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ON A RESCALED NONLOCAL DIFFUSION PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. In this work, we consider the rescaled nonlocal diffusion problem with Neumann Boundary Conditions

$$\begin{cases} u^{\epsilon}_t(x,t) = \dfrac{1}{\epsilon^2}\displaystyle\int_{\Omega} J_{\epsilon}(x-y)(u^{\epsilon}(y,t)-u^{\epsilon}(x,t))dy \\ \qquad\qquad + \dfrac{1}{\epsilon}\displaystyle\int_{\partial\Omega} G_{\epsilon}(x-y)g(y,t)dS_y, \\ u^{\epsilon}(x,0) = u_0(x), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, connected and smooth domain, g a positive continuous function, $J_{\epsilon}(z) = C_1\frac{1}{\epsilon^N}J(\frac{z}{\epsilon}), G_{\epsilon}(x) = C_1\frac{1}{\epsilon^N}G(\frac{x}{\epsilon})$, J and G well defined kernels, C_1 a normalization constant. The solutions of this model have been used without prove to approximate the solutions of a family of nonlocal diffusion problems to solutions of the respective analogous local problem. We prove existence and uniqueness of the solutions for this problem by using the Banach Fixed Point Theorem. Finally, some conclusions are given.

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1. Introduction

Equations of the form

$$\begin{cases} u_t(x,t) = J * u - u = \displaystyle\int_{\Omega} J(x-y)[u(y,t) - u(x,t)]dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where J is a non-negative, smooth, symmetric radially and strictly decreasing function, with $\displaystyle\int_{\mathbb{R}^n} J(x)dx = 1$, supported in the unitary ball, $\Omega \subseteq \mathbb{R}$, are considered as non local diffusion equations. The equation (1.1) has been derived in the reference [1], as an analogous of the classic heat equation

$$\begin{cases} u_t(x,t) = \triangle u(x,t), & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

About (1.1), the non local concept refers to the fact that the density u, does not depend only of the point (x,t) locally, but also on all values of u in a neighborhood of x through the convolution term

$$\int_{\Omega} J(x-y)u(y,t)dy.$$

In this model, if u is thought as the density of a population in a point x in a time t and $J(x-y)$ as the probability distribution of jumping from a point y to x, then the previous convolution is the rate at which the individuals are arriving at the location x from all other positions y.

Remark 1.1. *In this case, it is not permitted jump to x from outside of Ω.*

In a similar way,

$$-\int_{\Omega} J(y-x)u(x,t)dy$$

is the rate at which individuals are leaving the position x to go to other places y. Then in the absence of external forces, we conclude that the density u satisfies the equation (1.1).

Remark 1.2. *In this case, individuals are not allowed to jump out of Ω.*

The Remarks 1.1 and 1.2, guarantees the mass conservation

$$\int_{\Omega} u(x,t)dx = \int_{\Omega} u_0(x)dx.$$

Variants of the equation (1.1) have been used to model phenomena in various branches of the pure and applied sciences [2], [3], [4] y [5]. A wide variety of analytical studies have been performed on various non-local models, [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. For example, the authors in [23] have studied the nonlocal diffusion problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega} J(x-y)(u(y,t)-u(x,t))dy + \int_{\partial\Omega} G(x-y)g(y,t)dS_y, \\ u(x,0) = u_0(x), \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth, bounded and connected domain, the kernel $J : \mathbb{R}^N \to \mathbb{R}$ is a continuous, nonnegative and radially symmetric function, with compact support in the unit ball such that

$$\int_{\mathbb{R}^N} J(z)dz = 1,$$

the kernel $G(x)$ with the same characteristics of the kernel J, the initial data $u_0(x)$ non negative and $g(y,t) \in L^{\infty}_{loc}[(0,\infty); L^1(\partial\Omega)]$. In this model, the individuals can only jump within Ω (what is reflected in the first integral over Ω), and on $\partial\Omega$ is imposed that the number of individuals come in (if g is positive) is $G(x-y)g(x,y)$. This is what in the non-local case, is known as Neumann conditions. The authors studied the existence and uniqueness of solutions for $u_0 \in L^1(\Omega)$, they proved a comparison principle and studied the asymptotic behavior of the solutions as $t \to \infty$. In [24], the authors showed that the solution to the classic Neumman problem

$$\begin{cases} u_t(x,t) - \triangle u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u(x,t)}{\partial \eta} = g(x,t), & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

can be approximated by the solutions of the family of nonlocal diffusion equations

$$
(1.3) \qquad \begin{cases} u_t^\epsilon(x,t) = \dfrac{1}{\epsilon^2}\displaystyle\int_\Omega J_\epsilon(x-y)(u^\epsilon(y,t)-u^\epsilon(x,t))dy \\ \qquad\qquad\quad +\dfrac{1}{\epsilon}\displaystyle\int_{\partial\Omega} G_\epsilon(x-y)g(y,t)dS_y, \\ u^\epsilon(x,0) = u_0(x), \end{cases}
$$

where $J_\epsilon(z) = C_1\frac{1}{\epsilon^N}J(\frac{z}{\epsilon}), G_\epsilon(x) = C_1\frac{1}{\epsilon^N}G(\frac{x}{\epsilon})$, C_1 a normalization constant. They showed that the corresponding solutions u^ϵ, converge to the solution of (1.2), when the parameter ϵ go to zero, in the sense of the weak star convergence, in the L^∞ topology.

The goal of this work, is to prove that the problem (1.3) has an unique solution $u^\epsilon \in C[[0,\infty);L^1(\Omega)]$, for all $u_0 \in L^1(\Omega)$ and for each $g \in L^\infty_{loc}[(0,\infty);L^1(\partial\Omega)]$. Whit this end, we will use the Banach Fixed Point Theorem.

2. Preliminary results

We will use the Banach fixed point theorem to show the existence and uniqueness of solutions for the equation (1.3). For this purpose, given $t_0 > 0$ fix, we will consider the following Banach space defined by

$$B_{t_0} = C([0,t_0];L^1(\Omega)),$$

with the norm given by

$$|||w||| = \max_{0\le t\le t_0} ||w(\cdot,t)||_{L^1(\Omega)} = \max_{0\le t\le t_0}\int_\Omega |w(\cdot,t)|dx = \max_{0\le t\le t_0}\int_\Omega |w(x,t)|dx.$$

We put u instead of w and t instead of s in (1.3) to obtain,

$$
(2.1) \qquad \begin{cases} w_s^\epsilon(x,s) = \dfrac{1}{\epsilon^2}\displaystyle\int_\Omega J_\epsilon(x-y)[w^\epsilon(y,s)-w^\epsilon(x,s)]dy \\ \qquad\qquad\quad +\dfrac{1}{\epsilon}\displaystyle\int_{\partial\Omega} G_\epsilon(x-y)g(y,s)dS_y. \\ w^\epsilon(x,0) = w_0(x) = u_0(x), \end{cases}
$$

here dS_y it is the surface differential. Integrating (2.1) from 0 to t we have

$$w^\epsilon(x,s) = w_0(x) + \frac{1}{\epsilon^2}\int_0^t \int_\Omega J_\epsilon(x-y)[w^\epsilon(y,s) - w^\epsilon(x,s)]dyds$$
$$+ \frac{1}{\epsilon}\int_0^t \int_{\partial\Omega} G_\epsilon(x-y)g(y,s)dS_y ds.$$

Now, we define the operator $T: B_{t_0} \longrightarrow B_{t_0}$ as

$$T_{w_0,g} = w_0(x) + \frac{1}{\epsilon^2}\int_0^t \int_\Omega J_\epsilon(x-y)[w^\epsilon(y,s) - w^\epsilon(x,s)]dyds$$
$$+ \frac{1}{\epsilon}\int_0^t \int_{\partial\Omega} G_\epsilon(x-y)g(y,s)dS_y ds.$$

Lemma 2.1. *The operator $T_{w_0,g}$ is well defined as an application from B_{t_0} to B_{t_0}.*

Proof. Let $g \in L^\infty_{loc}[(0,\infty); L^1(\partial\Omega)]$, $0 < t_1 < t_2 \le t_0$, $w^\epsilon \in B_{t_0}$, $||J_\epsilon||_\infty = K_1$ y $||G_\epsilon||_\infty = K_2$. After calculations, we obtain

$$\left|\left|T_{w_0,g}[w^\epsilon(x,t_1)] - T_{w_0,g}[w^\epsilon(x,t_2)]\right|\right|_{L^1(\Omega)}$$
$$= \int_\Omega \left|T_{w_0,g}[w^\epsilon(x,t_1)] - T_{w_0,g}[w^\epsilon(x,t_2)]\right| dx$$
$$= \int_\Omega \left|\frac{1}{\epsilon^2}\int_{t_1}^{t_2} \int_\Omega J_\epsilon(x-y)[w^\epsilon(y,s) - w^\epsilon(x,s)]dyds\right.$$
$$\left. + \frac{1}{\epsilon}\int_{t_1}^{t_2} \int_{\partial\Omega} G_\epsilon(x-y)g(y,s)dS_y ds\right| dx$$
$$\le (t_2 - t_1)\max\left\{1, \frac{K_1}{\epsilon^2}|\Omega|, \frac{K_2}{\epsilon}|\Omega|\right\}\left\{2|||w||| + ||g||_{L^\infty[(0,t_0);L^1(\partial\Omega)]}\right\}.$$

From the above inequality it follows that the operator is continuous in $t \in (0, t_0]$. For continuity at 0, we see that

$$\left|\left|T_{w_0,g}[w^\epsilon(x,t)] - w_0(x)\right|\right|_{L^1(\Omega)} = \int_\Omega \left|T_{w_0,g}[w^\epsilon(x,t)] - w_0(x)\right| dx$$
$$= \int_\Omega \left|w_0(x) + \frac{1}{\epsilon^2}\int_0^t \int_\Omega J_\epsilon(x-y)[w^\epsilon(y,s) - w^\epsilon(x,s)]dyds\right.$$
$$\left. + \frac{1}{\epsilon}\int_0^t \int_{\partial\Omega} G_\epsilon(x-y)g(y,s)dS_y ds - w_0(x)\right| dx$$
$$\le (t)\max\left\{1, \frac{K_1}{\epsilon^2}|\Omega|, \frac{K_2}{\epsilon}|\Omega|\right\}\left\{2|||w||| + ||g||_{L^\infty[(0,t_0);L^1(\partial\Omega)]}\right\}.$$

Reasoning as before, we have

$$\begin{aligned}&\left|\left|T_{w_0,g}[w^\epsilon(x,t)]-w_0(x)\right|\right|_{L^1(\Omega)}\\&\leq (t)\max\left\{1,\frac{K_1}{\epsilon^2}|\Omega|,\frac{K_2}{\epsilon}|\Omega|\right\}\left\{2|||w|||+||g||_{L^\infty[(0,t_0);L^1(\partial\Omega)]}\right\}.\end{aligned}$$

We thus obtain that the operator is continuous in $t\in[0,t_0]$. In this way, $T_{w_0,g}$ apply B_{t_0} over B_{t_0} and then it is well defined. □

Lemma 2.2. *Let* $w_0,u_0\in L^1(\Omega)$*,* $g,h\in L^\infty[(0,t_0);L^1(\partial\Omega)]$ *y* $w^\epsilon,z^\epsilon\in B_{t_0}$*. Then, there exists a constant* $C=C(\Omega,J_\epsilon,G_\epsilon)$ *such that*

$$\begin{aligned}&|||T_{w_0,g}[w^\epsilon(x,t)]-T_{z_0,h}[z^\epsilon(x,t)]|||\\&\leq ||w_0-z_0||_{L^1(\Omega)}+Ct_0\{|||w^\epsilon-z^\epsilon|||+||g-h||_{L^\infty[(0,t_0);L^1(\partial\Omega)]}\}.\end{aligned}\tag{2.2}$$

Proof. Proceeding as in previous lemma, we have

$$\left|\left|T_{w_0,g}[w^\epsilon(x,t)]-T_{z_0,h}[w^\epsilon(x,t)]\right|\right|_{L^1(\Omega)}=\int_\Omega\left|T_{w_0,g}[w^\epsilon(x,t)]-T_{z_0,h}[z^\epsilon(x,t)]\right|dx$$

$$\begin{aligned}=\int_\Omega\Big|&(w_0(x)-z_0(x))\\&+\frac{1}{\epsilon^2}\int_0^t\int_\Omega J_\epsilon(x-y)[(w^\epsilon(y,s)-z^\epsilon(y,s)-(w^\epsilon(x,s)-z^\epsilon(x,s))]dyds\\&+\frac{1}{\epsilon}\int_0^t\int_{\partial\Omega}G_\epsilon(x-y)[g(y,s)-h(y,s)]dS_yds\Big|dx,\end{aligned}$$

and therefore

$$\begin{aligned}\leq&||w_0(x)-z_0(x)||_{L^1(\Omega)}\\&+\frac{1}{\epsilon^2}\int_\Omega\int_0^t\int_\Omega|J_\epsilon(x-y)||(w^\epsilon(y,s)-z^\epsilon(y,s))|dydsdx\\&+\frac{1}{\epsilon^2}\int_\Omega\int_0^t\int_\Omega|J_\epsilon(x-y)||(w^\epsilon(x,s)-z^\epsilon(x,s))|dydsdx\\&+\frac{1}{\epsilon}\int_\Omega\int_0^t\int_{\partial\Omega}|G_\epsilon(x-y)||g(y,s)-h(y,s)|dS_ydsdx,\end{aligned}$$

from which,

$$
\begin{aligned}
&\left|\left|T_{w_0,g}[w^\epsilon(x,t)] - T_{z_0,h}[w^\epsilon(x,t)]\right|\right|_{L^1(\Omega)}\\
&\leq ||w_0(x) - z_0(x)||_{L^1(\Omega)} + \max\{1, \frac{K_1}{\epsilon^2}|\Omega|\}(t_0)\bigg\{\int_\Omega |(w^\epsilon(y,s) - z^\epsilon(y,s))|dy\\
&\quad + \int_\Omega |(w^\epsilon(x,s) - z^\epsilon(x,s))|dx\bigg\} + \max\{1, \frac{K_2}{\epsilon}|\Omega|\}(t_0)\int_{\partial\Omega} |g(y,s) - h(y,s)|dS_y,\\
&\leq ||w_0(x) - z_0(x)||_{L^1(\Omega)} + Ct_0\bigg\{|||w^\epsilon - z^\epsilon||| + ||g - h||_{L^\infty[(0,t_0);L^1(\partial\Omega)]}\bigg\},
\end{aligned}
$$

with

$$
C = \max\bigg\{1, 2\frac{K_1}{\epsilon^2}|\Omega|, \frac{K_2}{\epsilon}|\Omega|\bigg\}.
$$

□

3. Existence and uniqueness of solutions of Eq.(1.3).

Theorem 3.1. *For $u_0 \in L^1(\Omega)$, there exists a unique solution $u^\epsilon \in C[[0,\infty); L^1(\Omega)]$ of the problem* (1.3).

Proof. In accordance with the Lemma (2.2), if we choose t_0 small enough, such that $Ct_0 < 1$ and we do $z_0 \equiv w_0 \equiv u_0$, $g = h$ in (2.2), we obtain

$$
|||T_{w_0,g}[w^\epsilon(x,t)] - T_{w_0,h}[z^\epsilon(x,t)]|||\leq Ct_0|||w^\epsilon - z^\epsilon|||.
$$

As we have taken Ct_0, such that $0 < Ct_0 < 1$, then, $T_{u_0,g}$ is a strict contraction in the Banach space B_{t_0}, and by the Banach Fixed Point Theorem, $T_{u_0,g}$ has a unique fixed point, in the interval $[0, t_0]$. To extend the solution to the interval $[0,\infty)$, we can take as initial data $u^\epsilon(x,t_0) \in L^1(\Omega)$ and if $u^\epsilon \in B_{t_0}$ is such $|||u^\epsilon||| < \infty$, an a similar argument exposed in the previous lemmas, allows to extend the solution up to interval $[0, t_1)$, with $t_1 > t_0$. Iterating this procedure, we obtain a definite solution in $[0,\infty)$. □

4. CONCLUSIONS

By using the Banach's Fixed Point Theorem we have proved the existence and uniqueness of solutions for a rescaled nonlocal diffusion problem equation (1.3). The results, join to those obtained in reference [24], show us that the nonlocal problems can be use to approximate solutions to respective local problem. Clearly, this is a very interesting fact. The relevance of the nonlocal problems is the widely new applications that are appearing in many branch of the pure and applied sciences. The calculations presented have been taken following the reference [25] and following the ideas presented here, we can to extend the study of nonlocal problems to other interesting models, for instance changing the domain Ω by other more general domains or decomposing $\partial\Omega$.

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