, \eta) \in C$ with $x \neq y$. Since

$$(1-\alpha)d(x, Tx) = 0 \le d(x, z),$$

we obtain

$$d(x,Tz) = d(Tx,Tz) \le \beta d(z,Tx) + \gamma d(x,Tz) + (1-(\gamma+\beta))d(x,z)$$

$$\Rightarrow d(x,Tz) \le d(x,z). \tag{3.2}$$

Similarly we obtain

$$d(y, Tz) \leq d(y, z). \tag{3.3}$$

Using (3.2) and (3.3) we obtain

$$\begin{aligned} d(x, y) = d(x, W(x, y, \eta)) &\leq d(x, T(W(x, y, \eta)) + d(T(W(x, y, \eta)), y) \\ &\leq d(x, W(x, y, \eta)) + d(W(x, y, \eta), y) \\ &\leq (1-\eta)d(x, x) + \eta d(x, y) + (1-\eta)d(x, y) + \eta d(y, y) \\ &= d(x, y). \end{aligned} \tag{3.4}$$

From (3.4) we can reach a conclusion that (3.2) and (3.3) are $d(x, Tz) = d(x, z)$ and $d(y, Tz) = d(y, z)$, if not, we get a contradict, that is $d(x, y) < d(x, y)$ in (3.4). Hence, we have that

$$Tz = z \Rightarrow z \in F(T).$$

Thus, $F(T)$ is convex. □

In view of Proposition 3.1, we have the following corollaries.

Corollary 3.1. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.2. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.3. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a mapping satisfying condition (C). Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.4. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a mapping satisfying condition (C_λ). Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.5. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a generalized mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.6. *Let C be a nonempty subset of a hyperbolic space X and $T : C \to C$ be a Suzuki mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Theorem 3.2. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of convexity η, $T : C \to C$ be a generalized nonexpansive mappig, and $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle - \lim_{n\to\infty} x_n = p$. Then $p \in F(T)$.*

Proof. Using the fact that $\{x_n\}$ is a bounded sequence and Lemma 2.1, it follows that $\{x_n\}$ has a unique asymptotic center in C. In addition, since $\triangle - \lim_{n\to\infty} x_n = p$, we have that $A(\{x_n\}) = \{p\}$. Now, observe that,

$$\begin{aligned} d(x_n, Tp) &\le d(x_n, Tx_n) + d(Tx_n, Tp) \\ &\le d(x_n, Tx_n) + \beta d(p, Tx_n) + \gamma d(x_n, Tp) + (1-\beta-\gamma) d(x_n, p) \\ &\le d(x_n, Tx_n) + \beta d(p, x_n) + \beta d(x_n, Tx_n) + \gamma d(x_n, Tp) \\ &\quad + (1-\beta-\gamma) d(x_n, p) \\ &= (1+\beta) d(x_n, Tx_n) + \gamma d(x_n, Tp) + (1-\gamma) d(x_n, p), \end{aligned} \tag{3.5}$$

this implies that

$$\begin{aligned} &(1-\gamma) d(x_n, Tp) \le (1+\beta) d(x_n, Tx_n) + (1-\gamma) d(x_n, p) \\ &\Rightarrow d(x_n, Tp) \le \frac{1+\beta}{1-\gamma} d(x_n, Tx_n) + d(x_n, p). \end{aligned} \tag{3.6}$$

Taking $\limsup_{n\to\infty}$ of both sides, we have

$$r(Tp, \{x_n\}) = \frac{1+\beta}{1-\gamma} \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, p) \tag{3.7}$$

$$= \limsup_{n\to\infty} d(x_n, p) = r(p, \{x_n\}). \tag{3.8}$$

Using the uniqueness of the asymptotic center of $\{x_n\}$, we obtain that $Tp = p$. Hence $p \in F(T)$. □

4. Convergence Results

In this section we establish some convergence results for generalized nonexpansive mapping mapping via a iterative algorithm in the framework of uniformly convex hyperbolic space. We define our iterative process as follows: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = W(x_n, Tx_n, \beta_n), \\ y_n = W(z_n, T^2 z_n, \gamma_n), \\ x_{n+1} = W(T^2 z_n, T^2 y_n, \alpha_n), \ n \geq 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

Lemma 4.1. *Let C be a nonempty closed and convex subset of a hyperbolic space X and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (4.1), then, the following hold:*

(i) $\{x_n\}$ *is bounded.*
(ii) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in F(T)$.

Proof. Let $p \in F(T)$. It is easy to see that

$$\begin{aligned} (1-\alpha)d(Tp,p) &= (1-\alpha)d(p,p) = 0 \leq d(x_n,p), \\ (1-\alpha)d(Tp,p) &= (1-\alpha)d(p,p) = 0 \leq d(y_n,p), \\ (1-\alpha)d(Tp,p) &= (1-\alpha)d(p,p) = 0 \leq d(z_n,p). \end{aligned}$$

Now, using (4.1) and Proposition 3.2, we obtain

$$\begin{aligned} d(z_n,p) &= d(W(x_n, Tx_n, \beta_n), p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d(Tx_n,p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{4.2}$$

Also, by using (4.1), (4.2) and Proposition 3.2 we obtain

$$\begin{aligned} d(y_n,p) &= d(W(z_n, T^2 z_n, \gamma_n), p) \\ &\leq (1-\gamma_n)d(z_n,p) + \gamma_n d(T(Tz_n),p) \\ &\leq (1-\gamma_n)d(z_n,p) + \gamma_n d(Tz_n,p) \\ &\leq (1-\gamma_n)d(z_n,p) + \gamma_n d(z_n,p) \\ &= d(z_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{4.3}$$

Lastly, by using (4.1), (4.3) and Proposition 3.2 we obtain

$$\begin{aligned}
d(x_{n+1},p) &= d(W(T^2z_n, T^2y_n, \alpha_n), p) \\
&\leq (1-\alpha_n)d(T^2z_n, p) + \alpha_n d(T^2y_n, p) \\
&= (1-\alpha_n)d(T(Tz_n), p) + \alpha_n d(T(Ty_n), p) \\
&\leq (1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, p) \\
&\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \qquad (4.4) \\
&= d(x_n, p).
\end{aligned}$$

This shows that $\{d(x_n,p)\}$ is bounded and non-increasing for all $x^* \in F(T)$. Thus $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,p)$ exists. □

Lemma 4.2. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (4.1), then* $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. Since $F(T) \neq \emptyset$, suppose that $p \in F(T)$. It follows from Lemma 4.1 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$. Suppose that $\lim_{n\to\infty} d(x_n,p) = c$. From (4.2), we obtain that $d(z_n,p) \leq d(x_n,p)$. Taking lim-sup of both sides, we have

$$\limsup_{n\to\infty} d(z_n,p) \leq c. \qquad (4.5)$$

In addition, using Proposition 3.2, we obtain that $d(Tx_n,p) \leq d(x_n,p)$, and that

$$\limsup_{n\to\infty} d(Tx_n,p) \leq c. \qquad (4.6)$$

From (4.4), we have

$$d(x_{n+1},p) \leq (1-\alpha_n)d(z_n,p) + \alpha_n d(x_n,p).$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$\begin{aligned}
c &\leq (1-\alpha_n)\limsup_{n\to\infty} d(z_n,p) + \alpha_n c \\
c &\leq \liminf_{n\to\infty} d(z_n,p). \qquad (4.7)
\end{aligned}$$

From (4.5) and (4.7), we obtain that $\lim_{n\to\infty} d(z_n, p) = c$. That is

$$\lim_{n\to\infty} d(W(x_n, Tx_n, \beta_n), p) = c.$$

Thus by Lemma 2.3 we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Theorem 4.1. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (4.1), then $\{x_n\}$ $\triangle$-converges to a fixed point of T.*

Proof. It has been established in Lemma 4.1 that $\lim_{n\to\infty} d(x_n, p)$ exists and that $\{x_n\}$ is bounded. Thus $\{x_n\}$ has a $\triangle$-convergent subsequence. In what follows, we are going to establish that every $\triangle$-convergent subsequence of $\{x_n\}$ has a unique $\triangle$-limit in $F(T)$. Let u and v be that $\triangle$-limit of the subsequences $\{u_{n_k}\}$ and $\{v_{n_j}\}$ of $\{x_n\}$. It follows from Lemma 2.1 that $A(C, \{u_n\}) = \{u\}$ and $A(C, \{v_n\}) = \{v\}$. In addition it follows Lemma 4.2 that $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\lim_{n\to\infty} d(v_n, Tv_n) = 0$. In what follows, we will establish that $u = v$. Now since T is a generalized nonexpansive mapping, observe that

$$\begin{aligned} d(u_n, Tu) &\leq d(u_n, Tu_n) + d(Tu_n, Tu) \\ &\leq d(u_n, Tu_n) + \beta d(u, Tu_n) + \gamma d(u_n, Tu) + (1-\beta-\gamma) d(u_n, u) \\ &\leq d(u_n, Tu_n) + \beta d(u, u_n) + \beta d(u_n, Tu_n) + \gamma d(u_n, Tu) \\ &\quad + (1-\beta-\gamma) d(u_n, u) \\ &= (1+\beta) d(u_n, Tu_n) + \gamma d(u_n, Tu) + (1-\gamma) d(u_n, u), \end{aligned} \tag{4.8}$$

this implies that

$$(1-\gamma) d(u_n, Tu) \leq (1+\beta) d(u_n, Tu_n) + (1-\gamma) d(u_n, u)$$

$$\Rightarrow d(u_n, Tu) \leq \frac{1+\beta}{1-\gamma} d(u_n, Tu_n) + d(u_n, u). \tag{4.9}$$

Taking $\limsup_{n\to\infty}$ of both sides, we have

$$r(Tp, \{u_n\}) = \frac{1+\beta}{1-\gamma} \limsup_{n\to\infty} d(u_n, Tu_n) + \limsup_{n\to\infty} d(u_n, u) \tag{4.10}$$

$$= \limsup_{n\to\infty} d(u_n, u) = r(u, \{u_n\}). \tag{4.11}$$

Using the uniqueness of the asymptotic center of $\{u_n\}$, we obtain that $Tu = u$. Hence, $u \in F(T)$. Using similar approach, we have that $v \in F(T)$. It follows from Lemma 4.1 that $\lim_{n\to\infty} d(x_n, y)$ exists. Now, suppose that $u \neq v$, then by the uniqueness of the asymptotic center, we have

$$\lim_{n\to\infty} d(x_n, u) = \lim_{k\to\infty} d(u_{n_k}, u) < \lim_{k\to\infty} d(u_{n_k}, v) = \lim_{n\to\infty} d(x_n, v)$$
$$= \lim_{j\to\infty} d(v_{n_j}, v) < \lim_{j\to\infty} d(v_{n_j}, u) = \lim_{n\to\infty} d(x_n, u).$$

This is a contradiction. So $u = v$. Hence $\{x_n\}$ $\triangle$-converges to a fixed point of $F(T)$ and this completes the proof. $\square$

Theorem 4.2. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (4.1), then, $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.*

Proof. Let $\{x_n\}$ converges to p a fixed point of T. Then $\lim_{n\to\infty} d(x_n, p) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, p)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 4.1 follows that $\lim_{n\to\infty} d(x_n, p)$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $p \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{u_k\}$ is a sequence in $F(T)$ such that for all $n \in \mathbb{N}$,

$$d(x_{n_k}, u_k) < \frac{1}{2^k}$$

it follows from (4.4) that $d(x_{n+1}, u_k) \leq d(x_n, u_k) < \frac{1}{2^k}$, hence

$$\begin{aligned} d(u_{k+1}, u_k) &\leq d(u_{k+1}, x_{n+1}) + d(x_{n+1}, u_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

Thus $\{u_k\}$ is a Cauchy sequence in $F(T)$. Also by Theorem 3.1, we have that $F(T)$ is closed. Thus $\{u_k\}$ is a convergent sequence in $F(T)$. Now, suppose that $\{u_k\}$ converges to $p* \in F(T)$. Therefore, since

$$d(x_{n_k}, p*) \leq d(x_{n_k}, u_k) + d(u_k, p*) \to 0 \text{ as } k \to \infty,$$

we obtain that $\lim_{k\to\infty} d(x_{n_k}, p*) = 0$ and so $\{x_{n_k}\}$ converges strongly to $p* \in F(T)$. Since $\lim_{n\to\infty} d(x_n, p*)$ exists, it follows that $\{x_n\}$ converges strongly to $p*$. □

Theorem 4.3. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \to C$ be a generalized nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (4.1). Let T satisfy condition (I), then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Using Lemma 4.1 and Theorem 4.2 we obtain that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Using the fact that

$$0 \leq \lim_{n\to\infty} f(d(x_n, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0, \ \forall x \in C,$$

and that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0,$ since, f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus by using Theorem 4.2 we obtain that $\{x_n\}$ converges strongly to $p \in F(T)$. □

5. Application to Nonlinear Integral Equation

In this section we present an application of our result to the nonlinear integral equation of the form:

(5.1)
$$x(t) = h(t) + \gamma \int_a^b M(t, s) f(t, x(s)) ds,$$

where $t \in I, \gamma \in (0, \infty), M : I \times I \to \mathbb{R}$, $h : I \to \mathbb{R}$ and $f : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $X = C(I)$ be the space of all continuous function defined on $I = [0, 1]$ and $d : C(I) \times C(I) \to \mathbb{R}$ defined by $d(x, y) = sup_{t\in I}|x(t) - y(t)|$ for all $x, y \in C(I)$. It is well-known that $(C(I), d)$ is a metric space and a hyperbolic space with modulus of uniform convexity. Let Ω be the set of function $\eta : (0, \infty) \to (0, \infty)$ such that η is nondecreasing and $\eta(t) \leq t$ for all $t \in (0, \infty)$.

Theorem 5.1. *Let $X = C(I)$ and $T : X \to X$ the operator given by*

$$Tx(t) = h(t) + \gamma \int_a^b M(t, s) f(t, x(s)) ds,$$

where $t \in I, \gamma \in (0, \infty), M : I \times I \to \mathbb{R}$, $h : I \to \mathbb{R}$ and $f : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions defined on $I = [0, 1]$. Furthermore, suppose the following conditions hold:

(1) *there exists a continuous mapping* $\upsilon : X \times X \to [0, \infty)$ *such that*

$$|f(s, x(s)) - f(s, y(s))| \leq \upsilon(x, y)|x(s) - y(s)|$$

for all $s \in [a, b]$ *and* $x, y \in X$.

(2) *there exists* $\omega \in [0, 1]$, *such that*

$$\int_a^b M(t, s)\upsilon(x, y) \leq \omega.$$

(3) *the sequence* $\{x_n\}$ *defined as in* (4.1) *is bounded and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Then the integral equation (5.1) *has a solution.*

Proof. Without loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s) - x(s)| : s \in [a, b]\} \geq \sup\{|Tx(s) - x(s)| : s \in [a, b]\},$$

which implies that

$$(1 - \lambda)d(Tx, x) \leq d(Tx, x) \leq d(y, x),$$

where $\lambda \in [0, 1)$. Thus, we have that

$$\begin{aligned}
|Ty(s) - Tx(s)| &= \left| h(t) + \gamma \int_a^b M(t, s) f(t, y(s)) \right. \\
&\qquad \left. - h(t) - \gamma \int_a^b M(t, s) f(t, x(s)) ds \right\| \\
&\leq \gamma \int_a^b |M(t, s)[f(t, y(s)) - f(t, x(s))]| ds \\
&\leq \gamma \int_a^b M(t, s)\upsilon(x, y)|y(s) - x(s)| ds \\
&\leq \sup_{s\in[a,b]} |y(s) - x(s)| \gamma \int_a^b M(t, s)\upsilon(x, y) ds \\
&\leq \gamma\omega\|y - x\| \\
&\leq \|y - x\|.
\end{aligned}$$

Thus

$$(1 - \alpha)d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Clearly by taking $\alpha = \frac{1}{2}$, it is easy to see that T is a condition (C) mapping and by Proposition 3.1, T is a generalized nonexpansive mapping and all the conditions in Theorem 4.1 are satisfied. As such T has a fixed point and consequently the integral equation (5.1) has a solution. □

6. Numerical Examples

Example 2. *Let $X = \mathbb{R}$ with metric defined as $d(x,y) = |x-y|$, $W : X^2 \times [0,1] \to X$ by $W(x,y,\beta) = \beta x + (1-\beta)y$ for each $x, y \in X$ and $\beta \in (0,1)$. Thus, (X,d,W) is a complete uniformly hyperbolic space with a monotone modulus of uniformly convexity and C is a nonempty compact convex subset of X. Define a mapping $T : [0,1] \to [0,1]$ as*

$$Tx = \begin{cases} 1 - x \textit{ if } x \in [0, \frac{1}{7}), \\ \frac{x+6}{7} \textit{ if } x \in [\frac{1}{7}, 1]. \end{cases} \tag{6.1}$$

It is easy to see that T satisfies condition (C) and thus it is a generalized nonexpansive mapping.

In what follows, we numerically compare our new iteration process with some existing iterative processes.

Case I: *Taking,* $\alpha_n = \frac{1}{2}, \gamma_n = \frac{1}{3}, \beta_n = \frac{1}{4}$ *and* $x_0 = 0.4$.

Case II: *Taking,* $\alpha_n = \frac{1}{67}, \gamma_n = \frac{1}{89}, \beta_n = \frac{1}{307}$ *and* $x_0 = 0.65$.

Case III: *Taking,* $\alpha_n = \frac{1}{\sqrt{n^{30}+40}}, \gamma_n = \frac{3}{3n^3}, \beta_n = \frac{1}{\sqrt{n^{12}+30}}$ *and* $x_0 = 0.25$.

Case IV: *Taking,* $\alpha_n = \frac{5}{31n^{30}}, \gamma_n = \frac{8}{51n^3}, \beta_n = \frac{7}{213n^2}$ *and* $x_0 = 0.9$.

We used $tol = 1 \times 10^{-7}$ for Case I and Case II and for Case III and IV it is $tol = 1 \times 10^{-9}$.

Comparison shows that the iterative processes (4.1) *converges faster than the iterative processes* (1.5) *and consequently converges faster than some exiting iterative schemes in the literature.*

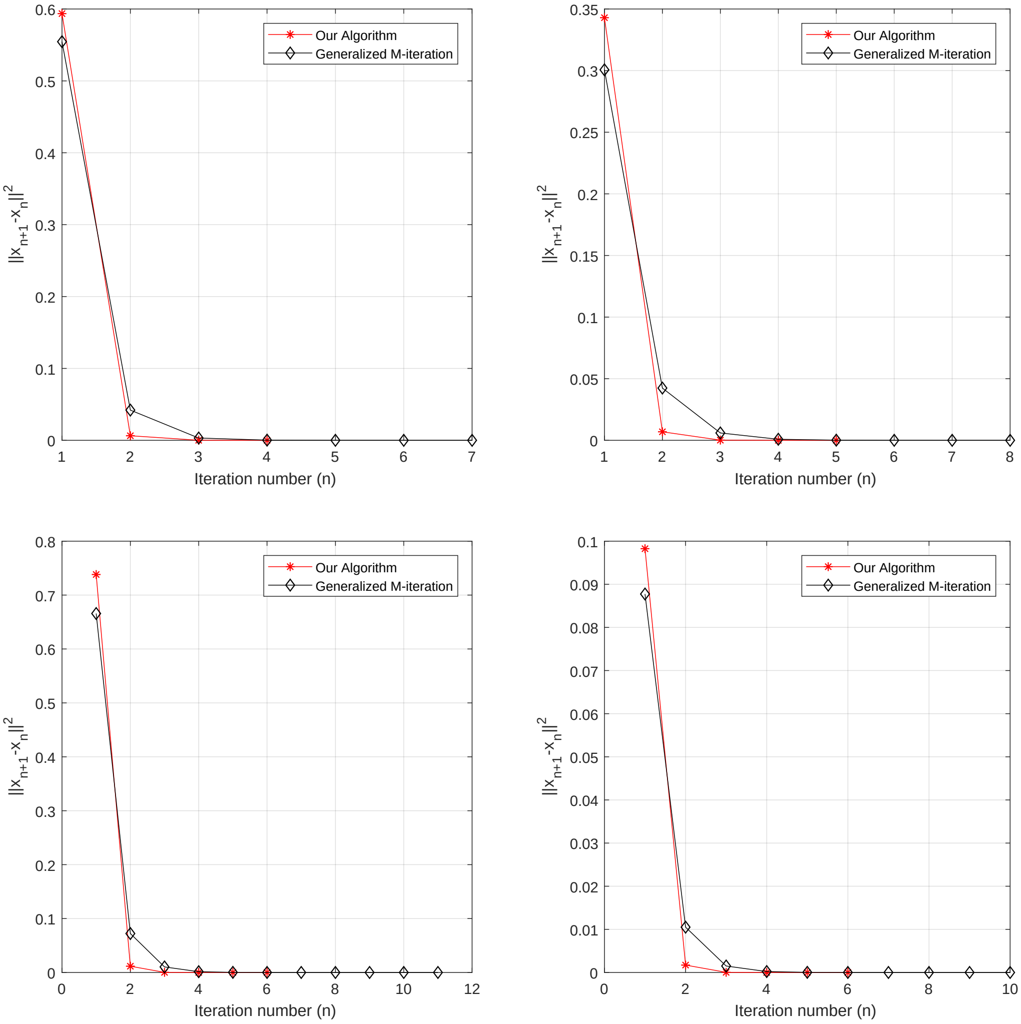

FIGURE 1. Example 2: Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

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