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# 2D ELASTICITY TENSOR INVARIANTS, INVARIANTS DEFINITE POSITIVE CRITERIA

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*To the memory of professor Claude VALLEE with whom this research was performed.*

ABSTRACT. In this paper, we constructed relationships with the differents 2D elasticity tensor invariants. Indeed, let $\mathbf{A}$ be a 2D elasticity tensor. Rotation group action leads to a pair of Lax in linear elasticity. This pair of Lax leads to five independent invariants chosen among six. The definite positive criteria are established with the determined invariants. We believe that this approach finds interesting applications, as in the one of elastic material classification or approaches in orbit space description.

## 1. INTRODUCTION

If, in relation to some orthogonal basis, the components of the stress and strain tensors are $\sigma_{ij}$ and $\epsilon_{ij}$, respectively, Hooke's law takes the form $\sigma_{ij} = \mathbf{A}_{ijkl}\epsilon_{kl}$ where $\mathbf{A}_{ijkl}$ are the components of the fourth-order elasticity tensor $\mathbf{A}$. The components of $\mathbf{A}$ satisfy the symmetry relations $\mathbf{A}_{ijkl} = \mathbf{A}_{jikl} = \mathbf{A}_{ijlk} = \mathbf{A}_{klij}$, which arise from the symmetry of the stress and strain tensors and the requierement that no net work be done by an elastic material in a closed loadind cycle.

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In elasticity, the materials are described by the orbits of the rotation group action on the space of elastic coefficients systems. Their description is done by determination of a finite system of polynomial invariants which separate the orbits.

Although the 3D problem has already been studied by Pratz [1], Cowin [2], Boehler, Kirillov Jr. and Onat [3], Ostrasablin [4], Bona, Bucataru and Slawinski [5], the aim is to solve it exhaustively for the 2D. Curiously, no many attention has been paid in the literature to it, except Vanucci [6] and Vanucci et al [7] following the polar method proposed by Verchery [8]. Recently, Auffray and Ropars [9], Auffray, kolev and Petito [10] de Saxcé and Vallée [11] proposed the study of the 2D case by an alternative method.

In the present work, we treat the planar case (d = 2) although some results are more general and, as we hope, our method could be generalized to the case d = 3. Thus, characterizing the elastic materials amounts to find a convenient parameterization of this set by local charts. In this paper, we have calculated the five invariants of **A** which can be used to designate a certain generic set of materials. Our method is structured as follows. we study the mapping provided from the action of rotation matrix on the elasticity tensor. The calculus are done on the basis of two deviatoric and one spheric matrices. The choice of this basis is not only judicious for the calculus of the invariants but also for the invariants definite positive criteria.

## 2. Definitions and Assignments

2.1. **Rotation group action on the elasticity tensor.** Let $\mathbb{E}L$ be a 2D strain space. This space is three-dimensional and it is a Euclidean space if we consider the scalar product:

$$\begin{aligned} \mathbb{E}L \times \mathbb{E}L &\longrightarrow \mathbb{R} \\ (\epsilon, \epsilon') &\longmapsto \text{tr}(\epsilon\epsilon') \end{aligned}$$

An elasticity tensor can be regarded as a self-adjoint linear mapping $A : \mathbb{E}L \longrightarrow \mathbb{E}L$.

We define a $SO(2)$ group action on the six-dimensional vector space of elasticity tensors by:

$$\forall\, \epsilon\ \in \mathbb{E}L, \forall\, R\ \in SO(2) : A_R(\epsilon) = RA(R^{-1}\epsilon R)R^{-1}.$$

We denote by $R(A) = A_R$ this action. For $R_1$ and $R_2$, two rotations, we can verify that

$$\begin{aligned}(R_1R_2)(A)(\epsilon) &= R_1R_2A\left(R_2^{-1}R_1^{-1}\epsilon R_1R_2\right)R_2^{-1}R_1^{-1}\\ &= R_1\left[R_2A(R_2^{-1}\left(R_1^{-1}\epsilon R_1\right)R_2)R_2^{-1}\right]R_1^{-1}\\ &= R_1\left[R_2(A)\left(R_1^{-1}\epsilon R_1\right)\right]R_1^{-1}\\ &= R_1\left(R_2\left(A\right)\right)(\epsilon).\end{aligned}$$

Then, for the elasticity tensors, the action satisfies the group action law

$$(R_1R_2)(A) = R_1(R_2(A)).$$

2.2. **Derivative of a rotation.** Consider in 2D a rotaion of $SO(2)$ with the angle $\theta$. This rotation can be expressed:

$$R = \cos\theta I_2 + \sin\theta J = \exp(\theta J),$$

where

$$I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

If we derive the above expression of $R$ with respect to $\theta$, we obtain

$$\frac{dR}{d\theta} = RJ = JR \quad \text{and} \quad \frac{dR^{-1}}{d\theta} = -JR^{-1} = -R^{-1}J.$$

2.3. **Pair of Lax in linear elasticity.** The derivative with respect to $\theta$ of the expression $A_R(\epsilon) = RA(R^{-1}\epsilon R)R^{-1}$ is

$$\begin{aligned}\frac{dA_R(\epsilon)}{d\theta} &= JRA(R^{-1}\epsilon R)R^{-1} - RA(R^{-1}\epsilon R)R^{-1}J + RA(R^{-1}(\epsilon J - J\epsilon)R)R^{-1}\\ &= JA_R(\epsilon) - A_R(\epsilon)J + A_R(\epsilon J - J\epsilon).\end{aligned}$$

We introduce the linear mapping $M$ of $\mathbb{E}L$ into $\mathbb{E}L$ defined by:

$$M\epsilon = \epsilon J - J\epsilon$$

then

$$\begin{cases} \frac{dA_R}{d\theta} & = A_R M - M A_R \\ A_R|_{\theta=0} & = A. \end{cases}$$

This pair of Lax proves that

$$A_R = e^{-\theta M} A e^{\theta M}.$$

Thereby, the group $SO(2)$ action is factorized on the elasticity tensors.

## 3. Properties of the linear mapping $M$

(1) *The linear mapping $M$ is anti-self-adjoint*

Indeed, if $\epsilon$ and $\epsilon'$ are two elements of $\mathbb{E}L$, then

$$\begin{aligned} \text{tr}\left[M(\epsilon)\epsilon'\right] &= \text{tr}\left[(\epsilon J - J\epsilon)\epsilon'\right] \\ &= \text{tr}\left[\epsilon(J\epsilon' - \epsilon' J)\right] \\ &= -\text{tr}\left[\epsilon M(\epsilon')\right]. \end{aligned}$$

Thus $M$ is the opposite of its adjoint $M^*$.

(2) *The application $e^{\theta M}$ is an element of $SO(2)$.*

Indeed,

(a)

$$\begin{aligned} \left(e^{\theta M}\right)^* e^{\theta M} &= e^{\theta M^*} e^{\theta M} \\ &= e^{-\theta M} e^{\theta M} = I_E \end{aligned}$$

where $I_E$ is the identity application of $\mathbb{E}L$.

(b) det $\left(e^{\theta M}\right) = e^{\theta \text{tr} M} = e^0 = 1.$

(3) *The kernel of the mapping $M$ is Ker$M = \mathbb{R}I_2$*

(4) *Mapping $M$ expression in the renormed basis of Voigt.*

If we consider the Voigt basis [12] of $\mathbb{E}L$, consisting of

$$E_1' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_2' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad E_3' = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

the matrix of $M$ is the skew-symmetric matrix

$$\begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{bmatrix}.$$

(5) *Expression of $M$ in the basis consisting of a spheric deformation and two deviatoric matrices*

We choose now two unitary orthogonal deviatoric matrices

$$E_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and a unitary spherical deformation

$$E_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

for the basis of $\mathbb{E}L$. This basis is orthonormed. In this basis:

$$ME_1 = E_1 J - JE_1 = 2E_2$$
$$ME_2 = E_2 J - JE_2 = -2E_1$$
$$ME_3 = E_3 J - JE_3 = 0.$$

The linear mapping $M$ is now represented by the matrix:

$$2\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & & 0 \\ & 2J & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(6) *Expression of $e^{\theta M}$ in the basis consisting of a spheric deformation and two deviatoric matrices*

In the basis $(E_1, E_2, E_3)$, the matrix of $e^{\theta M}$ is

$$\begin{bmatrix} & & 0 \\ & e^{2\theta J} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 4. ELASTICITY TENSOR INVARIANTS

Usually, the elasticity tensor $A$ is defined as a fourth order tensor.
We regard this tensor as a linear self-adjoint mapping of $\mathbb{E}L$ into $\mathbb{E}L$, i.e as a second order symmetric mixed tensor.

Let us choose $(E_1, E_2, E_3)$ as a basis of $\mathbb{E}L$ and let us denote by

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} & & A_{13} \\ & \hat{A} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

the matrix representing $A$ in this basis.

The invariants of $A$ are the functions of $A$ invariants for the action

$$\begin{bmatrix} & & 0 \\ & e^{-2\theta J} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & A_{13} \\ & \hat{A} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} & & 0 \\ & e^{2\theta J} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-2\theta J}\hat{A}e^{2\theta J} & e^{-2\theta J}\begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} \\ \begin{bmatrix} A_{13} & A_{23} \end{bmatrix} e^{2\theta J} & A_{33} \end{bmatrix}.$$

The matrices $\begin{bmatrix} & & 0 \\ & e^{-2\theta J} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the elements of an one-dimensional subgroup of $SO(3)$ and also of $GL(3)$ [13]. A subgroup of this subgroup leads to an important number of invariants.

Here, there are the three invariants of $SO(3)$:

- the trace of $A$: $A_{11} + A_{22} + A_{33}$;
- the trace of $A^2$ : $(A_{11})^2 + (A_{22})^2 + 2\,(A_{12})^2 + 2\,(A_{13})^2 + 2\,(A_{23})^2 + (A_{33})^2$;
- the determinant of $A$ : $A_{11}A_{22}A_{33} + 2A_{12}A_{23}A_{13} - (A_{23})^2 A_{11} - (A_{13})^2 A_{22} - (A_{13})^2 A_{33}$.

There are other invariants too:

- the trace of $\hat{A}$ : $A_{11} + A_{22}$;
- the determinant of $\hat{A}$: $A_{11}A_{22} - (A_{12})^2$;
- the coefficient $A_{33}$;
- the square of the length of $\begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix}$: $(A_{13})^2 + (A_{23})^2$ ;
- the square of the length of $\hat{A}\begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix}$: $(A_{11}A_{13} + A_{12}A_{23})^2 + (A_{12}A_{13} + A_{22}A_{23})^2$.

We can construct even more invariants:

- the trace of $A^3$ instead of the determinant of $A$;
- the trace of $\hat{A}^2$ instead of the determinant of the matrix $\hat{A}$;
- etc.

We have determined

- the first degree invariants:

$$A_{11} + A_{22} + A_{33}$$
$$A_{11} + A_{22}$$
$$A_{33}$$

- the second degree invariants:

$$(A_{11})^2 + (A_{22})^2 + (A_{33})^2 + 2\,(A_{12})^2 + 2\,(A_{13})^2 + 2\,(A_{23})^2$$
$$(A_{11})^2 + (A_{22})^2 + 2\,(A_{12})^2$$
$$A_{11}A_{22} - (A_{12})^2$$
$$(A_{13})^2 + (A_{23})^2$$

- the third degree invariants:

$$\det A$$
$$\operatorname{tr}\left(A^3\right)^2$$

- a fourth degree invariant

$$(A_{11}A_{13} + A_{12}A_{23})^2 + (A_{12}A_{13} + A_{22}A_{23})^2$$

Some algebraic relations, called syzygies, between the invariants are already apparent ( see [14–18] for other results).

## 5. Syzygies

(1) First degree syzygies

$$A_{11} + A_{22} + A_{33} = (A_{11} + A_{22}) + A_{33}$$

(2) Second degree syzygies
It is clear that

$$\begin{aligned} &(A_{11})^2 + (A_{22})^2 + (A_{33})^2 + 2\,(A_{12})^2 + 2\,(A_{13})^2 + 2\,(A_{23})^2 \\ &= \left[(A_{11})^2 + (A_{22})^2 + 2\,(A_{12})^2\right] + 2\left[(A_{13})^2 + (A_{23})^2\right] + (A_{33})^2 \end{aligned}$$

and that

$$\begin{aligned} & 2\left[A_{11}A_{22}-(A_{12})^2\right] \\ & = -\left[(A_{11})^2+(A_{22})^2+2\,(A_{12})^2\right]+(A_{11})^2+(A_{22})^2+2A_{11}A_{22} \\ & = \left(\text{tr}\hat{A}\right)^2-\text{tr}\left(\hat{A}^2\right) \end{aligned}$$

This is the classical syzygie.

$$2\text{det }\hat{A}=\left(\text{tr}\hat{A}\right)^2-\text{tr}\left(\hat{A}^2\right)$$

concerning the $2\times 2$ symmetric matrices which can be deduced from the Hamilton-Cayley theorem

$$\left(\text{det }\hat{A}\right)I_2=\left(\text{tr}\hat{A}\right)\hat{A}-\left(\hat{A}\right)^2$$

by taking the trace.

(3) Third degree syzygies

Considering the Hamilton-Cayley theorem for the $3\times 3$ symmetric matrices

$$(\text{det }A)\,I_3=\frac{(\text{tr}A)^2-\text{tr}\,(A^2)}{2}A-(\text{tr}A)\,A^2+A^3$$

and taking the trace, we establish a relation between det $A$, tr $A$, tr$(A^2)$, and tr$(A^3)$:

$$3\text{det }A=\frac{(\text{tr}A)^2-\text{tr}\,(A^2)}{2}\,(\text{tr}A)-(\text{tr}A)\,\text{tr}\left(A^2\right)+\text{tr}\left(A^3\right).$$

Let us remark that

$$\begin{aligned} & 2A_{12}A_{13}A_{23}-A_{11}\,(A_{23})^2-A_{22}\,(A_{23})^2 \\ & =\text{det }A-A_{33}\left(A_{11}A_{22}-(A_{12})^2\right) \\ & =\text{det }A-A_{33}\text{det }\hat{A} \end{aligned}$$

is a simple third degree invariant.

(4) An additional syzygie

It exists a syzygie between the fourth degree invariant and the other invariants.

Let us prove that the fourth degree invariant is

$$\left[\text{tr}\left(\hat{A}^2\right)+\text{det }\hat{A}\right]\left[(A_{13})^2+(A_{23})^2\right]+\left(\text{tr}\hat{A}\right)\left[\text{det }A-A_{33}\text{det }\hat{A}\right]$$

*Proof.* Let us develop the fourth degree invariant expression:

$$(A_{11}A_{13}+A_{12}A_{23})^2+(A_{12}A_{13}+A_{22}A_{23})^2$$
$$=\left[(A_{11})^2+(A_{13})^2\right]+\left[(A_{12})^2+(A_{22})^2\right](A_{23})^2+2\,(A_{11}+A_{22})\,A_{12}A_{13}A_{23}$$
$$=(A_{11}+A_{22})\left[2A_{12}A_{13}A_{23}-A_{11}\,(A_{23})^2-A_{22}\,(A_{13})^2\right]$$
$$+\left[(A_{11}+A_{22})\,A_{11}+(A_{12})^2+(A_{22})^2\right](A_{13})^2$$
$$+\left[(A_{11}+A_{22})\,A_{22}+(A_{12})^2+(A_{11})^2\right](A_{13})^2$$
$$=\left(\text{tr}\hat{A}\right)\left[\det A-A_{33}\det\hat{A}\right]\left[(A_{12})^2+(A_{11})^2+(A_{22})^2+A_{11}A_{22}\right]$$
$$\left[(A_{13})^2+(A_{23})^2\right]$$
$$=\left(\text{tr}\hat{A}\right)\left[\det A-A_{33}\det\hat{A}\right]+\left[\text{tr}\left(\hat{A}^2\right)+\det\hat{A}\right]\left[(A_{13})^2+(A_{23})^2\right]$$
$$=\left(\text{tr}\hat{A}\right)\left[\det A-A_{33}\det\hat{A}\right]+\frac{1}{2}\left[\left(\text{tr}\hat{A}\right)^2+\text{tr}\left(\hat{A}\right)^2\right]\left[(A_{13})^2+(A_{23})^2\right].$$

□

## 6. The choice of independent invariants

One can retain five invariants:

- two first degree invariants :

$$A_{11}+A_{22}$$
$$A_{33}$$

- two second degree invariants:

$$(A_{11})^2+(A_{22})^2+2\,(A_{12})^2$$
$$(A_{13})^2+(A_{23})^2$$

- one third degree invariant

$$\det A=A_{11}A_{22}A_{33}+2A_{12}A_{13}A_{23}-A_{11}\,(A_{23})^2-A_{22}\,(A_{13})]^2-A_{33}\,(A_{12}))^2$$

or

$$\det A-A_{33}\det\hat{A}=2A_{12}A_{13}A_{23}-A_{11}\,(A_{23})^2-A_{22}\,(A_{13})^2.$$

## 7. The isotropy of $A$

$L$ is isotropic if $A_R = A$, i.e.

$$Ae^{\theta M} = e^{\theta M} A.$$

The derivative of this expression with respect to $\theta$ for $\theta = 0$ leads to

$$AM - MA = 0.$$

This necessary condition of the $A$ to commute with $M$ is sufficient because $A$ commutes with $e^{\theta M}$.

So,

$$A \quad \text{is isotropic} \quad \Longleftrightarrow \quad AM - MA = 0.$$

The details of the calculus of the commutator of the matrices $A$ and $M$ leads to

$$\begin{bmatrix} \hat{A}J - J\hat{A} & -J \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} \\ \begin{bmatrix} A_{13} & A_{23} \end{bmatrix} J & 0 \end{bmatrix}.$$

This commutator is zero if and only if

$$A_{22} = A_{11}, \quad A_{12} = 0, \quad A_{13} = 0, \quad A_{23} = 0,$$

i.e., if, and only if, $A$ has the uniaxial form

$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

or, if, and only if,

$$A\epsilon = A_{11} \left( \epsilon - \frac{\text{tr}\epsilon}{2} I_3 \right) + A_{33} \frac{\text{tr}\epsilon}{2} I_3.$$

We remark that in the Voigt basis the matrices of the isotropic tensors are diagonal.

Another advantage of our choice of Voigt renormed basis is : the positivity of the isotropic law is expressed by the inequalities

$$A_{11} \geq 0 \qquad A_{33} \geq 0.$$

When $A$ is isotropic, it is rotationally invariant: its six coefficients are invariants.

## 8. Orbit of an anisotropic elasticity tensor

Let $L$ be an anisotropic elasticity tensor. Consider the vector space of linear self-adjoint $A_Z$ of $\mathbb{E}L$ in $\mathbb{E}L$ defined by

$$A_Z(\epsilon) = D(R \longmapsto A_R(\epsilon))(I_2)(Z)$$

when $Z = \theta J$ describes the Lie algebra of $SO(2)$ ( [20, 21]).

This vector space is $\mathbb{R}(AM - MA)$. Since the elasticity tensor is assumed nonisotropic, this space has dimension 1. The orbit of $A$ is

$$\text{Orb}A = \{A_R | R \in SO(2)\}$$

is an one-dimensional variety plunged in a vector space (dimension 6) of elasticity tensors. Its tangent vector space at $A_R$ is $\mathbb{R}(A_R M - MA_R)$.

## 9. Invariant criteria of the positivity of the elasticity tensor

An elasticity tensor is invariant if its matrix $A$ is positive. This will occur if the invariant $A_{33}$ is positive and if the $2 \times 2$ symmetric matrix

$$A_{33} \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} \begin{bmatrix} A_{13} & A_{23} \end{bmatrix}$$

is positive [22]. The determinant of this $2 \times 2$ matrix is

$$\begin{aligned} &\left[A_{33}A_{11} - (A_{13})^2\right]\left[A_{33}A_{22} - (A_{23})^2\right] - (A_{33}A_{12} - A_{23}A_{13})^2 \\ &= A_{33}\left[A_{11}A_{22}A_{33} - A_{11}(A_{23})^2 - A_{22}(A_{13})^2 - A_{33}(A_{12})^2 + 2A_{12}A_{23}A_{13}\right] \\ &= A_{33}\det A. \end{aligned}$$

This determinant is an invariant. The trace of this above $2 \times 2$ matrix is

$$A_{33}(A_{11} + A_{22}) - \left((A_{13})^2 + (A_{23})^2\right);$$

it is also an invariant. Conditions in terms of invariants necessary for the positivity of the elasticity tensor $A$ are

$$\begin{aligned} &A_{33} \geq 0 \\ &A_{33}\det A \geq 0 \\ &A_{33}\text{tr}\hat{A} \geq (A_{13})^2 + (A_{23})^2. \end{aligned}$$

These conditions are also sufficient.

## 10. Josef Betten method

Let $A$ be an elasticity tensor and Hooke be an isotropic tensor. The Hooke tensor matrix is

$$\begin{bmatrix} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & m \end{bmatrix},$$

where $l$ and $m$ are two scalars.

### 10.1. Invariant polynomial. **Property**: The two variables polynomial

$$P(l,m) = \text{det }(A - \text{Hooke})$$

is invariant under the action of $SO(2)$ defined at paragraph 2.1. Therefore its coefficients are invariant.

*Proof.* As

$$A_R = e^{-\theta M} A e^{\theta M} \quad \text{and} \quad \text{Hooke} = e^{-\theta M} \text{Hooke } e^{\theta M}$$

we deduce

$$\text{det }(A_R - \text{Hooke}) = \text{det }(A - \text{Hooke}).$$

□

### 10.2. Invariant polynomial coefficients. If we develop the $P(l,m)$ polynomial, we have

$$\begin{aligned} P(l,m) = \text{det } A &+ \left[(A_{13})^2 + (A_{23})^2 - A_{33}(A_{11} + A_{22})\right] l \\ &+ \left[(A_{12})^2 - A_{11}A_{22}\right] m + A_{33} l^2 + (A_{11} + A_{22}) lm - ml^2. \end{aligned}$$

We then find the five invariants

$$A_{33}, A_{11} + A_{22}, (A_{12})^2 - A_{11}A_{22}, (A_{13})^2 + (A_{23})^2 \text{ and det } A.$$

**Remark 10.1.** *Josef Betten works in the renormed Voigt basis for which the Hooke tensor is non diagonal and there it becomes a little bit difficult to develop the invariant polynomial into power. The previous property validates his method.*

## 11. Conclusion

We have essentially determined the elasticity tensor invariants grace to the pair of Lax which is highlighted in the paragraph 2.1. This pair of Lax subsists in $3D$. We think that the applied method in 2D can be deployed for the invariants of the 3D elasticity tensor. For the 2D, we were fortunate that the method has put into play a subgroup to a parameter of $SO(2)$. For $3D$, the method will be a little difficult to implement because we must take into account a subgroup to three parameters of $SO(6)$. This technic effort won't be necessary because the method of Josef Betten is unable to provide a sufficient number of invariants.

## References

[1] J. Pratz: *Décomposition canonique des tenseurs de rang 4 de l'élasticité*, Journal de Mécanique Théorique et Appliquée, **2** (1983), 893-913.

[2] S.C. Cowin: *Properties of the anisotropic elasticity tensors*, Q. J. Mech. Appl. Math. **42** (1989), 249-266.

[3] J.-P. Boehler, A.A. Kirillov, E.T. Onat: *On the polynomial invariants of the elastic tensor*, J. of Elast. **34** (1994), 97-110.

[4] N.I. Ostrasablin: *On invariants of the fourth-rank tensor of elastic moduli*, Sib. Zh. Indust. Mat. **1** (1998), 155-163.

[5] I. Bòna, Bucataru, M.A. Slawinski: *Space of SO(3)-orbits of elastic tensors*, Arch. Mech. (Arch. Mech. Stos.) **60** (2008), 123-138.

[6] P. Vannucci: *Plane anisotropy by the polar method*, Meccanica, **40** (2005), 437-454.

[7] P. Vannucci, G. Verchery: *Anisotropy of plane complex elastic bodies*, Int. J. of Solids and Structures, **47** (2010), 1154-1166.

[8] G. Verchery: *Les invariants des tenseurs d'ordre 4 du type de l'élasticité* (1979), in, Proceedings of the Euromech Colloquium 115 Villard-de-Lans, Paris, (1982), 93-104.

[9] N. Auffray, P. Ropars: *Invariant-based reconstruction of bidimensional elasticity tensors*, Int. J. Solides and Structures, **87** (2016), 183–193.

[10] N. Auffray, B. Kolev, M. Petitot: *On anisotropic polynomial relations for the elasticity tensor*, J. Elast. **115**(1) (2014), 77–103.

[11] G. de Saxcé, C. Vallée: *Structure of the space of 2D elasticity tensors*, Discrete & Continuous Dynamical Systems - S, **6**(6) (2013), 1525-1537.

[12] W. Voigt: *Lehrbuch der Kristallphysics*, Teubner, Leipzig, 1910.

[13] H. Bacry: *Leçons sur les théories des groupes et les symétries des particules élémentaires*, Gordon & Breach, Paris-London-New York, 1967.

[14] L. Qi: *Eigenvalues and invariants of tensors*, J. Math. Anal. Appl. **325** (2007), 1363-1377.

[15] F. AHMAD: *Invariants and structural invariants of the anisotropic elasticity tensor*, Q. JI Mech. Appl. Math., **55**(4) (2008), 597-606.

[16] I. BUCATARU, M.A. SLAWINSKI: *Invariant properties for finding distance in space of elasticity tensors*, J. Elast. **94** (2009), 97-114.

[17] M.A. RASHID, F. AHMAD, N. AMIR: *Linear Invariants of a Cartesian Tensor Under SO(2), SO(3) and SO(4)*, Int. J. Theor. Phys., **50**(2) (2011), 479-487.

[18] J.C. NADEAU, M. FERRARI: *Invariant Tensor-to-Matrix Mappings for Evaluation of Tensorial Expressions*, J. Elast., **52**(1) (1998), 43-61.

[19] G. DE SAXCÉ, C. VALLÉE: *Invariant measures of the lack of symmetry with respect to the symmetry groups of 2D elasticity tensors*, J. Elast., **111** (2013), 21–39.

[20] J.-M. SOURIAU: *Structure des systèmes dynamiques*, Dunod, Paris, 1970.

[21] D. MUMFORD, J. FOGARTY, F. KIRWAN: *Geometry invariant theory, Results in Mathematics and Related Areas*, **34**, Springer, Berlin, 1994.

[22] J.-M. SOURIAU: *Calcul linéaire*, PUF, Paris, 1965.

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