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STRONG CONVERGENCE OF IMPLICIT ITERATIVE ALGORITHMS FOR STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. The class of strictly pseudo-contractive mappings is known to have more powerful applications than the class of nonexpansive mappings in solving nonlinear equations such as inverse and equilibrium problems. Motivated by the potency of the class of strictly pseudo-contractive mappings, a generalized viscosity implicit algorithm is constructed for finding their fixed points in the framework of Banach spaces. The strong convergence of the newly constructed sequence to a fixed point of a strictly pseudo-contractive mapping is obtained under some mild conditions on the parameters and the fixed point is shown to solve some variational inequality problems. An example is given to illustrate the convergence analysis of the newly constructed generalized viscosity implicit algorithm for the class of strictly pseudo-contractive mappings. The example also shows that the algorithm and the conditions which are imposed on the parameters are not just optical illusion.

1. INTRODUCTION

Let E be a real Banach space with dual space E^* and let C be a nonempty subset of E. The duality mapping $J : E \to 2^{E^*}$ is defined as

$$J(x) = \{\varphi \in E^* : \langle x, \varphi \rangle = \|x\|\|\varphi\|, \|x\| = \|\varphi\|\},$$

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where $\langle .,. \rangle$ is the duality pairing between E and E^*. $F(T)$ will denote the set of fixed point of a mapping $T : C \to C$, which is said to be

(i) Lipschitzian if there exists a constant $c > 0$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq c\|x - y\|;$$

(ii) nonexpansive if $c = 1$;

(iii) a contraction if $c \in [0, 1)$;

(iv) λ-strictly pseudo-contractive if for each $x, y \in C$, there exists a constant $\lambda > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

where I is the identity operator (See Browder and Petryshyn [5]). A reform of the inequality (1.1) is

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \lambda\|(I - T)x - (I - T)y\|^2. \tag{1.2}$$

The equivalence of (1.1) (and so (1.2)) in the Hilbert spaces is

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + c\|(I - T)x - (I - T)y\|^2, \tag{1.3}$$

where $c = (1 - 2\lambda) < 1$.

Remark 1.1. *Obviously, the class of nonexpansive mappings is a subset of the class of strictly pseudo-contractive mappings.*

Many mathematical models for real life analysis fall under the initial value problem of the form

$$x'(t) = f\left(x(t)\right),\ x(t_0) = x_0. \tag{1.4}$$

Most ordinary differential equations are known to defy the analytical methods for finding their solutions. Numerical methods emerge as essential ways of dealing with time-dependent ordinary and partial differential equations. Analysis of the physical processes by computer simulation involves numerical methods. Most famous among numerical methods are the implicit procedures. Let T be a nonexpansive mapping associated with a contraction operator f and $\{\sigma_n\}_{n=1}^{\infty} \subset (0, 1)$. The semi-implicit sequence,

$$x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.5}$$

was introduced by Xu et al. [20]. The sequence (1.5) was shown to converge to a fixed point p of T which also solves the variational inequality

$$\langle (I-f)p, x-p\rangle \geq 0,\ \forall\, x \in F(T), \tag{1.6}$$

where $\langle , \rangle$ is the inner product. Recently, the convergence of the semi-implicit sequence

$$\begin{cases} y_n = \sigma_n f(x_n) \oplus (1-\sigma_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n, \qquad\qquad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, was considered in a complete CAT(0) spaces by Xiong and Lan [17]. A generalized form of the semi-implicit sequence (1.5) is

$$x_{n+1} = \sigma_n^1 f(x_n) + \sigma_n^2 x_n + \sigma_n^3 T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right),\ n \in \mathbb{N}, \tag{1.8}$$

where $\left\{\{\sigma_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}, \{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ and $\sum_{i=1}^{3} \sigma_n^i = 1$. The sequence (1.8) was introduced by Ke and Ma [7]. It appears that a lot of research efforts for over a decade have been devoted on the implicit algorithms for the class of nonexpansive mappings (See e.g, Aibinu [1,3], Xiong and Lan [15,16], Cai et al. [6], Luo et al. [9] and references therein).

We are motivated by the previous works on the implicit iterative sequence to study a generalized form of (1.7) for the class of strictly pseudo-contractive mappings. The strict contraction f in (1.7) is replaced by the generalized contraction. Also, the semi-implicit sequence in (1.7) is changed to an arbitrary real real sequence in $(0,1)$. Precisely, for a nonempty closed convex subset C of a uniformly smooth Banach space E and real sequences $\{\delta_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [0,1]$, $\{\gamma_n\}_{n=1}^{\infty} \subset [0,1)$ and $\left\{\{\sigma_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset (0,1)$ such that $\sum_{i=1}^{3} \sigma_n^i = 1$, the implicit viscosity algorithm is defined from an arbitrary $x_1 \in K$ by

$$\begin{cases} y_n = \sigma_n^1 f(x_n) + \sigma_n^2 x_n + \sigma_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \qquad\qquad n \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $S_n x := \gamma_n x + (1-\gamma_n)Tx$, f is a generalized contraction and T is a λ-strictly pseudo-contractive mapping. The conditions are established for the strong convergence of the sequence (1.9) to a fixed point p of T and in relation to the solution of the variational inequality

$$\langle (I-f)p, J(x-p)\rangle \geq 0, \text{ for all } x \in F(T). \tag{1.10}$$

The importance and application of the class of strictly pseudo-contractive mappings in solving several nonlinear problems justify the efforts for this research. The iteration procedures are succinct and easy to follow. An example is given to illustrate the convergence analysis of (1.9) for the class of strictly pseudo-contractive mappings.

2. Preliminaries

The definitions and some known results which are essential in obtaining the main results of this paper are recalled in this section.

Definition 2.1. *Let E and E^* respectively denote a real Banach space and its dual. The modulus of smoothness of E is the function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ defined by*

$$\omega(k) = \sup\left\{ \frac{\|x+ky\| + \|x-ky\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}.$$

E is said to be uniformly smooth if $\lim_{k\to 0} \frac{\omega(k)}{k} = 0$. In a uniformly smooth Banach space, the duality mapping J is known to be single valued and uniformly continuous on any bounded subset of E.

Definition 2.2. *Let (E,d) be a metric space and C a subset of E. $f : C \to C$ is a mapping defined on C.*

- (i) *f is said to be a Meir-Keeler contraction if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in C$, with $\epsilon \leq d(x,y) < \epsilon + \delta$, we have $d(f(x), f(y)) < \epsilon$.*
- (ii) *Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{R}^+$ the set of all positive real numbers. A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an L-function if $\psi(0) = 0,\ \psi(k) > 0$ for all $k > 0$ and for every $s > 0$, there exists $u > s$ such that $\psi(k) \leq s$ for each $k \in [s,u]$.*
- (iii) *$f : E \to E$ is called a (ψ, L)-contraction if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x,y))$, for all $x, y \in E,\ x \neq y$.*

The generalized contraction mappings in this paper will refer to Meir-Keeler contractions or (ψ, L)-contractions. It is assumed that the L-function from the definition of (ψ, L)-contraction is continuous, strictly increasing and $\lim_{k\to\infty} \phi(k) = \infty$, where $\phi(k) = k - \psi(k)$ for all $k \in \mathbb{R}^+$ (Ke and Ma [7]). Whenever there is no confusion, $\phi(k)$ and $\psi(k)$ will be written as $\phi\, k$ and $\psi\, k$, respectively.

The following interesting results about the Meir-Keeler contraction are readily available.

Proposition 2.1. *Let (E, d) be a complete metric space and let f be a Meir-Keeler contraction on E. Then f has a unique fixed point in E (See Meir and Keeler [10]).*

Proposition 2.2. *Let E be a Banach space, C a convex subset of E and $f : C \to C$ a Meir-Keeler contraction. Then $\forall\ \epsilon > 0$, there exists $c \in (0, 1)$ such that*

$$\|f(u) - f(v)\| \leq c\|u - v\| \tag{2.1}$$

for all $u, v \in C$ with $\|u - v\| \geq \epsilon$ (See Suzuki [13]).

Proposition 2.3. *Let (E, d) be a metric space and $f : E \to E$ be a mapping. The following assertions are equivalent (See Lim [8]):*

(i) *f is a Meir-Keeler type mapping;*
(ii) *there exists an L-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a (ψ, L)-contraction.*

Proposition 2.4. *Let C be a nonempty convex subset of a Banach space E, $T : C \to C$ a nonexpansive mapping and $f : C \to C$ a Meir-Keeler contraction. Then Tf and fT are Meir-Keeler contractions (See Lim [8]).*

The following Lemmas are needed in the sequel.

Lemma 2.1. *Let E be a real smooth Banach space. Suppose one of the followings holds:*

(i) *j is uniformly continuous on any bounded subset of E.*
(ii) *$\langle u - v, ju - jv\rangle \leq \|u - v\|^2$, for all $u, v \in E$.*
(iii) *For any bounded subset C_1 of E, there is a ω such that*

$$\langle u - v, ju - jv\rangle \leq \omega\left(\|u - v\|\right),\ \vee\ u, v \in C_1,$$

where ω satisfies $\lim_{k\to 0^+} \frac{\omega(k)}{k} = 0$.

Then, for any $\epsilon > 0$ and any bounded subset C_2, there is δ such that

$$\|ku + (1 - k)v\|^2 \leq 2k\langle u, jv\rangle + 2k\epsilon + (1 - 2k)\|v\|^2$$

for any $u, v \in C_2$ and $k \in [0, \delta)$ (See Park [11]).

Lemma 2.2. *Let* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *be bounded sequences in a Banach space* E *and* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$. *Suppose that* $u_{n+1} = (1-\lambda_n)u_n + \lambda_n v_n$ *for all* $n \geq 0$ *and* $\limsup_{n\to\infty} (\|u_{n+1} - u_n\| - \|v_{n+1} - v_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|u_n - v_n\| = 0$ *(See Suzuki [14]).*

Lemma 2.3. *Let* C *be a nonempty closed and convex subset of a uniformly smooth Banach space* E. *Let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$ *and* $f : C \to C$ *be a generalized contraction mapping. Assume that* $\{x_k\}$ *defined by* $x_k = kf(x_k) + (1-k)Tx_k$ *for* $k \in (0,1)$, *converges strongly to* $p \in F(T)$ *as* $k \to 0$. *Suppose that* $\{x_n\}$ *is a bounded sequence such that* $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$. *Then (See Sunthrayuth and Kumam [12]),*

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle \leq 0.$$

Lemma 2.4. *Let* C *be a nonempty closed and convex subset of a uniformly smooth Banach space* E. *Let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$ *and* $f : C \to C$ *be a generalized contraction mapping. Then* $\{x_k\}$ *defined by* $x_k = kf(x_k) + (1-k)Tx_k$ *for* $k \in (0,1)$, *converges strongly to* $p \in F(T)$, *which solves the following variational inequality (See Sunthrayuth and Kumam [12]):*

$$\langle f(p) - p, J(z - p)\rangle \leq 0, \ \forall\, z \in F(T).$$

Lemma 2.5. *Let* $\{\sigma_n\}$ *be a sequence of nonnegative real numbers satisfying the property*

$$\sigma_{n+1} = (1-\gamma_n)\sigma_n + \gamma_n\beta_n, \ n \in \mathbb{N},$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset \mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then $\{\sigma_n\}$ *converges to zero, as* $n \to \infty$ *(See Xu [19]).*

3. Main Results

Assumption 3.1. *Let* E *be a uniformly smooth Banach space and* C *be a nonempty bounded closed convex subset* E. *Let* $f : C \to C$ *be a generalized contraction mapping and* T *a* λ*-strictly pseudo-contractive mapping defined on* C *such that*

$F(T) \neq \emptyset$. The real sequences $\left\{\{\sigma_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}$ are in $(0,1)$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are in $[0,1]$ and $\{\gamma_n\}_{n=1}^{\infty}$ is in $[0,1)$. The boundedness and convergence of the iterative sequence (1.9) are studied under the following conditions:

(i) $\sum_{i=1}^{3} \sigma_n^i = 1;$

(ii) $\lim_{n\to\infty} \sigma_n^1 = 0,\ \sum_{n=1}^{\infty} \sigma_n^1 = \infty;$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

(iv) $\lim_{n\to\infty} \sigma_n^3 = 0,\ \lim_{n\to\infty} |\sigma_{n+1}^2 - \sigma_n^2| = 0;$

(v) $0 < \delta_n \leq \delta_{n+1} \leq \delta < 1$ *for all* $n \in \mathbb{N}$.

Under the conditions (i)-(v) of Assumption 3.1 stated above, this study establishes the convergence of the iterative scheme (1.9).

Firstly, S_n is shown to be nonexpansive for all $n \in \mathbb{N}$. Indeed, by taking

$$0 < \epsilon \leq \lambda \|Tx - Ty - (x-y)\|^2$$

for all $x, y \in C$ and applying Lemma 2.1,

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \|(1-\gamma_n)x + \gamma_n Tx - (1-\gamma_n)y - \gamma_n Ty\|^2 \\
&= \|(1-\gamma_n)(x-y) + \gamma_n(Tx - Ty)\|^2 \\
&\leq 2\gamma_n \langle Tx - Ty, j(x-y)\rangle + 2\epsilon\gamma_n + (1-2\gamma_n)\|x-y\|^2 \\
&\leq 2\gamma_n \left(\|x-y\|^2 - \lambda\|Tx - Ty - (x-y)\|^2\right) \\
&\quad +2\epsilon\gamma_n + (1-2\gamma_n)\|x-y\|^2 \\
&\leq \|x-y\|^2 - 2\gamma_n\lambda\|Tx - Ty - (x-y)\|^2 + 2\epsilon\gamma_n \\
&\leq \|x-y\|^2.
\end{aligned} \tag{3.1}$$

Next is to show that for all $v \in C$, the mapping defined by

$$\begin{aligned}
u &\mapsto T_v(u) \\
&=: \beta_n v + (1-\beta_n)\left[\sigma_n^1 f(v) + \sigma_n^2 x + \sigma_n^3 S_n\left(\delta_n v + (1-\delta_n)u\right)\right],
\end{aligned} \tag{3.2}$$

for all $u \in C$, is a contraction.

Obviously, for all $x, y \in C$,

$$\begin{aligned}
\|T_v(x) - T_v(y)\| &= \sigma_n^3(1-\beta_n)\|S_n(\delta_n v + (1-\delta_n)x) - S_n(\delta_n v + (1-\delta_n)y)\| \\
&\leq \sigma_n^3(1-\beta_n)(1-\delta_n)|x-y\|.
\end{aligned} \tag{3.3}$$

T_v is therefore a contraction with coefficient $\sigma_n^3(1-\beta_n)(1-\delta_n) \in (0,1)$ and Banach's contraction mapping principle ascertains that T_v has a fixed point. This indicates that the sequence of iteration (1.9) is well defined. Observe that for each $n \in \mathbb{N}, x \in F(T) \Leftrightarrow x \in F(S_n)$. Indeed, suppose $x \in F(T)$, then

$$S_n x = \gamma_n x + (1-\gamma_n)Tx = \gamma_n x + (1-\gamma_n)x = x.$$

Thus, $x \in F(S_n)$. Also, suppose $x \in F(S_n)$, then

$$\begin{aligned} 0 &= x - S_n x \\ &= x - \gamma_n x - (1-\gamma_n)Tx \\ &= (1-\gamma_n)(x - Tx). \end{aligned} \tag{3.4}$$

Since $(1-\gamma_n) \neq 0$, (3.4) holds if and only if $Tx = x$. Thus, $x \in F(T)$. Hence $F(T) = F(S_n) \neq \emptyset$.

The proof of the following lemmas which are useful in establishing the main result are given.

Lemma 3.1. *Let E be a uniformly smooth Banach space and C be a nonempty bounded closed convex subset E. Let $f : C \to C$ be a generalized contraction mapping and T a λ-strictly pseudo-contractive mapping defined on C such that $F(T) \neq \emptyset$. From an arbitrary $x_1 \in C$, an iterative sequence $\{x_n\}_{n=1}^\infty$ which is defined by (1.9) is shown to bounded under the conditions (i)-(v) of Assumption 3.1.*

Proof. The boundedness of the sequence $\{x_n\}_{n=1}^\infty$ is being established here. For $p \in F(T)$,

$$\begin{aligned} \|y_n - p\| &= \|\sigma_n^1 f(x_n) + \sigma_n^2 x_n + \sigma_n^3 S_n(\delta_n x_n + (1-\delta_n)x_{n+1}) - p\| \\ &\leq \sigma_n^1\|f(x_n) - p\| + \sigma_n^2\|x_n - p\| + \sigma_n^3\|S_n(\delta_n x_n + (1-\delta_n)x_{n+1}) - p\| \\ &\leq \sigma_n^1\|f(x_n) - f(p)\| + \sigma_n^1\|f(p) - p\| + \sigma_n^2\|x_n - p\| \\ &\quad + \sigma_n^3\|\delta_n x_n + (1-\delta_n)x_{n+1} - p\| \\ &= \sigma_n^1\|f(x_n) - f(p)\| + \sigma_n^1\|f(p) - p\| + \sigma_n^2\|x_n - p\| \\ &\quad + \sigma_n^3\|\delta_n(x_n - p) + (1-\delta_n)(x_{n+1} - p)\| \end{aligned}$$

$$
\begin{aligned}
&\leq \sigma_n^1\psi\|x_n-p\|+\sigma_n^1\|f(p)-p\|+\sigma_n^2\|x_n-p\| \\
&\quad +\sigma_n^3\delta_n\|x_n-p\|+\sigma_n^3(1-\delta_n)\|x_{n+1}-p\| \\
&\leq \left(\sigma_n^1\psi+\sigma_n^2+\sigma_n^3\delta_n\right)\|x_n-p\|+\sigma_n^1\|f(p)-p\| \\
&\quad +\sigma_n^3(1-\delta_n)\|x_{n+1}-p\| \\
&= \left(\sigma_n^1\psi+(1-\sigma_n^1-\sigma_n^3)+\sigma_n^3\delta_n\right)\|x_n-p\| \\
&\quad +\sigma_n^1\|f(p)-p\|+\sigma_n^3(1-\delta_n)\|x_{n+1}-p\| \\
&= \left(1-\sigma_n^1(1-\psi)-\sigma_n^3(1-\delta_n)\right)\|x_n-p\| \\
&\quad +\sigma_n^1\|f(p)-p\|+\sigma_n^3(1-\delta_n)\|x_{n+1}-p\| \\
&= \left(1-\sigma_n^3(1-\delta_n)-\sigma_n^1\phi\right)\|x_n-p\| \\
&\quad +\sigma_n^1\|f(p)-p\|+\sigma_n^3(1-\delta_n)\|x_{n+1}-p\|.
\end{aligned}
$$

Since $\big\{\{\sigma_n^i\}_{n=1}^\infty\big\}_{i=1}^3\subset(0,1)$ and $\{\delta_n\}_{n=1}^\infty\subset[0,1]$, it obvious that $1-\sigma_n^3(1-\delta_n)>0.$ So, it is obtained that

$$
\|y_n-p\| \leq \frac{1-\sigma_n^3(1-\delta_n)-\sigma_n^1\phi}{1-\sigma_n^3(1-\delta_n)}\|x_n-p\|+\frac{\sigma_n^1}{1-\sigma_n^3(1-\delta_n)}\|f(p)-p\|.
$$

It is further known from (1.9) that

$$
\begin{aligned}
\|x_{n+1}-p\| &\leq \beta_n\|x_n-p\|+(1-\beta_n)\|y_n-p\| \\
&= \frac{1-\sigma_n^3(1-\delta_n)-\sigma_n^1(1-\beta_n)\phi}{1-\sigma_n^3(1-\delta_n)}\|x_n-p\| \\
&\quad +\frac{\sigma_n^1(1-\beta_n)}{1-\sigma_n^3(1-\delta_n)}\|f(p)-p\| \\
&= \left(1-\frac{\sigma_n^1(1-\beta_n)\phi}{1-\sigma_n^3(1-\delta_n)}\right)\|x_n-p\| \\
&\quad +\frac{\sigma_n^1(1-\beta_n)\phi}{1-\sigma_n^3(1-\delta_n)}.\phi^{-1}\|f(p)-p\| \\
&\leq \max\left\{\|x_n-p\|,\ \phi^{-1}\|f(p)-p\|\right\}. \qquad (3.5)
\end{aligned}
$$

Then by induction, we have

$$
\|x_{n+1}-p\|\leq\max\left\{\|x_1-p\|,\ \phi^{-1}\|f(p)-p\|\right\}.
$$

Thus, $\{x_n\}_{n=1}^\infty$ is bounded. It can be inferred that $\{f(x_n)\}_{n=1}^\infty$ and $\{S_nw_n\}_{n=1}^\infty$ are bounded since $\{x_n\}_{n=1}^\infty$ is bounded, where $w_n=:\delta_nx_n+(1-\delta_n)x_{n+1}$. Indeed, for

$p \in F(T)$,

$$\begin{aligned}
\|f(x_n)\| &= \|f(x_n) - f(p) + f(p)\| \\
&\leq \|f(x_n) - f(p)\| + \|f(p)\| \\
&\leq \psi\|x_n - p\| + \|f(p)\| \\
&\leq \max\left\{\psi\|x_1 - p\|,\ \psi\phi^{-1}\|f(p) - p\|\right\} + \|f(p)\| \text{ (by induction).}
\end{aligned}$$

Also,

$$\begin{aligned}
\|S_n(\delta_n x_n + (1-\delta_n)x_{n+1})\| &= \|S_n(\delta_n x_n + (1-\delta_n)x_{n+1}) - p + p\| \\
&\leq \|S_n(\delta_n x_n + (1-\delta_n)x_{n+1}) - S_n p\| + \|p\| \\
&\leq \|\delta_n x_n + (1-\delta_n)x_{n+1} - p\| + \|p\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)\|x_{n+1} - p\| + ||p|| \\
&\leq \max\left\{\|x_1 - p\|,\ \phi^{-1}\|f(p) - p\|\right\} + \|p\| \\
&\quad \text{(by induction).}
\end{aligned}$$

□

Lemma 3.2. *Let E be a uniformly smooth Banach space and C be a nonempty bounded closed convex subset E. Let $f : C \to C$ be a generalized contraction mapping and T a λ-strictly pseudo-contractive mapping defined on C such that $F(T) \neq \emptyset$. $\{\delta_n\}_{n=1}^{\infty} \subset [0,1]$ is a real sequences and $w_n =: \delta_n x_n + (1-\delta_n)x_{n+1}$. Let $Q_1 = \sup_n \|w_n - S_n(w_n)\|$, then*

$$\begin{aligned}
\|S_{n+1}(w_{n+1}) - S_n(w_n)\| &\leq (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\| \\
&\quad +\delta_n\|x_{n+1} - x_n\| + (\gamma_{n+1} - \gamma_n)Q_1.
\end{aligned} \tag{3.6}$$

Proof.

$$\begin{aligned}
\|S_{n+1}(w_{n+1}) - S_n(w_n)\| &= \|S_{n+1}(w_{n+1}) - S_{n+1}(w_n) + S_{n+1}(w_n) - S_n(w_n)\| \\
&\leq \|S_{n+1}(w_{n+1}) - S_{n+1}(w_n)\| + \|S_{n+1}(w_n) - S_n(w_n)\| \\
&= \|w_{n+1} - w_n\| + \|\gamma_{n+1}w_n + (1-\gamma_{n+1})Tw_n \\
&\quad -\gamma_n w_n - (1-\gamma_n)Tw_n\| \\
&= \|w_{n+1} - w_n\| + \|(\gamma_{n+1} - \gamma_n)(w_n - Tw_n)\|
\end{aligned}$$

$$\begin{aligned}
&= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2} - \delta_n x_n - (1-\delta_n)x_{n+1}\| \\
&\quad + \|(\gamma_{n+1} - \gamma_n)(w_n - Tw_n)\| \\
&= \|(1-\delta_{n+1})(x_{n+2} - x_{n+1}) + \delta_n(x_{n+1} - x_n)\| \\
&\quad + \|(\gamma_{n+1} - \gamma_n)(w_n - Tw_n)\| \\
&\leq (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\| + \delta_n\|x_{n+1} - x_n\| \\
&\quad + (\gamma_{n+1} - \gamma_n)Q_1.
\end{aligned}$$

□

Theorem 3.1. *Let E be a uniformly smooth Banach space and C be a nonempty bounded closed convex subset E. Let $f : C \to C$ be a generalized contraction mapping and T a λ-strictly pseudo-contractive mapping defined on C such that $F(T) \neq \emptyset$. The iterative sequence $\{x_n\}_{n=1}^{\infty}$ is defined from an arbitrary $x_1 \in K$ by (1.9). The sequence $\{x_n\}_{n=1}^{\infty}$ converges in norm to $p \in F(T)$ which solves the variational inequality (1.10), given by*

$$\langle (I-f)p, J(x-p)\rangle \geq 0, \text{ for all } x \in F(T).$$

Proof. The first thing to do here is to show that
$\limsup_{n\to\infty} (\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \leq 0$ and then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Let $Q_2 = \max\left\{\sup_n \|x_n\|, \sup_n \|y_n\|, \sup_n \|S_n w_n\|, \sup_n \|f(x_n)\|\right\}$. It can be obtained from (1.9) that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\beta_{n+1}x_{n+1} + (1-\beta_{n+1})y_{n+1} - \beta_n x_n - (1-\beta_n)y_n\| \\
&= \|\beta_{n+1}(x_{n+1} - x_n) + (1-\beta_{n+1})(y_{n+1} - y_n) \\
&\quad + (\beta_{n+1} - \beta_n)x_n + (\beta_n - \beta_{n+1})y_n\| \\
&= \beta_{n+1}\|x_{n+1} - x_n\| + (1-\beta_{n+1})\|y_{n+1} - y_n\| + 2|\beta_{n+1} - \beta_n|Q_2. \qquad (3.7)
\end{aligned}$$

Also,

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\sigma_{n+1}^1 f(x_{n+1}) + \sigma_{n+1}^2 x_{n+1} + \sigma_{n+1}^3 S_{n+1} w_{n+1} \\
&\quad - \sigma_n^1 f(x_n) - \sigma_n^2 x_n - \sigma_n^3 S_n w_n\|
\end{aligned}$$

$$
\begin{aligned}
&= \|\sigma^1_{n+1}\left(f(x_{n+1})-f(x_n)\right)+\sigma^2_{n+1}(x_{n+1}-x_n) \\
&\quad +\sigma^3_{n+1}(S_{n+1}w_{n+1}-S_nw_n)+(\sigma^1_{n+1}-\sigma^1_n)f(x_n) \\
&\quad +(\sigma^2_{n+1}-\sigma^2_n)x_n+(\sigma^3_{n+1}-\sigma^3_n)S_nw_n\| \\
&\leq \sigma^1_{n+1}\|f(x_{n+1})-f(x_n)\|+\sigma^2_{n+1}\|x_{n+1}-x_n\| \\
&\quad +\sigma^3_{n+1}\|S_{n+1}w_{n+1}-S_nw_n\|+|\sigma^1_{n+1}-\sigma^1_n|\|f(x_n)\| \\
&\quad +|\sigma^2_{n+1}-\sigma^2_n|\|x_n\|+|\sigma^3_{n+1}-\sigma^3_n|\|S_nw_n\| \\
&\leq \sigma^1_{n+1}\psi\|x_{n+1}-x_n\|+\sigma^2_{n+1}\|x_{n+1}-x_n\| \\
&\quad +\sigma^3_{n+1}\left((1-\delta_{n+1})\|x_{n+2}-x_{n+1}\|+\delta_n\|x_{n+1}-x_n\|+(\gamma_{n+1}-\gamma_n)Q_1\right) \\
&\quad +|\sigma^1_{n+1}-\sigma^1_n|\|f(x_n)\|+|\sigma^2_{n+1}-\sigma^2_n|\|x_n\|+|\sigma^3_{n+1}-\sigma^3_n|\|S_nw_n\| \\
&= \left(\sigma^1_{n+1}\psi+\sigma^2_{n+1}+\sigma^3_{n+1}\delta_n\right)\|x_{n+1}-x_n\| \\
&\quad +\sigma^3_{n+1}(1-\delta_{n+1})\|x_{n+2}-x_{n+1}\|+\sigma^3_{n+1}|\gamma_{n+1}-\gamma_n|Q_1 \qquad (3.8)\\
&\quad +\left(|\sigma^1_{n+1}-\sigma^1_n|+|\sigma^2_{n+1}-\sigma^2_n|+|\sigma^3_{n+1}-\sigma^3_n|\right)Q_2.
\end{aligned}
$$

Substituting (3.7) into (3.8) gives

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \qquad (3.9)\\
&\leq \left(\sigma^1_{n+1}\psi+\sigma^2_{n+1}+\sigma^3_{n+1}\delta_n\right)\|x_{n+1}-x_n\| \\
&\quad +\sigma^3_{n+1}(\gamma_{n+1}-\gamma_n)Q_1+\left(|\sigma^1_{n+1}-\sigma^1_n|+|\sigma^2_{n+1}-\sigma^2_n|+|\sigma^3_{n+1}-\sigma^3_n|\right)Q_2 \\
&\quad +\sigma^3_{n+1}(1-\delta_{n+1})\beta_{n+1}\|x_{n+1}-x_n\| \\
&\quad +\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})\|y_{n+1}-y_n\| \\
&\quad +2\sigma^3_{n+1}(1-\delta_{n+1})|\beta_{n+1}-\beta_n|Q_2 \\
&\leq \Bigg(\sigma^1_{n+1}\psi+\sigma^2_{n+1}+\sigma^3_{n+1}\delta_n+\sigma^3_{n+1}(1-\delta_{n+1})\beta_{n+1}\Bigg)\|x_{n+1}-x_n\| \\
&\quad +\Bigg(|\sigma^1_{n+1}-\sigma^1_n|+|\sigma^2_{n+1}-\sigma^2_n|+|\sigma^3_{n+1}-\sigma^3_n| \\
&\quad +2\sigma^3_{n+1}(1-\delta_{n+1})|\beta_{n+1}-\beta_n|\Bigg)Q_2 \qquad (3.10)\\
&\quad +\sigma^3_{n+1}|\gamma_{n+1}-\gamma_n|Q_1+\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})\|y_{n+1}-y_n\|.
\end{aligned}
$$

Let $\zeta_n =: |\sigma^1_{n+1}-\sigma^1_n|+|\sigma^2_{n+1}-\sigma^2_n|+|\sigma^3_{n+1}-\sigma^3_n|+2\sigma^3_{n+1}(1-\delta_{n+1})|\beta_{n+1}-\beta_n|$. Then (3.9) gives

$$
\begin{aligned}
& \|y_{n+1}-y_n\| \\
\leq\ & \frac{\sigma^1_{n+1}\psi+\sigma^2_{n+1}+\sigma^3_{n+1}\delta_n+\sigma^3_{n+1}(1-\delta_{n+1})\beta_{n+1}}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}\|x_{n+1}-x_n\| \\
& +\frac{\zeta_n}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_2+\frac{\sigma^3_{n+1}|\gamma_{n+1}-\gamma_n|}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_1 \\
=\ & \left(1-\frac{\sigma^3_{n+1}(\delta_{n+1}-\delta_n)+\sigma^1_{n+1}(1-\psi)}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}\right)\|x_{n+1}-x_n\| \\
& +\frac{\zeta_n}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_2+\frac{\sigma^3_{n+1}|\gamma_{n+1}-\gamma_n|}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_1 \\
=\ & \left(1-\frac{\sigma^3_{n+1}(\delta_{n+1}-\delta_n)+\sigma^1_{n+1}\phi}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}\right)\|x_{n+1}-x_n\| \\
& +\frac{\zeta_n}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_2+\frac{\sigma^3_{n+1}|\gamma_{n+1}-\gamma_n|}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_1,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \leq\ & -\frac{\sigma^3_{n+1}(\delta_{n+1}-\delta_n)+\sigma^1_{n+1}\phi}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}\|x_{n+1}-x_n\| \\
& +\frac{\zeta_n}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_2 \\
& +\frac{\sigma^3_{n+1}|\gamma_{n+1}-\gamma_n|}{1-\sigma^3_{n+1}(1-\delta_{n+1})(1-\beta_{n+1})}Q_1.
\end{aligned}
$$

According to the conditions of Assumption 3.1,

$$
\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\leq 0. \tag{3.11}
$$

Lemma 2.2 is implored to get

$$
\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.12}
$$

A consequence of (3.12) is that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|\beta_n x_n+(1-\beta_n)y_n-x_n\| \\
&= \|(1-\beta_n)y_n-(1-\beta_n)x_n\| \\
&= \|(1-\beta_n)(y_n-x_n)\| \\
&\leq (1-\beta_n)\|y_n-x_n\|\to 0 \text{ as } n\to\infty. \qquad (3.13)
\end{aligned}
$$

By invoking (3.12) and (3.13), it can be deduced from (1.9) that

$$\begin{aligned}\|x_n - S_n(x_n)\| &\le \|x_n - y_n\| + \|y_n - S_n(x_n)\| \\ &= \|y_n - x_n\| + \|\sigma_n^1 f(x_n) + \sigma_n^2 x_n \\ &\quad +\sigma_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) - S_n(x_n)\| \\ &\le \|y_n - x_n\| + \sigma_n^1\|f(x_n) - S_n(x_n)\| + \sigma_n^2\|x_n - S_n(x_n)\| \\ &\quad +\sigma_n^3\|S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right) - S_n(x_n)\| \\ &\le \|y_n - x_n\| + \sigma_n^1\|f(x_n) - S_n(x_n)\| + \sigma_n^2\|x_n - S_n(x_n)\| \\ &\quad +\sigma_n^3\|\delta_n x_n + (1-\delta_n)x_{n+1} - x_n\| \\ &\le \|y_n - x_n\| + \sigma_n^1\|f(x_n) - S_n(x_n)\| + \sigma_n^2\|x_n - S_n(x_n)\| \\ &\quad +\sigma_n^3(1-\delta_n)\|x_{n+1} - x_n\|. \end{aligned} \tag{3.14}$$

Let $0 < \sigma_n^2 \le \sigma < 1$ since $\{\sigma_n^2\}_{n=1}^{\infty} \subset (0,1)$. Factorizing (3.14) gives

$$\begin{aligned}\|x_n - S_n(x_n)\| &\le \frac{1}{1-\sigma_n^2}\|y_n - x_n\| + \frac{\sigma_n^1}{1-\sigma_n^2}\|f(x_n) - S_n(x_n)\| \\ &\quad +\frac{\sigma_n^3(1-\delta_n)}{1-\sigma_n^2}\|x_{n+1} - x_n\| \\ &\le \frac{1}{1-\sigma}\|y_n - x_n\| + \frac{\sigma_n^1}{1-\sigma}\|f(x_n) - S_n(x_n)\| \\ &\quad +\frac{\sigma_n^3(1-\delta_n)}{1-\sigma}\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.15}$$

According to Lemma 2.4, define a sequence $\{x_k\}$ by $x_k = kf(x_k) + (1-k)S_n(x_k)$ for $k \in (0,1)$. Strong convergence of $\{x_k\}$ to $p \in F(T)$ is a solution to the variational inequality:

$$\langle f(p) - p, J(x-p)\rangle \le 0,\ \forall\, x \in F(T),$$

which is equivalent to

$$\langle (I-f)p, J(x-p)\rangle \ge 0,\ \forall\, x \in F(T).$$

It is fundamental to establish that

$$\limsup_{n\to\infty} \langle f(p) - p,\ J(x_{n+1} - p)\rangle \le 0, \tag{3.16}$$

where $p \in F(T)$ is the unique fixed point of the generalized contraction $P_{F(T)}f(p)$ (Proposition 2.4), that is, $p = P_{F(T)}f(p)$.

Recall that $\lim_{n\to\infty} \|x_n - S_n(x_n)\| = 0$ by (3.15), Lemma 2.3 upholds that

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle \leq 0.$$

Since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ (3.13) and by the uniform continuity property of the duality mapping,

$$\begin{aligned} \limsup_{n\to\infty} \langle f(p) - p,\ J(x_{n+1} - p)\rangle &= \limsup_{n\to\infty} \langle f(p) - p,\ J(x_{n+1} - x_n + x_n - p)\rangle \\ &= \limsup_{n\to\infty} \langle f(p) - p,\ J(x_n - p)\rangle \leq 0. \end{aligned} \tag{3.17}$$

The next important step is to show that $x_n \to p \in F(T)$ as $n \to \infty$.
To use a method of contradiction, it is assumed that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge strongly to $p \in F(T)$. Thus, for a $\{x_n\}_{n=1}^{\infty}$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ and $\epsilon > 0$ such that $\|x_{n_j} - p\| \geq \epsilon$, for all $j \in \mathbb{N}$. Therefore, there exists for this ϵ an $\alpha \in (0,1)$ such that

$$\|f(x_{n_j}) - f(p)\| \leq c\|x_{n_j} - p\|.$$

Consequently,

$$\begin{aligned} & ||x_{n_{j+1}} - p||^2 \\ = \; & \beta_{n_j} \left\langle x_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle + (1-\beta_{n_j}) \left\langle y_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle \\ = \; & \beta_{n_j} \left\langle x_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle + \sigma^1_{n_j}(1-\beta_n) \left\langle f(x_{n_j}) - p, J(x_{n_{j+1}} - p)\right\rangle \\ & +\sigma^2_{n_j}(1-\beta_{n_j}) \left\langle x_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle \\ & +\sigma^3_{n_j}(1-\beta_{n_j}) \left\langle S_{n_j}(w_{n_j}) - p, J(x_{n_{j+1}} - p)\right\rangle \\ = \; & \beta_{n_j} \left\langle x_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle + \sigma^1_{n_j}(1-\beta_{n_j}) \left\langle f(x_{n_j}) - f(p), J(x_{n_{j+1}} - p)\right\rangle \\ & +\sigma^1_{n_j}(1-\beta_{n_j}) \left\langle f(p) - p, J(x_{n_{j+1}} - p)\right\rangle \\ & +\sigma^2_{n_j}(1-\beta_{n_j}) \left\langle x_{n_j} - p, J(x_{n_{j+1}} - p)\right\rangle \\ & +\sigma^3_{n_j}(1-\beta_{n_j}) \left\langle S_{n_j}(w_{n_j}) - p, J(x_{n_{j+1}} - p)\right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \beta_{n_j}\|x_{n_j}-p\|\|x_{n_{j+1}}-p\|+\sigma^1_{n_j}(1-\beta_{n_j})\|f(x_{n_j})-f(p)\|\|x_{n_{j+1}}-p\| \\
&\quad +\sigma^1_{n_j}(1-\beta_{n_j})\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle \\
&\quad +\sigma^2_{n_j}(1-\beta_{n_j})\|x_{n_j}-p\|\|x_{n_{j+1}}-p\| \\
&\quad +\sigma^3_{n_j}(1-\beta_{n_j})\|S_{n_j}(w_{n_j})-p\|\|x_{n_{j+1}}-p\| \\
&\leq \beta_{n_j}\|x_{n_j}-p\|\|x_{n_{j+1}}-p\|+\alpha\sigma^1_{n_j}(1-\beta_{n_j})\|x_{n_j}-p\|\|x_{n_{j+1}}-p\| \\
&\quad +\sigma^1_{n_j}(1-\beta_{n_j})\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle \\
&\quad +\sigma^2_{n_j}(1-\beta_{n_j})\|x_{n_j}-p\|\|x_{n_{j+1}}-p\| \\
&\quad +\sigma^3_{n_j}(1-\beta_{n_j})\|x_{n_{j+1}}-p\|\left(\delta_{n_j}\|x_{n_j}-p\|+(1-\delta_{n_j})\|x_{n_{j+1}}-p\|\right) \\
&= \Big(\beta_{n_j}+\alpha\sigma^1_{n_j}(1-\beta_{n_j})+\sigma^2_{n_j}(1-\beta_{n_j})+\delta_{n_j}\sigma^3_{n_j}(1-\beta_{n_j})\Big) \\
&\quad \times\|x_{n_j}-p\|\|x_{n_{j+1}}-p\|+\sigma^1_{n_j}(1-\beta_{n_j})\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle \\
&\quad +\sigma^3_{n_j}(1-\beta_{n_j})(1-\delta_{n_j})\|x_{n_{j+1}}-p\|^2 \\
&= \frac{\beta_{n_j}+\alpha\sigma^1_{n_j}(1-\beta_{n_j})+\sigma^2_{n_j}(1-\beta_{n_j})+\delta_{n_j}\sigma^3_{n_j}(1-\beta_{n_j})}{2} \\
&\quad \times\left(\|x_{n_j}-p\|^2+\|x_{n_{j+1}}-p\|^2\right)+\sigma^1_{n_j}(1-\beta_{n_j})\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle \\
&\quad +\sigma^3_{n_j}(1-\beta_{n_j})(1-\delta_{n_j})\|x_{n_{j+1}}-p\|^2 \\
&= \frac{1-(1-\beta_{n_j})\Big((1-\alpha)\sigma^1_{n_j}+(1-\delta_{n_j})\sigma^3_{n_j}\Big)}{2}\|x_{n_j}-p\|^2 \\
&\quad +\frac{1-(1-\beta_{n_j})\Big((1-\alpha)\sigma^1_{n_j}-(1-\delta_{n_j})\sigma^3_{n_j}\Big)}{2}\|x_{n_{j+1}}-p\|^2 \\
&\quad +(1-\beta_{n_j})\sigma^1_{n_j}\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle.
\end{aligned}$$

Collect the like terms and simplify to get

$$\begin{aligned}
&||x_{n_{j+1}}-p||^2 \\
&\leq \frac{1-(1-\beta_{n_j})\Big((1-\alpha)\sigma^1_{n_j}+(1-\delta_{n_j})\sigma^3_{n_j}\Big)}{1+(1-\beta_{n_j})\Big((1-\alpha)\sigma^1_{n_j}-(1-\delta_{n_j})\sigma^3_{n_j}\Big)}\|x_{n_j}-p\|^2 \\
&\quad +\frac{2(1-\beta_{n_j})\sigma^1_{n_j}}{1+(1-\beta_{n_j})\Big((1-\alpha)\sigma^1_{n_j}-(1-\delta_{n_j})\sigma^3_{n_j}\Big)}\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle
\end{aligned}$$

$$= \left(1 - \frac{2(1-\beta_{n_j})\sigma_{n_j}^1}{1+(1-\beta_{n_j})\left((1-\alpha)\sigma_{n_j}^1 - (1-\delta_{n_j})\sigma_{n_j}^3\right)}\right)\|x_{n_j} - p\|^2$$
$$+\frac{2(1-\beta_{n_j})\sigma_{n_j}^1}{1+(1-\beta_{n_j})\left((1-\alpha)\sigma_{n_j}^1 - (1-\delta_{n_j})\sigma_{n_j}^3\right)}\left\langle f(p)-p, J(x_{n_{j+1}}-p)\right\rangle.$$

By Lemma 2.5, $x_{n_j} \to p$ as $j \to \infty$. This is a contradiction. Thus, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F(T)$. □

Remark 3.1. *The following results are deduction from Theorem 3.1 and they also extend and improve some existing results.*

Theorem 3.2. *Let E be a uniformly smooth Banach space and C be a nonempty bounded closed convex subset E. Let $f : C \to C$ be a generalized contraction mapping and T_j a λ_j-strictly pseudo-contractive mapping defined on C such that $\cap_{j=1}^{M-1} F(T_j) \neq \emptyset$, where M is an integer and $0 \le j \le M-1$. The iterative sequence $\{x_n\}_{n=1}^{\infty}$ is defined from an arbitrary x_1 by*

$$\begin{cases} y_n = \sigma_n^1 f(x_n) + \sigma_n^2 x_n + \sigma_n^3 S_n\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, n \in \mathbb{N}, \end{cases} \tag{3.18}$$

where $S_n x := \gamma_n x + (1-\gamma_n)T_{[n]}$ and $T_{[n]} = T_j$ with $j = n$Mod M, $0 \le j \le M-1$. The real sequences $\left\{\{\sigma_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}$ in $(0,1)$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in $[0,1]$ and $\{\gamma_n\}_{n=1}^{\infty}$ in $[0,1)$ are assumed to satisfy the following condition:

(i) $\sum_{i=1}^{3} \sigma_n^i = 1$;

(ii) $\lim_{n\to\infty} \sigma_n^1 = 0$, $\sum_{n=1}^{\infty} \sigma_n^1 = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} \sigma_n^3 = 0$, $\lim_{n\to\infty} |\sigma_{n+1}^2 - \sigma_n^2| = 0$;

(v) $0 < \delta_n \le \delta_{n+1} \le \delta < 1$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in norm to $p \in \cap_{j=1}^{M-1} F(T_j)$ which solves the variational inequality

$$\langle (I-f)p, J(x-p)\rangle \ge 0, \text{ for all } x \in \cap_{j=1}^{M-1} F(T_j).$$

Proof. It suffices to show that S_n is nonexpansive for all $n \in \mathbb{N}$. Indeed, for all $x, y \in C$ and any $0 \leq j \leq M-1$, take

$$0 < \epsilon \leq \lambda_j \|T_j x - T_j y - (x-y)\|^2$$

and apply Lemma 2.1 to have

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \|(1-\gamma_n)x + \gamma_n T_{[n]}x - (1-\gamma_n)y - \gamma_n T_{[n]}y\|^2 \\
&= \|(1-\gamma_n)(x-y) + \gamma_n(T_{[n]}x - T_{[n]}y)\|^2 \\
&\leq 2\gamma_n \left\langle T_{[n]}x - T_{[n]}y, j(x-y)\right\rangle + 2\epsilon\gamma_n + (1-2\gamma_n)\|x-y\|^2 \\
&\leq 2\gamma_n \left(\|x-y\|^2 - \lambda\|T_{[n]}x - T_{[n]}y - (x-y)\|^2\right) \\
&\quad +2\epsilon\gamma_n + (1-2\gamma_n)\|x-y\|^2 \\
&\leq \|x-y\|^2 - 2\gamma_n\lambda\|T_{[n]}x - T_{[n]}y - (x-y)\|^2 + 2\epsilon\gamma_n \\
&\leq \|x-y\|^2.
\end{aligned} \tag{3.19}$$

The rest of the proof follows from Theorem 3.1. □

Theorem 3.3. *Let E be a uniformly smooth Banach space and C be a nonempty bounded closed convex subset E. Let $f : C \to C$ be a generalized contraction mapping and T a λ-strictly pseudo-contractive mapping defined on C such that $F(T) \neq \emptyset$. The iterative sequence $\{x_n\}_{n=1}^{\infty}$ is defined from an arbitrary $x_1 \in K$ by (1.9).*

$$\begin{cases} y_n = \sigma_n^1 f(x_n) + \sigma_n^2 x_n + \sigma_n^3 S_n\left(\frac{x_n + x_{n+1}}{2}\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.20}$$

where $S_n x := \gamma_n x + (1-\gamma_n)T$. The real sequences $\left\{\{\sigma_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}$ in $(0,1)$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in $[0,1]$ and $\{\gamma_n\}_{n=1}^{\infty}$ in $[0,1)$ are assumed to satisfy the following condition:

(i) $\sum_{i=1}^{3} \sigma_n^i = 1$;

(ii) $\lim_{n\to\infty} \sigma_n^1 = 0$, $\sum_{n=1}^{\infty} \sigma_n^1 = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} \sigma_n^3 = 0$, $\lim_{n\to\infty} |\sigma_{n+1}^2 - \sigma_n^2| = 0$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges in norm to* $p \in F(T)$ *which solves the variational inequality*

$$\langle (I - f)p, J(x - p)\rangle \geq 0, \textit{ for all } x \in F(T).$$

Proof. The sequence (3.20) is a semi-implicit iteration which is obtained from (1.9) by setting $\delta_n := 2$ for all $n \in \mathbb{N}$. Hence, the result follows from Theorem 3.1. The result of Theorem 3.3 extends and improves the results of Xiong and Lan [17]. For instance, Theorem 3.3 admits a generalized contraction operator which is more broad than the associated contraction operator in (1.2) of Xiong and Lan [17]. Also, the main results of Xiong and Lan [17] are stated for a nonexpansive mapping while Theorem 3.3 holds for the class of λ-strictly pseudo-contractive mappings which are more general. □

Theorem 3.4. *Let* E *be a uniformly smooth Banach space and* C *be a nonempty bounded closed convex subset* E*. Let* $f : C \to C$ *be a generalized contraction mapping and* T *a nonexpansive mapping defined on* C *such that* $F(T) \neq \emptyset$*. The iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *is defined from an arbitrary* x_1 *by*

$$\begin{cases} y_n = \sigma_n^1 f(x_n) + \sigma_n^2 x_n + \sigma_n^3 T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, & n \in \mathbb{N}. \end{cases} \tag{3.21}$$

The real sequences $\left\{\{\sigma_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3}$ *in* $(0,1)$*,* $\{\delta_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *in* $[0,1]$ *are assumed to satisfy the following condition:*

(i) $\sum_{i=1}^{3} \sigma_n^i = 1$;

(ii) $\lim_{n\to\infty} \sigma_n^1 = 0$, $\sum_{n=1}^{\infty} \sigma_n^1 = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} \sigma_n^3 = 0$;

(v) $0 < \delta_n \leq \delta_{n+1} \leq \delta < 1$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges in norm to* $p \in F(T)$ *which solves the variational inequality*

$$\langle (I - f)p, J(x - p)\rangle \geq 0, \textit{ for all } x \in F(T).$$

Proof. The class of λ-strictly pseudo-contractive mappings is more general than the class of nonexpansive mappings. Therefore, the result follows from Theorem

3.1. The result of Theorem 3.4 extends and improves the results of Ke and Ma [7], which was obtained in Hilbert spaces and whose algorithm is obtained from (3.21) when $\beta_n =: 0$. □

4. Numerical Examples

An example of a λ-strictly pseudo-contractive mapping is constructed and the convergence analysis of Theorem 3.1 is illustrated numerically. $\mathbb{R}$ is a real line endowed with the Euclidean norm and $T : \mathbb{R} \to \mathbb{R}$ is a mapping defined by

$$Tx = \begin{cases} -2x + \frac{1}{3}, & x \in (-\infty, 0], \\ \frac{1}{3}(x+1), & x \in (0, \infty). \end{cases} \tag{4.1}$$

The first step is to show that T is a λ-strictly pseudo-contractive mapping with $\lambda \in (0,1)$.

Case (i) : Notice that for all $x, y \in (-\infty, 0]$,

$$|Tx - Ty|^2 = 4|x-y|^2, \ |(I-T)x - (I-T)y|^2 = 9|x-y|^2.$$

Therefore,

$$|Tx - Ty|^2 = |x-y|^2 + \lambda|(I-T)x - (I-T)y|^2,$$

for $\lambda_1 := \frac{1}{3}$.

Case (ii) : For all $x, y \in (0, \infty)$,

$$|Tx - Ty|^2 = \frac{1}{9}|x-y|^2, \ |(I-T)x - (I-T)y|^2 = \frac{4}{9}|x-y|^2.$$

Therefore,

$$|Tx - Ty|^2 < |x-y|^2 + \lambda|(I-T)x - (I-T)y|^2,$$

for $\lambda_2 := \frac{1}{4}$.

Case (iii) : For all $x \in (-\infty, 0]$ and $y \in (0, \infty)$ with $\lambda_1 := \frac{1}{3}$,

$$\begin{aligned} |x-y|^2 + \frac{1}{3}|(I-T)x - (I-T)y|^2 &= |x-y|^2 + \frac{1}{3}\left|3x - \frac{1}{3} - (y - \frac{1}{3}(y+1))\right|^2 \\ &= |x-y|^2 + \frac{1}{27}|9x - 2y|^2 \\ &= x^2 - 2xy + y^2 + \frac{1}{27}\left(81x^2 - 36xy + 4y^2\right) \\ &= 4x^2 - \frac{10}{3}xy + \frac{31}{27}y^2. \end{aligned}$$

Moreover, for all $x \in (-\infty, 0]$ and $y \in (0, \infty)$ with $\lambda_1 := \frac{1}{3}$,

$$\begin{aligned}
|Tx - Ty|^2 &= \left| -2x + \frac{1}{3} - \frac{1}{3}(y+1) \right|^2 \\
&= \frac{1}{9}|-6x - y|^2 \\
&= \frac{1}{9}\left(36x^2 + 12xy + y^2\right) \\
&= 4x^2 + \frac{4}{3}xy + \frac{1}{9}y^2 \\
&\leq 4x^2 + \frac{4}{3}xy + \frac{1}{9}y^2 - \frac{14}{3}xy + \frac{29}{27}y^2 \\
&\quad \text{(Since } x \in (-\infty, 0] \text{ and } y \in (0, \infty),) \\
&= 4x^2 - \frac{10}{3}xy + \frac{31}{27}y^2 \\
&= |x - y|^2 + \frac{1}{3}|(I - T)x - (I - T)y|^2,
\end{aligned}$$

Hence, T is a λ-strictly pseudo-contractive mapping with $\lambda := \min\{\lambda_1, \lambda_2\}$ and $F(T) = \left\{\frac{1}{2}\right\}$.

Let $\{\sigma_n^1\}_{n=1}^{\infty} := \left\{\frac{1}{n+4}\right\}_{n=1}^{\infty}, \{\sigma_n^2\}_{n=1}^{\infty} := \left\{\frac{n+1}{n+4}\right\}_{n=1}^{\infty}, \{\sigma_n^3\}_{n=1}^{\infty} := \left\{\frac{2}{n+4}\right\}_{n=1}^{\infty}$ and notice that $\sigma_n^1 + \sigma_n^2 + \sigma_n^3 = 1$ and that they satisfy the conditions of Theorem 3.1. Furthermore, let $f(x) = \frac{1}{4}x$ and $\beta_n = \delta_n = \gamma_n = \frac{1}{2}$ for all $n \in \mathbb{N}$. The analysis of convergence of the sequence $\{x_n\}_{n=1}^{\infty}$ are displayed in Figures 1, 2 & 3 for $x_1 = -1.5,\ 0$ and 2.5 respectively.

5. CONCLUSION

The research efforts for over a decade have been devoted on the class of nonexpansive mappings (See e.g, Aibinu et al. [2], Alghamdi et al. [4], Xiong [18] and references therein). The class of strictly pseudo-contractive mappings is known to have more powerful applications than the class of nonexpansive mappings. The efficacy of the class of strictly pseudo-contractive mappings in dealing with nonlinear problems such as inverse and equilibrium problems, motivated these research efforts. Some mild conditions are imposed on the parameters to obtain the strong convergence of the newly constructed algorithm to a fixed point of a strictly pseudo-contractive mapping in the framework of Banach spaces. The given numerical example illustrates the convergence analysis of the

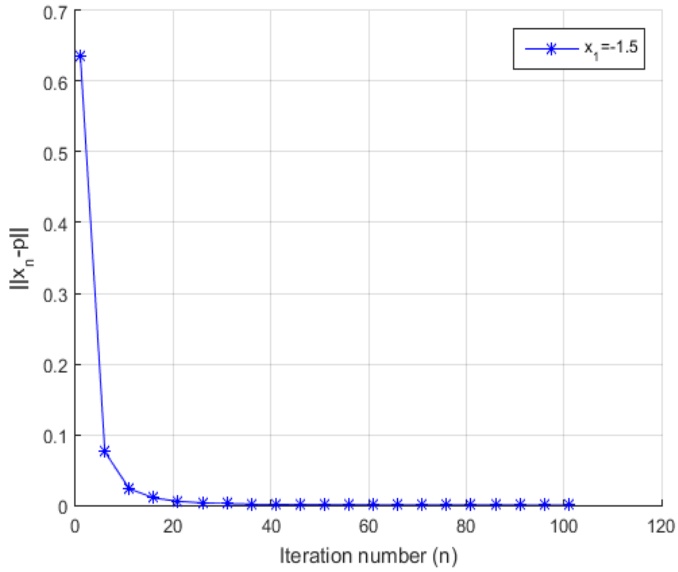

FIGURE 1. Convergence of sequence of iteration with $x_1 = -1.5$.

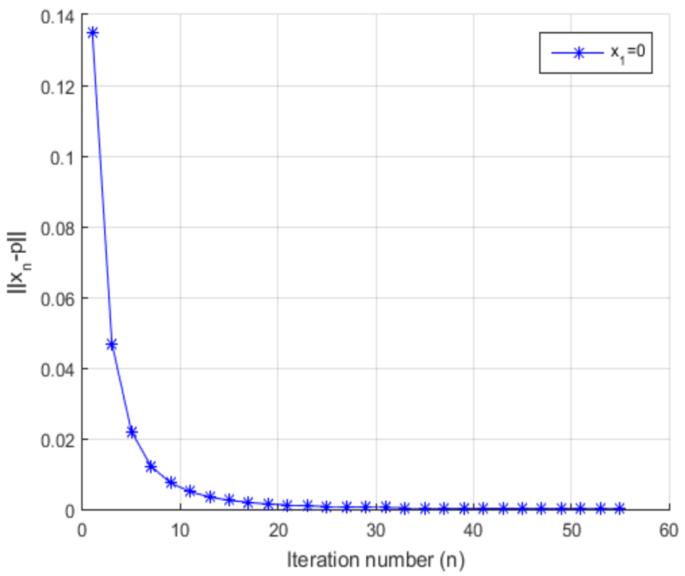

FIGURE 2. Convergence of sequence of iteration with $x_1 = 0$.

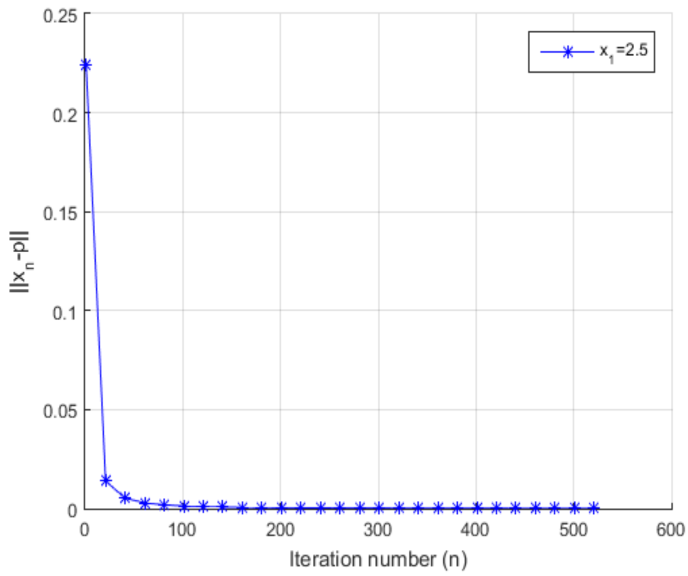

FIGURE 3. Convergence of sequence of iteration with $x_1 = 2.5$.

newly proposed generalized viscosity implicit sequence. It is also to eliminate skepticism about the sequence of iteration and the conditions which are imposed on the parameters.

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