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# ON SOME CHARACTERISTIC CAUCHY PROBLEMS

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ABSTRACT. By means of some regularizations for an ill posed Cauchy problem, we define an associated generalized problem and discuss the conditions for solvability of it. To illustrate this, starting from the semilinear unidirectional wave equation with data given on a characteristic curve, we show existence and uniqueness of the solution in convenient generalized algebras.

## 1. INTRODUCTION

Many obstructions can be encountered when trying to solve a Cauchy problem for PDEs with the data given on a characteristic manifold and, a fortiori, to obtain uniqueness or well posedness in Hadamard sense We can refer to many works inspired in the real field by the ideas of Egorov [1], Hörmander [2] and others on the distribution solutions of some Cauchy problems supported in a half space whose boundary is a characteristic hyperplane.

Here we propose another method, based on a parametric family of geometric transformations of the characteristic manifold, in continuation of previous ideas developed in [3, 4, 5, 6, 7] . In order to concentrate on the method and not on the technicalities, we consider the Cauchy problem for a simple equation,

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namely the transport equation in basic form $(P_c)$ : $\partial u/\partial t = F(.,.,u), u\left|_{\gamma} = v\right|_{\gamma} = v$ where $\gamma$ of equation $x = 0$ is obviously globally characteristic for the Cauchy problem. $(P_c)$ is ill posed but can be associated to a generalized problem $P(D)\,u = \mathcal{F}(u)\,, \mathcal{R}(u) = v$ well formulated in convenient algebras of generalized functions by means of generalized operators $\mathcal{F}$ associated to $F$ and $\mathcal{R}$ obtained by replacing the characteristic curve by a family $(\gamma_\varepsilon)_\varepsilon$ of non characteristic ones of equation $x = l_\varepsilon(t)$ where $(l_\varepsilon)_\varepsilon$ is a regularizing family.

## 2. General overview on $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-type algebras

2.1. **Algebraic and topological structures.** We suppose that

- $\mathbb{K}$ is the real or complex field and $\Lambda$ a set of indices left-filtering for a partial order $\prec$;
- $\mathcal{C}$ is a factor ring $A/I$ where $I$ is an ideal of $A$, a given subring of $\mathbb{K}^\Lambda$;
- $A$ and $I$ are both solid i.e equal to their solid hull;
- $(\mathcal{E}$,$\mathcal{P})$ is a sheaf of topological $\mathbb{K}$-algebras on a topological space $X$,the topology on $\mathcal{E}(\Omega)$ being given for any open set $\Omega$ in $X$ by a family $\mathcal{P}(\Omega)$ of seminorms.

Then we set

$$\begin{aligned}
\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) &= \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda\, \forall p \in \mathcal{P}(\Omega)\,, (p\,(u_\lambda))_\lambda \in A\right\},\\
\mathcal{J}_{(A,\mathcal{E},\mathcal{P})}(\Omega) &= \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda\, \forall p \in \mathcal{P}(\Omega)\,, (p\,(u_\lambda))_\lambda \in I\right\}.
\end{aligned}$$

Under some more technical conditions dtailed in [4] we have the following

**Theorem 2.1.** *The factor space $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(A,\mathcal{E},\mathcal{P})}$ is a presheaf with localsation principle.*

The proof is given in [4]. In [7] it is shown that if $\mathcal{E}$ is a fine sheaf, then $\mathcal{A}$ is also a fine sheaf.

2.2. **Generalized operators and general restrictions.** Let $\Omega$ be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$. We say that the algebra $\mathcal{A}(\Omega)$ is *stable under* $F$ if for all $(u_\varepsilon)_\varepsilon \in \mathcal{H}(\Omega)$ and all $(i_\varepsilon) \in \mathcal{J}(\Omega)$ we have $(F(.,.,u_\varepsilon))_\varepsilon \in \mathcal{H}(\Omega)$ and $(F(.,.,u_\varepsilon + i_\varepsilon))_\varepsilon - (F(.,.,u_\varepsilon))_\varepsilon \in \mathcal{J}(\Omega)$. If $\mathcal{A}(\mathbb{R}^2)$ is *stable under* $F$, for $u = [u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$, $[F(.,.,u_\varepsilon)]$ is a well defined element of $\mathcal{A}(\mathbb{R}^2)$ (i.e. not depending on $(u_\varepsilon)_\varepsilon \in u$).

**Definition 2.1.** *If $\mathcal{A}(\mathbb{R}^2)$ is stable under $F$, the map $\mathcal{F} : \mathcal{A}(\mathbb{R}^2) \to \mathcal{A}(\mathbb{R}^2) : u = [u_\varepsilon] \to [F(.,.,u_\varepsilon)]$ is called the generalized map corresponding to $F$.*

Now, consider $(l_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})^\Lambda$ and set $R_\varepsilon : \mathrm{C}^\infty(\mathbb{R}^2) \to \mathrm{C}^\infty(\mathbb{R}), g \to \mathbb{R}_\varepsilon(g)$, with $\mathbb{R}_\varepsilon(g) : t \to g(t, l_\varepsilon(t))$. We say that $(l_\varepsilon)_\varepsilon$ is compatible with the generalized restriction if, for all $(u_\varepsilon)_\varepsilon \in \mathcal{H}(\mathbb{R}^2)$ (resp.$(i_\varepsilon) \in \mathcal{J}(\mathbb{R}^2)$, $(u_\varepsilon(., l_\varepsilon(.)))_\varepsilon \in \mathcal{H}(\mathbb{R})$ (resp. $(i_\varepsilon(., l_\varepsilon(.)))_\varepsilon \in \mathcal{H}(\mathbb{R}) \in \mathcal{J}(\mathbb{R})$).

**Definition 2.2.** *If the family of smooth functions $(l_\varepsilon)_\varepsilon$ is compatible with the generalized restriction, the map $\mathcal{R} : u = [u_\varepsilon] \to [u_\varepsilon(., l_\varepsilon(.))] = [R_\varepsilon(u_\varepsilon)]$ is called the restriction mapping corresponding to the family $(l_\varepsilon)_\varepsilon$.*

**Proposition 2.1.** *Assume that $(l_\varepsilon)_\varepsilon$ belongs to $\mathcal{H}(\mathbb{R})$ and is c-bounded. Then the family $(l_\varepsilon)_\varepsilon$ is compatible with generalized restriction.*

## 3. Application: a characteristic Cauchy problem

We deal with the characteristic Cauchy problem for the transport equation formally written in characteristic coordinates : $P_c : \partial u/\partial t = F(.,.,u), u\,\big|_{\{x=0\}} = f \in \mathrm{C}^\infty$.We have to formulate some assumptions which will allows us to associate to $P_c$ a generalized well posed $(P_g)$ :

### 3.1. From the ill-posed problem $(P_c)$ to a well-posed one .

We replace the characteristic curve by a family $(\gamma_\varepsilon)_\varepsilon$ of non characteristic ones of equation $x = l_\varepsilon(t)$ where $(l_\varepsilon)_\varepsilon$ is a smooth function c-bounded with strictly positive derivative and image $\mathbb{R}$, and we suppose that for each $K \Subset \mathbb{R}$ and $l \in \mathbb{N}$ the family $(P_{K,l}(l_\varepsilon))_\varepsilon$ is in $|A|$.

**Theorem 3.1.** *Under the above hypothesis*

- *The algebra $\mathcal{A}(\mathbb{R}^2)$ is stable under $F$*
- *The generalized restriction operator $\mathcal{R}$ is well defined for $u = [u_\varepsilon]_{\mathcal{A}(\mathbb{R}^2)}$ by*

$$\mathcal{R}(u) := [t \to u_\varepsilon(t, l_\varepsilon(t))]_{\mathcal{A}(\mathbb{R})}$$

We are now enable to associate $(P_c)$ with the generalized problem

$$(P_g) : \partial u/\partial t = \mathcal{F}(u), \mathcal{R}(u) = f.$$

3.2. **Existence of a solution to** $(P_g)$**.** In order to solve $(P_g)$, we begin to solve in $\mathrm{C}^\infty(\mathbb{R}^2)$ the regularized problem

$$(P_\infty) : \partial u_\varepsilon/\partial t\,(t,x)) = F\left(t,x,u_\varepsilon(t,x)\right), u_\varepsilon\left(t,l_\varepsilon(t)\right) = f(t)\,.$$

**Proposition 3.1.** *Under the above hypothesis the problem* $(P_\infty)$ *admits an unique smooth solution such that*

$$u_\varepsilon(t,x) = f\left(l_\varepsilon^{-1}(x)\right) + \int_{l_\varepsilon^{-1}(x)}^{t} F\left(\tau,x,u_\varepsilon(\tau,x)\right) d\tau.$$

The proof uses the classical Picard iteration procedure.

**Theorem 3.2.** *Under the above hypothesis the problem* $(P_g)$ *admits* $[u_\varepsilon]_{\mathcal{A}(\mathbb{R}^2)}$ *as solution where* $u_\varepsilon$ *is the solution given in proposition 6*

The proof follows the same steps as the existence results which can be found in [6]

3.3. **Independance from the regularizing process.** The solution of all the problems regularized by the Colombeau method depends *a priori* on the choice of the regularizing process. Indeed in the preceding section we have built the solution $u$ to $(P_g)$ by making use in a crucial way of the representative $(l_\varepsilon)_\varepsilon$. So even though the map $\mathcal{R}$ itself does not depends on the representative of $l = [(l_\varepsilon)_\varepsilon]$, we need to prove that our solution is independant of the representative. A first step in this direction is done by [3] in which the purely characreristic case is studied. Here we have an analogous result whose proof follows essentially the same lines.

**Theorem 3.3.** *In addition to the previous assumptions suppose that* $(l_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$ *and* $(l_\varepsilon^{-1})_\varepsilon \in \mathcal{M}_\tau(\mathbb{R})$*. Then, the generalized funtion* $u = [u_\varepsilon]$ *where* $u_\varepsilon(t,x) = f\left(l_\varepsilon^{-1}(x)\right) + \int_{l_\varepsilon^{-1}(x)}^{t} F\left(\tau,x,u_\varepsilon(\tau,x)\right) d\tau$ *depends solely on* $l = [l_\varepsilon] \in \mathcal{G}_\tau(\mathbb{R})$ *as generalized function and not on the representative* $(l_\varepsilon)_\varepsilon$*.*

3.4. **The framework** $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ **and uniqueness.** The natural topology of $\mathcal{O}_M$ permits to define a new algebra of tempered generalized functions $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ which differs from $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ but permits a point value characterization [8] and an extension $\mathcal{A}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ in the framework of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras. As $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ is of $(\mathcal{C},\mathcal{E},\mathcal{P})$-type and endowed with the sharp topology, our goal is at least to recover uniqueness of the solution of $(P_g)$ in this context.

Define $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ as the quotient algebras $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)/\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ where

$$\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)=\left\{\begin{array}{c}(u_\varepsilon)_\varepsilon\in\mathcal{O}_M\left(\mathbb{R}^d\right)^{(0,1]}:\left(\forall\varphi\in S\left(\mathbb{R}^d\right)\right)\left(\forall\alpha\in\mathbb{N}^d\right)\left(\exists M\in\mathbb{N}\right)\\ \left(\exists\varepsilon_0\right)\left(\forall\varepsilon<\varepsilon_0\right)\left(\sup_{x\in\mathbb{R}^d}\left|\varphi\left(x\right)\partial^\alpha u_\varepsilon\left(x\right)\leq\varepsilon^{-M}\right|\right)\end{array}\right\},$$

$$\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)=\left\{\begin{array}{c}(u_\varepsilon)_\varepsilon\in\mathcal{O}_M\left(\mathbb{R}^d\right)^{(0,1]}:\left(\forall\varphi\in S\left(\mathbb{R}^d\right)\right)\left(\forall\alpha\in\mathbb{N}^d\right)\left(\forall m\in\mathbb{N}\right)\\ \left(\exists\varepsilon_0\right)\left(\forall\varepsilon<\varepsilon_0\right)\left(\sup_{x\in\mathbb{R}^d}\left|\varphi\left(x\right)\partial^\alpha u_\varepsilon\left(x\right)\leq\varepsilon^{m}\right|\right)\end{array}\right\}.$$

This definition can be compared to the one of $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$. We can prove that $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)=\mathcal{M}_\tau\left(\mathbb{R}^d\right)$ but we only have $\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)\supseteq\mathcal{N}_\tau\left(\mathbb{R}^d\right)$, thus$\in\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ differs from $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$. On the other hand, along the same lines as $[10,$ Prop. 3.2$]$ we get

$$\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)=\left\{\begin{array}{c}(u_\varepsilon)_\varepsilon\in\mathcal{O}_M\left(\mathbb{R}^d\right)^{(0,1]}\left(\forall\alpha\in\mathbb{N}^d\right)\left(\forall m\in\mathbb{N}\right)\left(\exists p\in\mathbb{N}\right)\left(\exists\varepsilon_0\right)\\ \left(\left(\forall\varepsilon<\varepsilon_0\right)\left(\sup_{x\in\mathbb{R}^d}\left(1+|x|\right)^{-p}\left|\partial^\alpha u_\varepsilon\left(x\right)\right|\leq\varepsilon^m\right)\right)\end{array}\right\}.$$

By the same Taylor argument as in $[11,$ Thm. 1.2.25$]$ we obtain

**Theorem 3.4.**

$$\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)=\left\{\begin{array}{c}(u_\varepsilon)_\varepsilon\in\mathcal{M}_\tau\left(\mathbb{R}^d\right)\left(\forall m\in\mathbb{N}\right)\left(\exists p\in\mathbb{N}\right)\left(\exists\varepsilon_0\right)\\ \left(\left(\forall\varepsilon<\varepsilon_0\right)\left(sup_{x\in\mathbb{R}^d}\left(1+|x|\right)^{-p}\left|u_\varepsilon\left(x\right)\right|\leq\varepsilon^m\right)\right)\end{array}\right\}.$$

We refer to generalized points and point values as developed in $[10,\S1,2,4]$ We recall that $\widetilde{\mathbb{K}}=\mathcal{M}_K/\mathcal{N}_K$ is the ring of Colombeau generalized numbers ($\mathbb{K}=\mathbb{R}$,$\mathbb{C}$), and similarly $\widetilde{\mathbb{K}^d}=\widetilde{\mathbb{K}}^d$ the set of generalized points.

**Definition 3.1.** *An element $\widetilde{x}=\left[(x_\varepsilon)_\varepsilon\right]\in\widetilde{\mathbb{R}}^d$ is of slow scale if for all $n\in\mathbb{N}$ there exists $\varepsilon_0$ such that for all $\varepsilon<\varepsilon_0$, we have $|x_\varepsilon|\leq\varepsilon^{-1/n}$.*

**Theorem 3.5.** *Let $u=\left[(u_\varepsilon)_\varepsilon\right]\in\widetilde{\mathbb{R}}^d\in\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ and let $\widetilde{x}=\left[(x_\varepsilon)_\varepsilon\right]$ be of slow scale. Then the point value $u\left(\widetilde{x}\right):=\left[\left(u_\varepsilon\left(x_\varepsilon\right)\right)_\varepsilon\right]\in\widetilde{\mathbb{C}}$ is well defined.*

*Proof.* Let $(u_\varepsilon)_\varepsilon\in\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)=\mathcal{M}_\tau\left(\mathbb{R}^d\right)$ be a representative of $u$. By [11,Prop. 1.2.45], $(u_\varepsilon)_\varepsilon\in\mathcal{M}_\tau\left(\mathbb{R}^d\right)$ implies that $\left(u_\varepsilon\left(x_\varepsilon\right)\right)_\varepsilon\in\mathcal{M}_\mathbb{R}$, and that $\left(u_\varepsilon\left(x_\varepsilon\right)\right)-\left(u_\varepsilon\left(x'_\varepsilon\right)\right)_\varepsilon\in\mathcal{N}_\mathbb{R}$ if $\left(x'_\varepsilon\right)_\varepsilon$ is another representative of $\widetilde{x}$. It remains to show that the definition of the point value does not depends on the choice of representative of $u$. So let $(u_\varepsilon)_\varepsilon\in\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ and $m\in\mathbb{N}$. Choose $p\in\mathbb{N}$ as in the statement of theorem 9. Then, for sufficiently small $\varepsilon$

$$\left|u_\varepsilon\left(x_\varepsilon\right)\right|\leq\varepsilon^m\left(1+|x_\varepsilon|\right)^p\leq\varepsilon^m\left(2\left|x_\varepsilon\right|\right)^p\leq\varepsilon^m\left(2\varepsilon^{-1/p}\right)^p=2^p\varepsilon^{m-1}$$

.Since $m\in\mathbb{N}$ is arbitrary, $\left(u_\varepsilon\left(x_\varepsilon\right)\right)_\varepsilon\in\mathcal{N}_\mathbb{C}$. $\square$

**Theorem 3.6.** *Let $u \in \mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$. Then $u = 0$ iff $u(\widetilde{x}) = 0$ for each slow scale point $\widetilde{x}$.*

*Proof.* If $u = 0$,then clearly $u(\widetilde{x}) = 0$ for each slow scale point (since the definition of point value does not depend on the representative of $u$). Conversely let $u(\widetilde{x}) = 0$ for each slow scale point $\widetilde{x}$. We first show by contradiction that

$$(\forall m \in \mathbb{N})\,(\exists n \in \mathbb{N})\,(\exists \varepsilon_0)\,(\forall \varepsilon < \varepsilon_0)\left(\sup_{|x| \leq \varepsilon^{-1/n}} |u_\varepsilon(x)| \leq \varepsilon^m\right).$$

Assuming the contrary we find $M \in \mathbb{N}$, a decreasing sequence $(\varepsilon_n)_n$ tending to $0$ and $x_{\varepsilon_n} \in \mathbb{R}^d$ with $|x_{\varepsilon_n}| \leq \varepsilon^{-1/n}$ and $|u_{\varepsilon_n}(x_{\varepsilon_n})| > \varepsilon_n^M$, for each $n$. Let $x_\varepsilon = 0$ if $\varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\}$ Then $\widetilde{x} := [(x_\varepsilon)_\varepsilon]$ is of slow scale and $(u_\varepsilon(x_\varepsilon))_\varepsilon \notin \mathcal{N}_{\mathbb{R}}$, contradicting $u(\widetilde{x}) = 0$. Now, let $m \in \mathbb{N}$ arbitrary. Choose n as in the previous equation, Since $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right) = \mathcal{M}_\tau\left(\mathbb{R}^d\right)$, there exists $N \notin \mathbb{N}$ such that, for small $\varepsilon$, $\sup_{x \in \mathbb{R}^d} (1 + |x|)^{-p} |u_\varepsilon(x)| \ \leq \varepsilon^{-N}$. Let $p := nm + nN + N$. Then, for small $\varepsilon$,

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^d} (1 + |x|)^{-p} |u_\varepsilon(x)| \\
= \ & \max\left(\sup_{|x| \leq \varepsilon^{-1/n}} (1 + |x|)^{-p} |u_\varepsilon(x)|, \sup_{|x| \geq \varepsilon^{-1/n}} (1 + |x|)^{-p} |u_\varepsilon(x)|\right) \\
\leq \ & \max\left(\sup_{|x| \leq \varepsilon^{-1/n}} (1 + |x|)^{-p} |u_\varepsilon(x)|, \sup_{x \in \mathbb{R}^d} (1 + |x|)^{-N} |u_\varepsilon(x)|,\right. \\
& \left.\sup_{|x| \geq \varepsilon^{-1/n}} (1 + |x|)^{N-p} |u_\varepsilon(x)|\right) \\
\leq \ & \max\left(\varepsilon^m, \varepsilon^{-N}\left(\varepsilon^{-1/n}\right)^{N-p}\right) = \varepsilon^m
\end{aligned}$$

Hence $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ by Theorem 9. □

3.5. **The main theorem.** We begin to give two technical lemmas, the proof of the first one being a simple adaptation of [11, Thm. 1.2.29].

**Lemma 3.1.** *Let $(f_\varepsilon)_\varepsilon$, $(g_\varepsilon)_\varepsilon$, $\left(\widetilde{f}_\varepsilon\right)_\varepsilon$, $(\widetilde{g}_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ be such that $[f_\varepsilon] = \left[\widetilde{f}_\varepsilon\right]$, $[g_\varepsilon] = [\widetilde{g}_\varepsilon]$. We have that $[f_\varepsilon \circ g_\varepsilon] = [f_\varepsilon \circ \widetilde{g}_\varepsilon]$. If $g_\varepsilon$ preserves slow scale points then $\left[\widetilde{f}_\varepsilon \circ g_\varepsilon\right] = [f_\varepsilon \circ g_\varepsilon]$.*

**Lemma 3.2.** *Consider $(f_\varepsilon)_\varepsilon$, $(g_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ with $f_\varepsilon$ and $g_\varepsilon$ bijectives, $(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ and $(f_\varepsilon^{-1})_\varepsilon$, $(g_\varepsilon^{-1})_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$. Suppose moreover that $(g_\varepsilon^{-1})_\varepsilon$ preserves slow scale points. Then $(f_\varepsilon^{-1} - g_\varepsilon^{-1})_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$.*

*Proof.* We have $\left(\left(f_\varepsilon^{-1}-g_\varepsilon^{-1}\right)\circ g_\varepsilon\right)_\varepsilon = \left(f_\varepsilon^{-1}\circ g_\varepsilon - Id\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ because $\left(g_\varepsilon - f_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ which implies that $[f_\varepsilon^{-1}\circ g_\varepsilon] = [f_\varepsilon^{-1}\circ f_\varepsilon] = [Id]$. But then as $f_\varepsilon^{-1}\circ f_\varepsilon = ((f_\varepsilon^{-1}\circ f_\varepsilon)\circ g_\varepsilon)\circ g_\varepsilon^{-1}$ and $\left(g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ and preserves slow scale points, then, using the previous Lemma we find that $\left(f_\varepsilon^{-1}-g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. □

**Theorem 3.7.** *Suppose that $(l_\varepsilon)_\varepsilon$ is taken in the subset $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R})$ in $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ of families $(g_\varepsilon)_\varepsilon$ such that $g_\varepsilon' > 0, \left(g_\varepsilon^{-1}\right)_\varepsilon$ preserves slow scale points, $lim_{\varepsilon\to 0,\mathcal{D}'(\mathbb{R})} g_\varepsilon = 0$. Then if $f \in \mathcal{O}_M(\mathbb{R})$ and $F=0$, the generalized function $u = [1_\varepsilon \otimes f \circ l_\varepsilon^{-1}]_{\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)}$ depends only on $l = [l_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}(\mathbb{R})}$. Moreover, $u$ is the unique solution to $(P_g)$ in $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$*

*Proof.* Take $(l_\varepsilon)_\varepsilon$, $(h_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ such that $[l_\varepsilon] = [h_\varepsilon]$ and let $u=[u_\varepsilon]$, $v=[v_\varepsilon]$ (with $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^2)$) be the corresponding solutions of $(P_g)$. For all $\varepsilon$, we have

$$
\begin{aligned}
u_\varepsilon(t,x) &= f\left(l_\varepsilon^{-1}(x)\right) + \mu_\varepsilon\left(l_\varepsilon^{-1}(x)\right)\int_{l_\varepsilon^{-1}(x)}^{t} i_\varepsilon(\tau,x)\,d\tau,\\
v_\varepsilon(t,x) &= f\left(h_\varepsilon^{-1}(x)\right) + \nu_\varepsilon\left(l_\varepsilon^{-1}(x)\right)\int_{h_\varepsilon^{-1}(x)}^{t} j_\varepsilon(\tau,x)\,d\tau,
\end{aligned}
$$

where $(i_\varepsilon)_\varepsilon$, $(j_\varepsilon)_\varepsilon$,$(\mu_\varepsilon)_\varepsilon$,$(\nu_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. First we know that $\left(l_\varepsilon^{-1}-h_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ and $f \in \mathcal{O}_M(\mathbb{R})$ so that $\left(f\circ l_\varepsilon^{-1} - f\circ h_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. Futhermore, as $(\mu_\varepsilon)_\varepsilon$,$(\nu_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$, $\left(l_\varepsilon^{-1}\right)_\varepsilon$, $\left(l_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ and they preserves slow scale points, we have that $\left(\mu\circ l_\varepsilon^{-1}\right)_\varepsilon$, $\left(\nu\circ h_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. To finish the proof we have to check that

$$
\int_{l_\varepsilon^{-1}(x)}^{t} i_\varepsilon(\tau,x)\,d\tau - \int_{h_\varepsilon^{-1}(x)}^{t} j_\varepsilon(\tau,x)\,d\tau \in \rfloor.
$$

We will do that only for the first integal part as they are almost identical. First, we set, for all $\varepsilon$, $k_\varepsilon(t,x) = \int_{l_\varepsilon^{-1}(x)}^{t} i_\varepsilon(\tau,x)\,d\tau$. Let $(t_\varepsilon,x_\varepsilon) \in \widetilde{\mathbb{R}}^2$ be a slow scale point. Then $(x_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}$ is a slow scale point and $(y_\varepsilon)_\varepsilon = \left(l_\varepsilon^{-1}(x_\varepsilon)\right)_\varepsilon$ is also a slow scale point. We have

$$
\forall\varepsilon, \exists c_\varepsilon \in [y_\varepsilon, t_\varepsilon] : k_\varepsilon(t_\varepsilon,x_\varepsilon) = \int_{y_\varepsilon}^{t_\varepsilon} i_\varepsilon(\tau,x)\,d\tau = (t_\varepsilon - y_\varepsilon)\, i_\varepsilon(c_\varepsilon,x_\varepsilon),
$$

but as $|c_\varepsilon| \le \max(|y_\varepsilon|, |t_\varepsilon|)$, $(c_\varepsilon)_\varepsilon$ is also a slow scale point. But then $[(c_\varepsilon,x_\varepsilon)_\varepsilon]$ is a slow scale point of $\widetilde{\mathbb{R}}^2$ so that $\left(i_\varepsilon(c_\varepsilon,x_\varepsilon)\right)_\varepsilon \in \mathcal{N}_\mathbb{R}$ and finally $\left(k_\varepsilon(c_\varepsilon,x_\varepsilon)\right)_\varepsilon \in \mathcal{N}_\mathbb{R}$. □

**Remark 3.1.** *However, we cannot prove the existence of a solution to* $(P_g)$ *in* $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ *if* $F \neq 0$ *as can be seen by taking* $F(.,.,u) = u$*. Indeed the regularizd problem becomes* $(P_\infty) : \partial u_\varepsilon / \partial t\,(t,x) = u_\varepsilon(t,x)$*,* $u_\varepsilon(t, \varepsilon t) = v(t)$ *whose solution is* $u_\varepsilon(t,x) = v(x/\varepsilon)\, e^{-x/\varepsilon} e^t$ *which clearly is not in* $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^2)$*.*

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