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WEIGHTED MOSTAR INDICES OF CERTAIN GRAPHS

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ABSTRACT. Chemical graph theory is a mixture of chemistry and mathematics both play an important role in chemical graph theory. Chemistry provides a chemical compound and graph theory transform this chemical compound into a molecular graph, which are associated with some numerical values these values are known as topological indices. In this study we consider the weighted modification of new bond-additive Mostar indices that appear to provide quantitative measures of peripheral shapes of molecules. We have computed the Additively Weighted Mostar Index and Multiplicatively Weighted Mostar Index for Conical and Generalized gear graph.

1. INTRODUCTION AND PRELIMINARIES

In graph theory, it is a branch of mathematics but at present graph theory is mostly used in chemistry, engineering and physics departments for their research studies. In the fields of chemical sciences, mathematical chemistry, chemical graph theory, and pharmaceutical science, topological invariants are of significant importance because of their definitional use, bond connectivity topological indices are used to measure the branching, compactness, centrality, regularity, variability, bipartivity, etc. In reticular chemistry, quantitative

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structure-property / quantitative structure-activity relationship (QSPR/QSAR) schemes are correlation/regression models used to associate with various biological and physicochemical activities. The Harold Wiener initiated this study; in 1947, he introduced a distance-based structure invariant to find out the boiling point of alkanes. In this process, the topologically invariant quantities change the chemical structure into a specific mathematical number named as topological descriptor or index. In chemical graph theory, the molecular structure descriptors establish, which models organic compounds into hydrogen-suppressed graphs with a relationship of vertices (resp. edges) to atoms (resp. bonds). The structure descriptors give efficient regression models, which correspond to different biological and physicochemical properties, including the critical temperature, critical volume, critical pressure, the boiling point, the heat of formation, enthalpy of vaporizationof organic compounds. There are several classes of topologicaldescriptors, including counting-based descriptors, valency-based descriptors, spectrum-based descriptors.

Apart from these, yet another interesting class of indices strives to capture some relevant properties of whole graphs by summing contributions of individual vertices and/or edges, such indices are called vertex and bond-additive indices or distance based, respectively. It attracted considerable attention, both in the context of complex networks and in more classical applications of chemical graph theory. Some of the familiar bond-additive distance-based indices like szeged index put forward by Gutman [20], Padmakar-Ivan(PI) index introduced by Khalifeh et al. [25] and $Do\check{s}li\acute{c}$ et al. [14] recently introduced a Mostar index that measures the peripherality of individual bonds (i.e., edges) and then sums the contributions of all edges and produces a global measure of peripherality of a given graph, which are defined respectively as follows:

$$Sz(G) = \sum_{e=uv\epsilon E(G)} n_e^G(u) n_e^G(v),$$

$$PI_v(G) = \sum_{e=uv\epsilon E(G)} \left(n_e^G(u) + n_e^G(v) \right),$$

$$Mo(G) = \sum_{e=uv\epsilon E(G)} \left| n_e^G(u) - n_e^G(v) \right|.$$

Here $n_e^G(u)$ denotes the number of vertices of G closer to u than to v and $n_e^G(v)$ is defined analogously. Nowadays there is a vast literature presenting scientific researches deeply related to the szeged and PI index (e.g. for some recent results see [23] and references cited therein). For a comprehensive study on the success of Mostar index, the interested reader is referred to [3–5, 8, 11, 12, 15, 16, 19, 26, 28]. After the introduction of szeged and PI index which are attracted much attention in the mathematical chemistry community, $Ili\check{c}$ and $Milosavljevi\acute{c}$ proposed a modification of the Szeged index and PI index [21] get inspired after extension of the Wiener index. This quantatity is named as weighted Szeged index and weighted PI index, defined as

$$PI_w(G) = \sum_{e=uv\epsilon E(G)} (deg_e^G(u) + deg_e^G(v)) \left(n_e^G(u) + n_e^G(v)\right),$$

$$Sz_w(G) = \sum_{e=uv\epsilon E(G)} (deg_e^G(u) + deg_e^G(v)) n_e^G(u) n_e^G(v).$$

These quantities are some of the central and most commonly studied weighted version of distance-based topological descriptors. For example, see recent research on weighted Szeged index and weighted PI index [22, 31, 32]. In [1] the simplest, to define the weight as the sum of the degree, $deg_e^G(u) + deg_e^G(v)$ and the product degree $deg_e^G(u).deg_e^G(v)$, these two weights are known as an additive and multiplicative degree-weighting respectively. Recently, in this direction a new topological indices called Additively weighted(Weighted plus) Mostar index and Multiplicatively weighted(Weighted product) Mostar index have been proposed by Arokiaraj et al. [6] defined as

$$Mo_A(G) = \sum_{e=uv\epsilon E(G)} (deg_e^G(u) + deg_e^G(v)) \left|n_e^G(u) - n_e^G(v)\right|,$$

$$Mo_M(G) = \sum_{e=uv\epsilon E(G)} (deg_e^G(u).deg_e^G(v)) \left|n_e^G(u) - n_e^G(v)\right|.$$

In his paper using cut method explicit formulae of Weighted plus Mostar indices and Weighted product Mostar indices have obtained to the chemically interested graphs like graphene, graphyne, graphdiyne and benzenoid system. In [7] Tratnik and Brezovnik have discussed various weighted version of mostar index for strength weighted graphs and some well known families of molecular structures like phenylenes, coronoid and benzenoid systems.

Through out this paper we consider a graph G be a simple, undirected, and connected graph with the vertex set $V(G)$ and the edge set $E(G)$. The degree of $u\epsilon V(G)$, represented as $deg^G(u) = deg_e^G(u)$, is described as the number of edges directly linked with u [29]. Graph operations play an important role in chemical graph theory. Different molecular graphs can be obtained by applying graph operations on some general or particular graphs. For example, the linear polynomial chain, nanotube, nanotorus, tetrameric 1,3-adamantane, truncated cube etc.. Hence it is important to study the various graph operations in order to understand how it is related to the corresponding topological indices of the original graphs. There are several other results regarding various topological indices under different graph operations, refer to recent paper [10, 24, 31].

In 2020, Ayache et al. [2] introduced a graph called the conical graph that consists of center u_o and (ℓ, k)-Cycles $C_k^1, C_k^2, \ldots, C_k^\ell$ interposed as it's illustrated in Figure 1. given below, and denoted by $G(\ell, k) = C(\ell, k)$. One can be seen that for $\ell = 1$, the graph $G(1, k)$ is not other of classic wheel graph W_k formed by connecting a single vertex u_o to all vertices of k-cycles C_k. Recently in [27, 30] the (a,b)-Zagreb index is obtained for co-normal, concentric wheel and intersection of graphs. Motivated by this structure, Kandan and Subramanian [22, 23] recently obtained the explicit formula of bond-additive indices like PI, szeged, Mostar and weighted version of PI and szeged index to the conical grap

Definition 1.1. *The Conical graph $C(\ell, k)$ is a graph which is obtained by taking adjacency from a center vertex u_o to the first layer of the Cartesian product of C_k and P_ℓ, with $\ell \geq 1$ and $k \geq 3$. (see Figure 1.)*

Let vertex set of $C(\ell, k)$ can be written as

$$V(G) = \{u_o, u_1^1, \ldots, u_k^1, u_1^2, \ldots, u_k^2, \ldots, u_1^\ell, \ldots, u_k^\ell\}$$

and for our convenience the edges set of $C(\ell, k)$ into four sets such that $E(C) = \bigcup_{n=0}^{3} E_n(C)$, where

$$E_0(C) = \{u_o u_1^1, u_o u_2^1, \ldots, u_o u_k^1\}, \quad \Rightarrow \quad |E_0| = k,$$
$$E_1(C) = \{u_1^{\ell-1} u_1^\ell, \ldots, u_k^{\ell-1} u_k^\ell\}, \quad \Rightarrow |E_1| = k,$$
$$E_2(C) = \{u_1^\ell u_2^\ell, u_2^\ell u_3^\ell, \ldots, u_k^\ell u_1^\ell\}, \quad \Rightarrow \quad |E_2| = k,$$
$$E_3(C) = E^+(C) \bigcup E'(C) \bigcup E^*(C),$$

where $E^+(C) = \{u_1^1 u_2^1, \ldots, u_k^1 u_1^1\}$, $E'(C) = \{u_1^j u_2^j, u_2^j u_3^j, \ldots, u_k^j u_1^j | j = 2, 3, \ldots, \ell - 1\}$, $E^*(C) = \{u_1^j u_1^{j+1}, u_2^j u_2^{j+1}, \ldots, u_k^j u_k^{j+1} | j = 1, 2, \ldots, \ell - 2\}$, $\Rightarrow |E_3(C)| = k + k(\ell - 2) + k(\ell - 2)$. Its clear for a conical graph $C(\ell, k)$, we have $|V(C(\ell, k))| = k\ell + 1$, $|E(C(\ell, k))| = 2k\ell$ and noticed that $deg^C(u_o) = k$, $deg^C(u_i^j) = 4$, if $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell - 1$, $deg^C(u_i^\ell) = 3$, if $i = 1, 2, \ldots, k$.

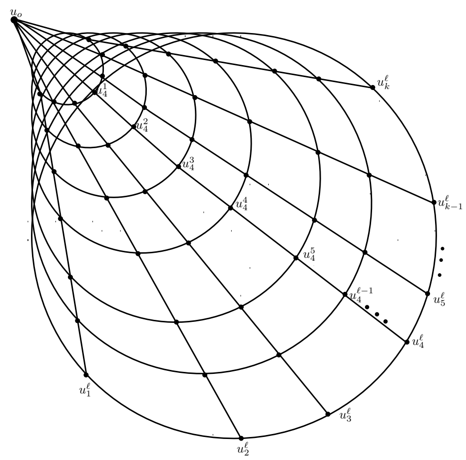

FIGURE 1. Conical graph $C(\ell, k)$

Inspired by the structure of conical graph and by the definition of subdivison graph, Kandan and Subramanian [23], recently introduced a Generalized gear graph and obtained the exact formula of PI, szeged, Mostar indices, moreover in [22] weighted PI and weighted szeged indices to it.

Definition 1.2. *For $\ell \geq 1$ and $k \geq 2$, the Generalized gear graph $C^*(\ell, 2k)$ is graph obtain from the conical graph with a vertex added between each pair adjacent vertices of the cycles. (see Figure 2.)*

Let vertex set of $C^*(\ell, 2k)$ can be written as $V(C^*) = \{u_o, \ldots, u_{k,1}^1, u_1^2, \ldots, u_{k,1}^2, \ldots, u_1^\ell, \ldots, u_{k,1}^\ell\}$ and our convenience the edges set of $C(\ell, 2k)$ into for sets such that $E(C^*) = \bigcup\limits_{n=0}^{3} E_n$, where $E_0 = \{u_o u_1^1, u_o u_2^1, \ldots, u_o u_k^1\}$, $E_1 = \{u_1^{\ell-1} u_1^\ell, u_2^{\ell-1} u_2^\ell, \ldots, u_k^{\ell-1} u_k^\ell\}$, $E_2 = \{u_1^\ell u_{1,2}^\ell, \ldots, u_{k,1}^\ell u_1^\ell\}$, $E_3(C^*) = E^+(C^*) \bigcup E^{'}(C^*) \bigcup E^*(C^*)$, where $E^+(C^*) = \{u_1^1 u_{1,2}^1, u_{1,2}^1 u_2^1, \ldots, u_k^1 u_{k,1}^1, u_{k,1}^1 u_1^1\}$ and $E^{'}(C^*) = \{u_1^j u_{1,2}^j, \ldots, u_k^j u_{k,1}^j, u_{k,1}^j u_1^j | j = 2, 3, \ldots, \ell - 1\}$, and $E^*(C^*) = \{u_1^j u_1^{j+1}, \ldots, u_k^j u_k^{j+1}\}$ such that $E(G) = \bigcup\limits_{n=o}^{3} E_n$.

Its clear that for a Generalized gear graph $C^*(\ell, 2k)$, we have $|V(C^*(\ell, 2k))| = 2k\ell + 1$ and $|E(C^*(\ell, 2k))| = |E_0(C^*)| + |E_1(C^*)| + |E_2(C^*)| + |E_3(C^*)| = k + k + 2k + 3k\ell - 4k = 3k\ell$ and notices that $deg^{C^*}(u_o) = k$, $deg^{C^*}(u_i^j) = 4$, if $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell - 1$ $deg^{C^*}(u_i^\ell) = 3$, if $i = 1, 2, \ldots, k$, $deg^{C^*}(u_{i,i+1}^\ell) = 2$, if $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell$ For $\ell = 1$ the Generalized gear graph is Gear graph of $C^*(\ell, 2k)$, also sometimes known as a bipartite wheel graph. More topological

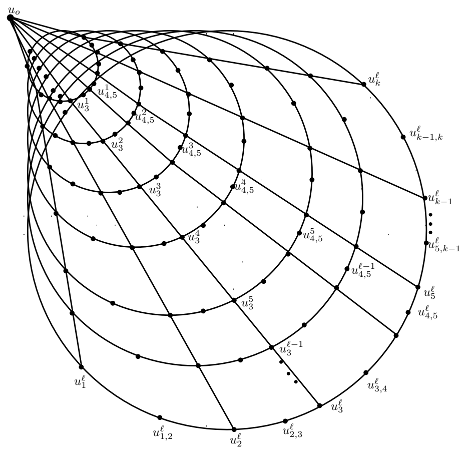

FIGURE 2. Generalized gear graph $C^*(\ell, 2k)$

indices has been studied to wheel and gear related graphs ref [9, 13, 17, 18, 23]. The following Lemma 2.1, is used to prove the main result of this section which follows immediately from the Figure 2.

2. MAIN RESULT

The main purpose of the present paper is to obtain the explicit formulae for the additively weighted Mostar index and multplicatively weighted Mostar index to the conical graph and Generalized gear graph.

2.1. **Conical graph.** As defined in the introduction part of the Conical graph, for our convenience the conical graph $C(\ell, k)$ is sometime simply denoted by C. The following Lemma is used in the proof of the main Theorem of this section, which follows immediately from the above defined edge partitions and structure of the conical graph.

Lemma 2.1. *[22] For a conical graph $C(\ell, k)$, with $\ell \geq 1$ and $k \geq 4$, we have*

(i) if $e = u_o u_i^1 \epsilon E_0(C)$, then $n_e^C(u_o) = \ell(k-3)+1$ and $n_e^C(u_i^1) = \ell$, for $i = 1, 2, \ldots, k$;

(ii) if $e = u_i^{\ell-1} u_i^{\ell} \epsilon E_1(C)$, then $n_e^C(u_i^{\ell-1}) = (k\ell+1)-k$ and $n_e^C(u_i^{\ell}) = k$, for $i = 1, 2, \ldots, k$;

(iii) if $e = u_i^{\ell} u_{i+1}^{\ell} \epsilon E_2(C)$, for $i = 1(= k+1), 2, \ldots, k$ then,

(a) if $n_e^C(u_i^{\ell}) = \frac{k\ell}{2} = n_e^C(u_{i+1}^{\ell})$, for k is even

(b) if $n_e^C(u_i^{\ell}) = \frac{(k-1)\ell}{2} = n_e^C(u_{i+1}^{\ell})$, for k is odd.

(iv)(a) for $i = 1(= k+1), 2, \ldots, k$, *if* $e = u_i^1 u_{i+1}^1 \epsilon E^+(C)$, *then* $n_e^C(u_i^1) = 2\ell = n_e^C(u_{i+1}^1)$.

(b) for $j = 2, 3, \ldots, \ell-1$ *and* $i = 1(= k+1), 2, \ldots, k$, *if* $e = u_i^j u_{i+1}^j \epsilon E'(C)$, *then*

(i) if $n_e^C(u_i^j) = \frac{k\ell}{2} = n_e^C(u_{i+1}^j)$, *k is even*

(ii) if $n_e^C(u_i^j) = \frac{(k-1)\ell}{2} = n_e^C(u_{i+1}^j)$, *k is odd*

(c) for $j = 1, 2, \ldots, \ell-2$ *and* $i = 1, 2, \ldots, k$, *if* $e = u_i^j u_i^{j+1} \epsilon E^*(C)$, *then* $n_e^C(u_i^j) = \sum_{j=1}^{\ell-2}(jk+1)$ *and* $n_e^C(u_i^{j+1}) = \sum_{j=1}^{\ell-2}(\ell-j)k$.

Using the Lemma 2.1, next we determine the exact value of Additively weighted Mostar index to the conical graph $C(\ell, k)$

Theorem 2.1. *For any conical graph* $C(\ell, k)$ *with* $\ell \geq 1$ *and* $k \geq 4$, *we have*

$$Mo_A(C(\ell,k)) = \begin{cases} k\left((k+4)(\ell(k-4)+1) + 7(k(\ell-2)+1)\right) + 4(k(\ell-2))^2 \\ \qquad \text{if } \ell \text{ is even} \\ k\left((k+4)(\ell(k-4)+1) + 7(k(\ell-2)+1)\right) + 8k\left((k\ell-1) + \frac{k(\ell-1)(\ell-5)}{2}\right) \\ \qquad \text{if } \ell \text{ is odd.} \end{cases}$$

Proof. By the definition of Additively weighted Mostar index, to obtain it for the conical graph, we have

$$Mo_A(C(\ell,k)) = \sum_{e=uv\epsilon E(C)} \left(deg_e^C(u) + deg_e^C(v)\right)\left|n_e^C(u) - n_e^C(v)\right|.$$

Using the edge partition, E_0, E_1, E_2 and E_3, of the conical graph $C(\ell,k)$, as define in the introduction and by the Lemma 2.1, we have the following four cases,

Case (i): For $i = 1, 2, \ldots, k$, if $e = u_o u_i^1 \epsilon E_0(C)$, then

$$\sum_{e\epsilon E_0(C)} (deg_e^C(u_o) + deg_e^C(u_i^1))\left|n_e^C(u_o) - n_e^C(u_i^1)\right|$$
$$= \sum_{e\epsilon E_0(C)} (k+4)\left|(\ell(k-3)+1) - \ell\right| = k(k+4)(\ell(k-4)+1), \quad since\ k \geq 4$$

Case (ii): For $i = 1, 2, \ldots, k$, if $e = u_i^{\ell-1} u_i^\ell \epsilon E_1(C)$, then

$$\sum_{e\epsilon E_1(C)} \left(deg_e^C(u_i^{\ell-1}) + deg_e^C(u_i^\ell)\right)\left|n_e^C(u_i^{\ell-1}) - n_e^C(u_i^\ell)\right|$$
$$= \sum_{e\epsilon E_1(C)} (4+3)\left|((\ell k+1)-k) - k\right| = 7k(k(\ell-2)+1).$$

Case (iii): For $i = 1(= k+1), 2, \ldots, k$, if $e = u_i^\ell u_{i+1}^\ell \epsilon E_2(C)$, then

$\sum_{e\epsilon E_2(C)} (deg_e^C(u_i^\ell) + deg_e^C(u_{i+1}^\ell)) \left|n_e^C(u_i^\ell) - n_e^C(u_{i+1}^\ell)\right| = 0.$

Case (iv): For $e \in E_3(C) = E^+(C) \bigcup E^{'}(C) \bigcup E^*(C)$, then the three sub-cases are

sub-case(a): For $i = 1(= k+1), 2, \ldots, k$, if $e = u_i^1 u_{i+1}^1 \epsilon E^+(C)$, then

$$\sum_{e\in E^+(C)} (deg_e^C(u_i^1) + deg_e^C(u_{i+1}^1)) \left|n_e^C(u_i^1) - n_e^C(u_{i+1}^1)\right| = 0$$

sub-case(b): For $i = 1(= k+1), 2, \ldots, k$ and $j = 2, 3, \ldots, \ell - 1$, if $e = u_i^j u_{i+1}^j \epsilon E^{'}(C)$, then

$$\sum_{e\epsilon E^{'}(C)} (deg_e^C(u_i^j) + deg_e^C(u_{i+1}^j)) \left|n_e^C(u_i^j) - n_e^C(u_{i+1}^j)\right| = 0$$

sub-case(c): For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell - 2$, if $e = u_i^j u_i^{j+1} \epsilon E^*(C)$, then

$$\begin{aligned}
&\sum_{e\epsilon E^*(C)} (deg_e^C(u_i^j) + deg_e^C(u_i^{j+1})) \left|n_e^C(u_i^j) - n_e^C(u_i^{j+1})\right| \\
&= \sum_{i=1}^{k} (4+4) \sum_{j=1}^{\ell-2} |(jk+1) - (\ell - j)k| \\
&= 8k \begin{cases} \sum_{j=1}^{\frac{\ell-2}{2}} k\ell - (2jk+1) + \sum_{j=\frac{\ell}{2}}^{\ell-2} k(2jk+1) - k\ell & \text{if } \ell \text{ is even} \\ \sum_{j=1}^{\frac{\ell-1}{2}} k\ell - (2jk+1) + \sum_{j=\frac{\ell+1}{2}}^{\ell-2} (2jk+1) - k\ell & \text{if } \ell \text{ is odd} \end{cases} \\
&= 8k \begin{cases} \frac{k(\ell-2)^2}{2} & \text{if } \ell \text{ is even} \\ k\ell - 1 + \frac{k(\ell-1)(\ell-5)}{2} & \text{if } \ell \text{ is odd} \end{cases}
\end{aligned}$$

Hence from the above four cases, we have the explicit formula of Additively weighted Mostar index to the conical graph $C(\ell, k)$ is

Case (i): For ℓ is even

$$Mo_A(C(\ell, k)) = k\left((k+4)(\ell(k-4)+1) + 7(k(\ell-2)+1)\right) + 4(k(\ell-2))^2.$$

Case (ii): For ℓ is odd

$$Mo_A(C(\ell, k)) = k\left((k+4)(\ell(k-4)+1) + 7(k(\ell-2)+1)\right) + 8k\left((k\ell - 1) + \frac{k(\ell-1)(\ell-5)}{2}\right).$$

□

Observe that Aditively weighted Mostar index of conical graph $C(\ell, k)$ can expressed interms of Mostar index as $Mo_A(C(\ell, k)) = (k+4)Mo(E_0(C)) +$

$7Mo(E_1(C)) + 8Mo(E^*(C))$ Using the above Theorem 2.1, we have following corollary.

Corollary 2.1. *For $\ell = 1$ and $k \geq 4$ the wheel graph W_k whose Additively weighted Mostar index $Mo_A(W_k) = k(k^2 - 9)$*

Next we have obtain the Multiplicatively weighted Mostar index of conical graph $C(\ell, k)$. as follow.

Theorem 2.2. *For a conical graph $C(\ell, k)$ with $\ell \geq 1$ and $k \geq 4$, we have*

$$Mo_M(C(\ell,k)) = \begin{cases} 4k\left(k(\ell(k-4)+1) + 3(k(\ell-2)+1)\right) & \\ +8(k(\ell-2))^2 & \text{if } \ell \text{ is even} \\ 4k\left(k(\ell(k-4)+1) + 3(k(\ell-2)+1)\right) & \\ +16k\left((k\ell-1) + \frac{k(\ell-1)(\ell-5)}{2}\right) & \text{if } \ell \text{ is odd} \end{cases}.$$

Proof. By the definition of Multiplicatively weighted Mostar index, to obtain it for the conical graph, we have

$$Mo_M(C(\ell,k)) = \sum_{e=uv\epsilon E(C)} \left(deg_e^C(u).deg_e^C(v)\right)\left|n_e^C(u) - n_e^C(v)\right|.$$

Using the edge partition E_0, E_1, E_2 and E_3, of the conical graph $C(\ell, k)$, as defined in the introduction and by the Lemma 2.1, we have the following four cases,

Case (i): For $i = 1, 2, \ldots, k$, if $e = u_o u_i^1 \epsilon E_0(C)$, then

$$\sum_{e\epsilon E_0(C)} \left(deg_e^C(u_o).deg_e^C(u_i^1)\right)\left|n_e^C(u_o) - n_e^C(u_i^1)\right|$$

$$= \sum_{e\epsilon E_0(C)} (4k)\left|(\ell(k-3)+1) - \ell\right| = k(4k)(\ell(k-4)+1), \ since\ k \geq 4$$

Case (ii): For $i = 1, 2, \ldots, k$, if $e = u_i^{\ell-1} u_i^\ell \epsilon E_1(C)$, then

$$\sum_{e\epsilon E_1(C)} \left(deg_e^C(u_i^{\ell-1}).deg_e^C(u_i^\ell)\right)\left|n_e^C(u_i^{\ell-1}) - n_e^C(u_i^\ell)\right|$$

$$= \sum_{e\epsilon E_1(C)} (4.3)\left|((\ell k+1) - k) - k\right| = 12k(k(\ell-2)+1)$$

Case (iii): For $i = 1(=k+1), 2, \ldots, k$, if $e = u_i^\ell u_{i+1}^\ell \epsilon E_2(C)$, then

$$\sum_{e\epsilon E_2(C)} (deg_e^C(u_i^\ell).deg_e^C(u_{i+1}^\ell))\left|n_e^C(u_i^\ell) - n_e^C(u_{i+1}^\ell)\right| = 0$$

Case (iv): For $e\epsilon E_3(C) = E^+(C) \bigcup E'(C) \bigcup E^*(C)$, then the three sub-cases are

sub-case(a): For $i = 1(=k+1), 2, \ldots, k$, if $u_i^1 u_{i+1}^1 \epsilon E^+(C)$, then

$\sum\limits_{e\in E^{+}(C)} (deg_e^C(u_i^1).deg_e^C(u_{i+1}^1)) \left|n_e^C(u_i^1) - n_e^C(u_{i+1}^1)\right| = 0$

sub-case(b): For $i = 1(= k+1), 2, \ldots, k$ and $j = 2, 3, \ldots, \ell - 1$, if $e = u_i^j u_{i+1}^j \epsilon E^{'}(C)$, then

$\sum\limits_{e\epsilon E^{'}(C)} (deg_e^C(u_i^j).deg_e^C(u_{i+1}^j)) \left|n_e^C(u_i^j) - n_e^C(u_{i+1}^j)\right| = 0$

sub-case(c): For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell - 2$, if $e = u_i^j u_i^{j+1} \epsilon E^{*}(C)$, then

$$\begin{aligned}
&\sum_{e\epsilon E^{*}(C)} (deg_e^C(u_i^j).deg_e^C(u_i^{j+1})) \left|n_e^C(u_i^j) - n_e^C(u_i^{j+1})\right| \\
&= 16k \sum_{j=1}^{\ell-2} |k\ell - (2jk+1)| \\
&= 16k \begin{cases} \sum\limits_{j=1}^{\frac{\ell-2}{2}} k\ell - (2jk+1) + \sum\limits_{j=\frac{\ell}{2}}^{\ell-2} (2jk+1) - k\ell & \text{if } \ell \text{ is even} \\ \sum\limits_{j=1}^{\frac{\ell-1}{2}} k\ell - (2jk+1) + \sum\limits_{j=\frac{\ell+1}{2}}^{\ell-2} (2jk+1) - k\ell & \text{if } \ell \text{ is odd} \end{cases} \\
&= 16k \begin{cases} \frac{k(\ell-2)^2}{2} & \text{if } \ell \text{ is even} \\ k\ell - 1 + \frac{k(\ell-1)(\ell-5)}{2} & \text{if } \ell \text{ is odd} \end{cases}.
\end{aligned}$$

By summing the above four cases, we have the explicit formula of Multiplicatively weighted Mostar index to the conical graph $C(\ell, k)$ is

Case (i): For ℓ is even

$Mo_M(C(\ell,k)) = 4k\left(k(\ell(k-4)+1) + 3(k(\ell-2)+1)\right) + 8(k(\ell-2))^2.$

Case (ii): For ℓ is odd

$Mo_M(C(\ell,k)) = 4k\left(k(\ell(k-4)+1) + 3(k(\ell-2)+1)\right)$
$+16k\left((k\ell-1) + \frac{k(\ell-1)(\ell-5)}{2}\right).$ □

Observe that Multiplicatively weighted Mostar Index of conical graph $C(\ell, k)$ can be expressed interms of Mostar index as follow:

$$Mo_M(C(\ell,k)) = 4kMo(E_0(C)) + 12Mo(E_1(C)) + 16Mo(E^{*}(C)).$$

Using the above Theorem 2.2, we have following corollary.

Corollary 2.2. *For $\ell = 1$ and $k \geq 3$ the wheel graph W_k whose Multiplicatively weighted Mostar index $Mo_M(W_k) = 3k^2(k-3)$.*

3. GENERALIZED GEAR GRAPH

In this section we have obtained the exact formula of Additively weighted Mostar index and Multiplicatively weighted Mostar index to the Generalized gear graph. The following Lemma is used in the proof of the main Theorem of this section, which follows immediately from the edge partitions as defined early and structure of the Generalized gear graph.

Lemma 3.1. *[22] For a Generalized gear graph $C^*(\ell, 2k)$, with $\ell \geq 1$ and $k \geq 2$, we have*

(i) for $i = 1, 2, \ldots, k$, if $e = u_o u_i^1 \epsilon E_0(C^*)$, then $n_e^{C^*}(u_o) = \ell(2k-3)+1$ and $n_e^{C^*}(u_i^1) = 3\ell$.

(ii) for $i = 1, 2, \ldots, k$, if $e = u_i^{\ell-1} u_i^{\ell} \epsilon E_1(C^*)$, then $n_e^{C^*}(u_i^{\ell-1}) = 2k(\ell-1)+1$ and $n_e^{C^*}(u_i^{\ell}) = 2k$.

(iii) for $i = 1(= k+1), 2, \ldots, k$ and let $E_2(C^*) = E^{'}(C^*) \bigcup E^{''}(C^*)$, where $e = u_i^{\ell} u_{i,i+1}^{\ell} \epsilon E^{'}(C^*)$ and $e = u_{i,i+1}^{\ell} u_{i+1}^{\ell} \epsilon E^{''}(C^*)$.

sub-case(a) if $e = u_i^{\ell} u_{i,i+1}^{\ell} \epsilon E^{'}(C^*)$, then $n_e^{C^*}(u_i^{\ell}) = \ell(k+1)$ and $n_e^{C^*}(u_{i,i+1}^{\ell}) = \ell(k-1)+1$.

sub-case(b) if $e = u_{i,i+1}^{\ell} u_{i+1}^{\ell} \epsilon E^{''}(C^*)$, then $n_e^{C^*}(u_{i,i+1}^{\ell}) = \ell(k-1)+1$ and $n_e^{C^*}(u_{i+1}^{\ell}) = \ell(k+1)$.

(iv) For $e \in E_3(C^*) = E^{+}(C^*) \bigcup E^{-}(C^*) \bigcup E^{*}(C^*)$, then the three cases are

Case (a) For $i = 1(= k+1), 2, \ldots, k$, and let $E^{+}(C^*) = E_a^{+}(C^*) \bigcup E_b^{+}(C^*)$, where $e = u_i^1 u_{i,i+1}^1 \epsilon E_a^{+}(C^*)$ and $e = u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^{+}(C^*)$.

sub-case(a) if $e = u_i^1 u_{i,i+1}^1 \epsilon E_a^{+}(C^*)$, then $n_e^{C^*}(u_i^1) = 2\ell(k-1)$ and $n_e^{C^*}(u_{i,i+1}^1) = 2\ell+1$.

sub-case(b) if $e = u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^{+}(C^*)$, then $n_e^{C^*}(u_{i,i+1}^1) = 2\ell+1$ and $n_e^{C^*}(u_{i+1}^j) = 2\ell(k-1)$.

Case (b) For $i = 1(= k+1), 2, \ldots, k$ and $j = 2, 3, \ldots, \ell-1$, let $E^{-}(C^*) = E_a^{-}(C^*) \bigcup E_b^{-}(C^*)$, where $e = u_i^j u_{i,i+1}^j \epsilon E_a^{-}(C^*)$ and $e = u_{i,i+1}^j u_{i+1}^j \epsilon E_b^{-}(C^*)$ then

sub-case(a) if $e = u_i^j u_{i,i+1}^j \epsilon E_a^{-}(C^*)$, then $n_e^{C^*}(u_i^j) = \ell(k+1)$ and $n_e^{C^*}(u_{i,i+1}^j) = \ell(k-1)+1$.

sub-case(b) if $e = u_{i,i+1}^j u_{i+1}^j \epsilon E_b^{-}(C^*)$, then $n_e^{C^*}(u_{i,i+1}^j) = \ell(k-1)+1$ and $n_e^{C^*}(u_{i+1}^j) = \ell(k+1)$.

Case (c) if $e = u_i^j u_i^{j+1} \epsilon E^*(C^*)$ *then,* $n_e^{C^*}(u_i^j) = \sum_{j=1}^{\ell-2}(2kj+1)$ *and* $n_e^{C^*}(u_i^{j+1}) = \sum_{j=1}^{\ell-2} 2k(\ell-j)$.

Using the Lemma 3.1, next we determine the explicit formula of Additively weighted Mostar index to the Generalized gear graph $C^*(\ell, 2k)$.

Theorem 3.1. *For a Generalized gear graph* $C^*(\ell, 2k)$ *with* $\ell \geq 1$ *and* $k \geq 3$*, we have*

$$Mo_A(C^*(\ell,2k)) = \begin{cases} k((k+4)(2\ell(k-3)+1) + 7(2k(\ell-2)+1)) + 12k(2k\ell-(4\ell+1)) \\ +2k(2\ell-1)(6\ell-7) + 8k^2(\ell-2)^2, \qquad \textit{if } \ell \textit{ is even} \\ k((k+4)(2\ell(k-3)+1) + 7(2k(\ell-2)+1)) + 12k(2k\ell-(4\ell+1)) \\ +2k(2\ell-1)(6\ell-7) + 8k\left((2k\ell-1) + k(\ell-1)(\ell-5)\right), \qquad \textit{if } \ell \textit{ is odd} \end{cases}.$$

Proof. By the definition of Additively weighted Mostar index, to obtain it for the Generalized gear graph $C^*(\ell, 2k)$, we have

$$Mo_A(C^*(\ell,2k)) = \sum_{e=uv\epsilon E(C^*)} (deg_e^{C^*}(u) + deg_e^{C^*}(v)) \left| n_e^{C^*}(u) - n_e^{C^*}(v) \right|.$$

Using the partition E_0, E_1, E_2 and E_3 of the Generalized gear graph $C^*(\ell, 2k)$, as defined in the introduction and by the Lemma 3.1, we have the following four cases,

Case (i): For $i = 1, 2, \ldots, k$, if $e = u_o u_i^1 \epsilon E_0(C^*)$, we have

$$\sum_{e\epsilon E_0(C^*)} (deg_e^{C^*}(u_o) + deg_e^{C^*}(u_i^1)) \left| n_e^{C^*}(u_o) - n_e^{C^*}(u_i^1) \right|$$
$$= k(k+4)\left|(\ell(2k-3)+1) - 3\ell\right| = k(k+4)(2\ell(k-3)+1), \ \ since\ k \geq 3$$

Case (ii): For $i = 1, 2, \ldots, k$, if $e = u_i^{\ell-1} u_i^{\ell} \epsilon E_1(C^*)$, we have

$$\sum_{e\epsilon E_1(C^*)} (deg_e^{C^*}(u_i^{\ell-1}) + deg_e^{C^*}(u_i^{\ell})) \left| n_e^{C^*}(u_i^{\ell-1}) - n_e^{C^*}(u_i^{\ell}) \right|$$
$$= \sum_{e\epsilon E_1(C^*)} (4+3)\left|(2k(\ell-1)+1) - 2k\right| = 7k(2k(\ell-2)+1)$$

Case (iii): For $i = (= k+1), 2, \ldots, k$, let $e = u_i^{\ell} u_{i+1}^{\ell} \in E_2(C^*) = E^{'}(C^*) \bigcup E^{''}(C^*)$, with $u_i^{\ell} u_{i,i+1}^{\ell} \epsilon E^{'}(C^*)$ and $u_{i,i+1}^{\ell} u_{i+1}^{\ell} \epsilon E^{''}(C^*)$, then we have

$$\sum_{e\epsilon E_2(C^*)} (deg_e^{C^*}(u_i^{\ell}) + deg_e^{C^*}(u_{i+1}^{\ell})) \left| n_e^{C^*}(u_i^{\ell}) - n_e^{C^*}(u_{i+1}^{\ell}) \right|$$
$$= k(3+2)\left|\ell(k+1) - (\ell(k-1)+1)\right| + k(3+2)\left|(\ell(k-1)+1) - \ell(k+1)\right|$$

$= 10k(2\ell - 1)$

Case (iv): For $e \in E_3(C^*) = E^+(C^*) \bigcup E^-(C^*) \bigcup E^*(C^*)$, then the three sub-cases are

sub-case(a): For $i = 1(= k+1), 2, \ldots, k$, let $e = u_i^1 u_{i+1}^1 \epsilon E^+(C^*)$, with $u_i^1 u_{i,i+1}^1 \epsilon E_a^+(C^*)$ and $u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^+(C^*)$, then we have

$$\sum_{e \epsilon E^+(C^*)} (deg_e^{C^*}(u_i^1) + deg_e^{C^*}(u_{i+1}^1)) \left| n_e^{C^*}(u_i^1) - n_e^{C^*}(u_{i+1}^1) \right|$$
$$= (4+2)k \left| 2\ell(k-1) - (2\ell+1) \right| + k(2+4) \left| (2\ell+1) - 2\ell(k-1) \right|$$
$$= 12k(2k\ell - (4\ell+1))$$

sub-case(b): For $i = 1(= k+1), 2, \ldots, k$ and $j = 2, 3, \ldots, \ell - 1$, let $e = u_i^j u_{i+1}^j \epsilon E^-(C^*)$, with $u_i^j u_{i+1}^j \epsilon E_a^-(C^*)$ and $u_{i,i+1}^j u_{i+1}^j \epsilon E_b^-(C^*)$, then we have

$$\sum_{e \epsilon E^-(C^*)} (deg_e^{C^*}(u_i^j) + deg_e^{C^*}(u_{i+1}^j)) \left| n_e^{C^*}(u_i^j) - n_e^{C^*}(u_{i+1}^j) \right|$$
$$= k(\ell-2)(4+2) \left| \ell(k+1) - (\ell(k-1)+1) \right| + k(\ell-2)(2+4) \left| (\ell(k-1)+1) - \ell(k+1) \right|$$
$$= 12k(2\ell - 1)(\ell - 2)$$

sub-case(c): For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell - 2$, if $e = u_i^j u_i^{j+1} \epsilon E^*(C^*)$, we have

$$\sum_{e \epsilon E^*(C^*)} (deg_e^{C^*}(u_i^j) + deg_e^{C^*}(u_i^{j+1}) \left| (n_e^{C^*}(u_i^j) - n_e^{C^*}(u_i^{j+1}) \right|$$
$$= \sum_{i=1}^{k} (4+4) \left| \sum_{j=1}^{\ell-2} (2kj+1) - \sum_{j=1}^{\ell-2} 2k(\ell - j) \right|$$
$$= 8k \sum_{j=1}^{\ell-2} \left| 2k\ell - (4kj+1) \right|$$
$$= 8k \begin{cases} \sum_{j=1}^{\frac{\ell-2}{2}} 2k\ell - (4jk+1) + \sum_{j=\frac{\ell}{2}}^{\ell-2} (4jk+1) - 2k\ell & \text{if } \ell \text{ is even} \\ \sum_{j=1}^{\frac{\ell-1}{2}} 2k\ell - (4jk+1) + \sum_{j=\frac{\ell+1}{2}}^{\ell-2} (4jk+1) - 2k\ell & \text{if } \ell \text{ is odd} \end{cases}$$
$$= 8k \begin{cases} k(\ell-2)^2 & \text{if } \ell \text{ is even} \\ (2k\ell - 1) + k(\ell-1)(\ell-5) & \text{if } \ell \text{ is odd} \end{cases}$$

By summing the above four cases, we have the explicit formula of Additively weighted Mostar index to the Generalized gear graph $C^*(\ell, 2k)$ is

Case (i): For ℓ is even: $Mo_A(C^*(\ell, 2k)) = k((k+4)(2\ell(k-3)+1) + 7(2k(\ell -$

$2)+1))+12k(2k\ell-(4\ell+1))+2k(2\ell-1)(6\ell-7)+8k^2(\ell-2)^2$

Case (ii): For ℓ is odd: $Mo_A(C^*(\ell,2k)) = k((k+4)(2\ell(k-3)+1)+7(2k(\ell-2)+1))+12k(2k\ell-(4\ell+1))+2k(2\ell-1)(6\ell-7)+8k\left((2k\ell-1)+k(\ell-1)(\ell-5)\right)$. $\square$

Observed that Additively weighted Mostar index of the Generalized gear graph $C^*(\ell,2k)$ can expressed interms of Mostar index as follow:

$$\begin{aligned} Mo_A(C^*(\ell,2k)) &= (k+4)Mo(E_0(C^*)) + 7Mo(E_1(C^*)) + 5Mo(E_2(C^*)) \\ &\quad +6Mo(E^+(C^*)) + 6Mo(E^-(C^*)) + 8Mo(E^*(C^*)) \end{aligned}$$

Using the Theorem 3.1, we have following corollary.

Corollary 3.1. *For $\ell = 1$ and $k \geq 2$, the gear graph $C^*(1,2k)$ whose Additively weighted Mostar index $Mo_A(C^*(1,2k)) = k(k+13)(2k-5)$.*

Next we have obtained the Multiplicatively weighted Mostar index the Generalized gear graph $C^*(\ell,2k)$, as follow.

Theorem 3.2. *For a Generalized gear graph $C^*(\ell,2k)$, with $\ell \geq 1$ and $k \geq 3$, we have*

$$Mo_M(C^*(\ell,2k)) = \begin{cases} 4k(k(2k(k-3)+1)+3(2k(\ell-2)+1)) & \\ +16k(2k\ell-(4\ell+1))+4k(2\ell-1)(4\ell-5) & \\ +16(k(\ell-2))^2 & \text{if } \ell \text{ is even} \\ 4k(k(2k(k-3)+1)+3(2k(\ell-2)+1)) & \\ +16k(2k\ell-(4\ell+1))+4k(2\ell-1)(4\ell-5) & \\ +16k\left((2k\ell-1)+k(\ell-1)(\ell-5)\right) & \text{if } \ell \text{ is odd} \end{cases}.$$

Proof. By the definition of Multiplicatively weighted Mostar index, to obtain it for the Generalized gear graph $C^*(\ell,2k)$, we have

$$Mo_M(C^*(\ell,2k)) = \sum_{e=uv\epsilon E(C^*)} (deg_e^{C^*}(u).deg_e^{C^*}(v)) \left|n_e^{C^*}(u) - n_e^{C^*}(v)\right|.$$

Using the edge partition E_0, E_1, E_2 and E_3 of the Generalized gear graph $C^*(\ell,2k)$, as define in the introduction and by Lemma 3.1, we have the following four cases,

Case (i): For $i = 1,2,\ldots,k$, if $e = u_o u_i^1 \epsilon E_0(C^*)$, we have

$$\begin{aligned} &\sum_{e\epsilon E_0(C^*)} (deg_e^{C^*}(u_o).deg_e^{C^*}(u_i^1)) \left|n_e^{C^*}(u_o) - n_e^{C^*}(u_i^1)\right| \\ &= k(4k)\left|(\ell(2k-3)+1)-3\ell\right| = k(4k)(2\ell(k-3)+1), \quad \text{since} \quad k \geq 3. \end{aligned}$$

Case (ii): For $i = 1, 2, \ldots, k$, if $e = u_i^{\ell-1} u_i^{\ell} \epsilon E_1(C^*)$, we have

$$\sum_{e \epsilon E_1(C^*)} (deg_e^{C^*}(u_i^{\ell-1}) + deg_e^{C^*}(u_i^{\ell})) \left| n_e^{C^*}(u_i^{\ell-1}) - n_e^{C^*}(u_i^{\ell}) \right|$$
$$= \sum_{e \epsilon E_1(C^*)} (4.3) \left| (2k(\ell-1)+1) - 2k \right| = 12k(2k(\ell-2)+1).$$

Case (iii): For $i = 1 (= k+1), 2, \ldots, k$, let $e = u_i^{\ell} u_{i+1}^{\ell} \in E_2(C^*) = E'(C^*) \bigcup E''(C^*)$, with $u_i^{\ell} u_{i,i+1}^{\ell} \epsilon E'(C^*)$ and $u_{i,i+1}^{\ell} u_{i+1}^{\ell} \epsilon E''(C^*)$, then we have

$$\sum_{e \epsilon E_2(C^*)} (deg_e^{C^*}(u_i^{\ell}).deg_e^{C^*}(u_{i+1}^{\ell})) \left| n_e^{C^*}(u_i^{\ell}) - n_e^{C^*}(u_{i+1}^{\ell}) \right|$$
$$= k(3.2) \left| \ell(k+1) - (\ell(k-1)+1) \right| + k(2.3) \left| (\ell(k-1)+1) - \ell(k+1) \right|$$
$$= 12k(2\ell-1)$$

Case (iv): For $e \in E_3(C^*) = E^+(C^*) \bigcup E^-(C^*) \bigcup E^*(C^*)$, then the three sub cases are

sub-case(a): For $i = 1 (= k+1), 2, \ldots, k$, let $e = u_i^1 u_{i+1}^1 \epsilon E^+$, with $u_i^1 u_{i,i+1}^1 \epsilon E_a^+(C^*)$ and $u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^+(C^*)$, then we have

$$\sum_{e \epsilon E^+(C^*)} (deg_e^{C^*}(u_i^1).deg_e^{C^*}(u_{i+1}^1)) \left| n_e^{C^*}(u_i^1) - n_e^{C^*}(u_{i+1}^1) \right|$$
$$= (4.2)k \left| (2\ell(k-1) - (2\ell+1) \right| + k(2.4) \left| (2\ell+1) - 2\ell(k-1) \right|$$
$$= 16k(2k\ell - (4\ell+1))$$

sub-case(b): For $i = 1 (= k+1), 2, \ldots, k$ and $j = 2, 3, \ldots, \ell-1$, let $e = u_i^j u_{i+1}^j \epsilon E^-(C^*)$, with $u_i^j u_{i,i+1}^j \epsilon E_a^-(C^*)$, and $u_{i,i+1}^j u_{i+1}^j \epsilon E_b^-(C^*)$, then we have

$$\sum_{e \epsilon E^-(C^*)} (deg_e^{C^*}(u_i^j).deg_e^{C^*}(u_{i+1}^j)) \left| n_e^{C^*}(u_i^j) - n_e^{C^*}(u_{i+1}^j) \right|$$
$$= k(\ell-2)(4.2) \left| \ell(k+1) - (\ell(k-1)+1) \right| + k(\ell-2)(2.4) \left| (\ell(k-1)+1) \right.$$
$$\left. -\ell(k+1) \right| = 16k(2\ell-1)(\ell-2)$$

sub-case(c): For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, \ell-2$, if $e = u_i^j u_i^{j+1} \epsilon E^*(C^*)$, we have

$$\sum_{e \epsilon E^*(C^*)} (deg_e^{C^*}(u_i^j).deg_e^{C^*}(u_i^{j+1}) \left| n_e^{C^*}(u_i^j) - n_e^{C^*}(u_i^{j+1}) \right|$$

$$= 16k \sum_{j=1}^{\ell-2} \left| (2kj+1) - 2k(\ell-j) \right| = 16k \sum_{j=1}^{\ell-2} \left| 2k\ell - (4kj+1) \right|$$

$$= 16k \begin{cases} \sum\limits_{j=1}^{\frac{\ell-2}{2}} 2k\ell - (4jk+1) + \sum\limits_{j=\frac{\ell}{2}}^{\ell-2} (4jk+1) - 2k\ell & \text{if } \ell \text{ is even} \\ \sum\limits_{j=1}^{\frac{\ell-1}{2}} 2k\ell - (4jk+1) + \sum\limits_{j=\frac{\ell+1}{2}}^{\ell-2} (4jk+1) - 2k\ell & \text{if } \ell \text{ is odd} \end{cases}$$

$$= 16k \begin{cases} k(\ell-2)^2 & \text{if } \ell \text{ is even} \\ (2k\ell-1)+k(\ell-1)(\ell-5) & \text{if } \ell \text{ is odd} \end{cases}$$

By summing the above four cases, we have the explicit formula of Multiplicatively weighted Mostar index of Generalized gear graph $C^*(\ell, 2k)$ is

Case (i): For ℓ is even:

$Mo_M(C^*(\ell, 2k)) = 4k(k(2k(k-3)+1)+3(2k(\ell-2)+1))+16k(2k\ell-(4\ell+1))+4k(2\ell-1)(4\ell-5)+16(k(\ell-2))^2$.

Case (ii): For ℓ is odd:

$Mo_M(C^*(\ell, 2k)) = 4k(k(2k(k-3)+1)+3(2k(\ell-2)+1))+16k(2k\ell-(4\ell+1))+4k(2\ell-1)(4\ell-5)+16k\left((2k\ell-1)+k(\ell-1)(\ell-5)\right)$. □

Observed that Multiplicatively weighted Mostar index of the Generalized gear graph $C^*(\ell, 2k)$ can expressed interms of Mostar index as follow

$Mo_M(C^*(\ell, 2k)) = 4kMo(E_0(C^*)) + 12Mo(E_1(C^*)) + 6Mo(E_2(C^*))$
$+8Mo(E^+(C^*)) + 8Mo(E^-(C^*)) + 16Mo(E^*(C^*))$.

Using the Theorem 3.2, we have following corollary.

Corollary 3.2. *For $\ell = 1$ and $k \geq 3$, the gear graph $C^*(1, 2k)$ whose multiplicatively weighted Mostar index $Mo_M(C^*(1, 2k)) = k(k+12)(2k-5)$.*

4. Conclusion

In this paper, we have evaluated the exact formula of vertex weighted version of Mostar index of Conical and Generalized Gear graph. Our exploration kept on determining new consequences of these graphs. Some of the topological indices have found applications, but others were given priority to the mathematical side in order to throw more clear on the relation between these various concepts. Further work may extended to special molecular structures and for some graph operations.

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REFERENCES

[1] A. ALI, T. $Doslic$: *Mostar index results and perspectives,* Application mathematics and computation, **404** (2021), art.no. 126245.

[2] A. AYACHE, A. ALUMERI, A. GHALLB: *Wiener polynomial and Wiener index of conical graph,* Sylwan, **3** (2020), 164.

[3] Y. ALIZADEH, K. XU, S. KLAVZAR: *On the Mostar index of trees and product graphs,* Filomat, **33**(19) (2019), 6453-6458.

[4] S. AKHTER, Z. IQBAL, I. A. ASLAM, W. GAO: *Computation of Mostar Index for some graph operations,* International Jouranal of Quantum Chemistry, **121**(15) (2021), e26674.

[5] M. AROCKIARAJ, J. CLEMENT, N. TRANIK: *Mostar indices of carbon nanostructures and circumscribed donut benzenoid systems,* Int. J. Quantum Chem., **119** (2019), Art e26043.

[6] M. AROCKIARAJ, J. CLEMENT, N. TRATNIK, S. MUSHTAQ, K. BALASUBRAMANIAN: *Weighted Mostar index as Measures of Molecular peripheral shapes with application to graphyne and graphdiyne nanribbon* , S.A. and QSAR in Environmental Research, **31** (2020), 187-208.

[7] S. BREZOVNIK, N. TRATNIK: *General cut method for computing Szeged-like topological indices with applications to molecular graphs,* Int. J. Quantum Chem. (2020), e26530, p.1-17.

[8] M. CANCAN, M. NAEEM, A. ASLAM, W. GAO, A.Q. BAIG: *Geometric arithmetic and Mostar indices of $P_{2n+F}P_{n+1}$* , J. Inf. Optim. Sci, **41** (2020), 1007-1024.

[9] O. COLAKOGLU HAVARE: *Mostar index(Mo) and edge (Mo) inedx for some cycle related graphs,* Rom.J.Math. Comput. Sci. **10** (2020), 53-66.

[10] O. COLAKOGLU HAVARE: *Mostar index of bridge graph,* TWMS J. App. and Eng. Math., **11**(2) (2021), 587-597.

[11] K. DEZA, S. LI: *On the extremal chemical trees respect to the mostar index,* MATCH Communications in Mathematical and in Computer Chemistry, **85** (2021), 161.

[12] N. DEHGARDI, M. AZARI: *More on Mostar index,* Appl. Math. E-Notes, **20** (2020), 316-322.

[13] J. DOU, Y. WANG, W. GAO: *Hyper-wiener index of hexagonal mobius graph*, BTAIJ, **10** (2014), 6230-6235.

[14] T. DOSILIC, I. MARTINJAK, R. SKREKOVSKI, S.T. SPUZEVIC, I. ZUBAC: *Mostar index,* journal of Mathematical Chemistry. **56**(10) (2018), 2995-3013.

[15] O. EGCIOGLU, E. SAYGI, Z. SAYGI: *The Mostar index of Fibonacci and lucas cubes,* arXiv:2101.04740v1[math.co].

[16] F. GAO, K. XU, T. DOSLIC: *On the difference of Mostar index and irregularity of graph,* Bull. Malays. Math. Sci. Soc., **44** (2021), 905-926.

[17] W. GAO, L. SHI: *Wiener index of gear fan graph and gear wheel graph,* Asian Journal of Chemistry, **26** (2014), 3397-3400.

[18] W. GAO, W. WANG: *The Vertex Version of Weighted Wiener Number for Bicyclic Molecular Structures,* Hindawi Publishing Corporation Computational and Mathematical Methods in Medicine, (2015), Article ID 418106, p.10.

[19] N. GHANBARI, S. ALIKHANI: *Mostar index and edge Mostar index of polymers,* arXiv:2106.06562v1 [math.CO].

[20] I. GUTMAN, A. DOBRYNIN: *The Szeged index - a success story,* Graph Theory Notes New York, **34** (1998), 37-44.

[21] A. ILIC AND N. MILOSAVLJEVIC: *The weighted vertex PI index,* Math. Comput. Modelling, **57** (2013), 623-631.

[22] P. KANDAN, S. SUBRAMAINAN: *Computation of Weighted PI and Szeged indices of Conical and Generalized Gear graphs*, (submitted).

[23] P. KANDAN, S. SUBRAMAINAN: *On bond-additive index of graph*, (submitted).

[24] P. KANDAN, S. SUBRAMAINAN: *on Mostar index of graph,* Advances in Mathematics: Scientific Journal, **10**(4) (2021), 2115–2129.

[25] M.H. KHALIFEH, H. YOUSEFI, A.R. ASHRAFI: *Vertex and edge PI-indices of cartesian Product graphs,* **156** (2008), 1780-1789.

[26] H. LIU: *On the Maximal Mostar Index of Tree-Type Phenylenes,* Polycyclic Aromatic Compounds, 2021.

[27] P. SARHAR, S. MONDAL, N. DE, A. PAL: *(a,b)-Zagreb index of some special graph,* Rev. Roum. Chim., **65** (2020), 1045-1055.

[28] F. YASMEEN, S. AKHTER, K. ALI, S. TAHIR R. RIZVI: *Edge Mostar Indices of Cacti Graph With Fixed Cycles,* Frontiers in Chemistry, **9** (2021), Article 693885.

[29] D.B. WEST: *Introduction of Graph Theory*, University of Illinois -Urbana 2edition, India, 2001.

[30] ADITYA PEGUI, BUDHESWAR DEKA, IDWEEP JYTI GOGOT, A.BHARALI: *Two generalized topological indices of some graph structures,* J. Math. Comput. Sci., **11**(5) (2021), 5549-5564.

[31] C. GOPIKA, J. GEETHA, K. SOMASUNDARAM: *Weighted PI index of tensor product and strong product of graphs,* Discrete Mathematics, Algorithms and Applications, **21** ((2021)), art.id. 2150019. .

[32] N. TRATNIK: *Computing weighted Szeged and PI indices from quotient graphs*, Int. J. Quantum. Chem., **119** (2019), e26006.

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