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EXACT PERIODIC SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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ABSTRACT. In the paper is given a method of finding periodical solutions of the differential equation of the form $x''(t) + p(t)x''(t-1) = q(t)x([t]) + f(t)$, where $[\cdot]$ denotes the greatest integer function, $p(t)$,$q(t)$ and $f(t)$ are continuous periodic functions of t. This reduces n-periodic soluble problem to a system of $n+1$ linear equations, where $n = 2, 3$. Furthermore, by using the well known properties of linear system in the algebra, all existence conditions for 2 and 3-periodical solutions are described, and the explicit formula for these solutions are obtained.

1. INTRODUCTION

Certain functional differential equation of neutral delay type with piecewise constant arguments exists in the form of:

$$x''(t) + px''(t-1) = qx([t]) + f(t), \tag{1.1}$$

where $[\cdot]$ denotes the greatest integer function, p and q are nonzero constants, and $f(t)$ is a periodic function with positive integer period of $2n$.

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In the past, many useful methods such as Hale [1] and Fink [2] were developed to study the almost periodic differential equations. Such equations have diversified application in the field of biology, neural networks, physics, chemistry, engineering, and so on [3], [4], [5], [6]. Besides, these equations have combined properties of both differential and difference type. The solutions of these equations are continuous with the continuous dynamical systems structure. Certain biomedical and disease dynamics models exploited these equations due to their resemblance with sequential continuous models [3].

The natural occurrence of these equations in approximating the partial differential equations via piecewice constant arguments has already been demonstrated [7]. Meanwhile, the uniqueness of almost periodic solutions to the second order neutral delay differential equations of the form (1.1) were studied in depth [8]- [16] Despite these studies, the uniqueness of the solution on such equation remains debatable.

A recently published papers [9], [10] has studied the second order DEPCA. The periodic solvable problem are reduced to the study a system of linear equations. Furthermore, by applying the well-known properties of linear system in algebra, all existence conditions are described for n-periodic solutions that yield explicit formula for the solutions of DEPCA.

In paper [11] was considered the boundary value problem (BVP) for forced diffusion equation with piecewise constant arguments. Applying the method used in [9] and [10] was obtained existence condition and explicit formula for the periodic solutions of DEPCA. That allowed to find the exact solutions of BVP.

In this view, this paper reports all conditions for the uniqueness, infiniteness and emptiness of 2 and 3-periodic solutions of the equation

$$x''(t) + p(t)x''(t-1) = q(t)x([t]) + f(t), \tag{1.2}$$

where p, q and f are continuous and n-periodic functions. Applying the method used in [9] and [10], an explicit formula for the exact periodic solutions of the equation is provided. The equivalence of equation (1.2) to the system of $n+1$ linear equations are also demonstrated. The existence condition for the periodic solution of (1.2) are described easily using the properties of linear algebraic system. Some equations having unique and infinite number of periodic solutions are emphasized.

2. THE MAIN RESULTS

A function x is said to be a solution of (1.2) if the following conditions are satisfied:

(i) x is differentiable on $\mathbf{R}$;

(ii) the second order derivative of $x(t)$ exists on $\mathbf{R}$ except possibly at the points $t = k, \quad k \in \mathbf{Z}$, where one-sided second order derivatives of $x(t)$ exist;

(iii) x satisfies (1.2) on each interval $(k-1, k)$ with integer $k \in \mathbf{Z}$.

We first give the existence conditions for 2-periodic solutions of equation (1.2) for the cases when p, q and f are 2-periodic functions.

On 2-periodic solutions. Let p, q and f be 2-periodic continuous functions and x be a 2-periodic solution of (1.2). Then by the definition of solution

$$x'(t) = x'(t+2) \quad \text{for all} \quad t \in \mathbf{R} \tag{2.1}$$

and $x''(t) = x''(t+2)$ on each interval $(k, k+1)$ with integer $k \in \mathbf{Z}$.

It follows from here and (1.2) that

$$\begin{aligned} x''(t) + p(t)x''(t-1) &= q(t)x([t]) + f(t), \\ x''(t+1) + p(t+1)x''(t) &= q(t+1)x([t+1]) + f(t+1). \end{aligned} \tag{2.2}$$

This gives

$$(1-p(t)p(t+1))x''(t) = q(t)x([t]) - p(t)q(t+1)x([t+1]) + f(t) - p(t)f(t+1). \tag{2.3}$$

Hence (2.3) yelds

$$x''(t) = \tfrac{q(t)x([t])}{\Delta(t)} - \tfrac{p(t)q(t+1)x([t+1])}{\Delta(t)} + \tfrac{f(t)-p(t)f(t+1)}{\Delta(t)}, \tag{2.4}$$

$$\Delta(t) = 1 - p(t)p(t+1).$$

Integrating (2.4) on $[0, t), t \leq 2$, we obtain

$$x(t) = x(0) + x'(0)t + \int_0^t \int_0^{t_1} L(s)dsdt_1. \tag{2.5}$$

where

$$L(t) = \frac{q(t)x([t])}{\Delta(t)} - \frac{p(t)q(t+1)x([t+1])}{\Delta(t)} + \frac{f(t) - p(t)f(t+1)}{\Delta(t)}. \tag{2.6}$$

or

$$x(t) = x(0) + x'(0)t + Q_2(t) + F_2(t), \tag{2.7}$$

where

$$Q_2(t)=\int_0^t\int_0^{t_1}\frac{q(s)x([s])}{\Delta(s)}\,ds\,dt_1-\int_0^t\int_0^{t_1}\frac{p(s)q(s+1)x([s+1])}{\Delta(s)}\,ds\,dt_1,$$

$$F_2(t)=\int_0^t\int_0^{t_1}\frac{f(s)-p(s)f(s+1)}{\Delta(s)}\,ds\,dt_1$$

The function $Q_2(t)$ on $t\in[0;\,1)$ can be written as

$$Q_2(t)=\int_0^t\int_0^{t_1}\frac{q(s)}{\Delta(s)}\,ds\,dt_1x(0)-\int_0^t\int_0^{t_1}\frac{p(s)q(s+1)}{\Delta(s)}\,ds\,dt_1x(1),t\in[0;\,1).$$

We represent the function $Q_2(t)$ on $t\in[1;\,2]$ via the unknown numbers $x(0)$ and $x(1)$ as

$$\begin{aligned}Q_2(t)=&\int_0^1\int_0^{t_1}\frac{q(s)x(0)-p(s)q(s+1)x(1)}{\Delta(s)}\,ds\,dt_1\\&+\int_1^t\int_0^1\frac{q(s)x(0)-p(s)q(s+1)x(1)}{\Delta(s)}\,ds\,dt_1\\&+\int_1^t\int_1^{t_1}\frac{-p(s)q(s+1)x(0)+q(s)x(1)}{\Delta(s)}\,ds\,dt_1.\end{aligned}$$

To find the unknown numbers $x(0),x(1)$ and $x'(0)$ from (2.7), we represent $x(t)$ as

$$\left\{\begin{array}{ll}x(t)=x(0)+x'(0)t+a_{00}(t)x(0)+a_{01}(t)x(1)+F_2(t), & t\in[0;\,1),\\ x(t)=x(0)+x'(0)t+a_{10}(t)x(0)+a_{11}(t)x(1)+F_2(t), & t\in[1;\,2],\end{array}\right.\tag{2.8}$$

where

$$a_{00}(t)=\int_0^t\int_0^{t_1}\frac{q(s)}{\Delta(s)}\,ds\,dt_1,$$

$$a_{01}(t)=-\int_0^t\int_0^{t_1}\frac{p(s)q(s+1)}{\Delta(s)}\,ds\,dt_1,$$

$$a_{10}(t)=\int_0^1\int_0^{t_1}\frac{q(s)}{\Delta(s)}\,ds\,dt_1+\int_1^t\int_0^1\frac{q(s)}{\Delta(s)}\,ds\,dt_1-\int_1^t\int_1^{t_1}\frac{p(s)q(s+1)}{\Delta(s)}\,ds\,dt_1,$$

$$\begin{aligned}a_{11}(t)=&-\int_0^1\int_0^{t_1}\frac{p(s)q(s+1)}{\Delta(s)}\,ds\,dt_1-\int_1^t\int_0^1\frac{p(s)q(s+1)}{\Delta(s)}\,ds\,dt_1\\&+\int_1^t\int_1^{t_1}\frac{q(s)}{\Delta(s)}\,ds\,dt_1.\end{aligned}$$

The second equation in (2.8) gives

$$x'(2)=x'(0)+a_{20}x(0)+a_{21}x(1)+F_2'(2),$$

where

$$a_{20} = a'_{10}(2) = \int_0^1 \frac{q(s)}{\Delta(s)}ds - \int_1^2 \frac{p(s)q(s+1)}{\Delta(s)}ds,$$
$$a_{21} = a'_{11}(2) = -\int_0^1 \frac{p(s)q(s+1)}{\Delta(s)}ds + \int_1^2 \frac{q(s)}{\Delta(s)}ds.$$

The periodicity of x and continuity of x' gave $x(0) = x(2)$ and $x'(0) = x'(2)$. Therefore, (2.8) gives the system of equations for $x(0), x(1)$ and $x'(0)$:

$$\text{(2.9)} \qquad \begin{cases} (1 + a_{00}(1))x(0) + (-1 + a_{01}(1))x(1) + x'(0) = -F_2(1), \\ a_{10}(2)x(0) + a_{11}(2)x(1) + 2x'(0) = -F_2(2), \\ a_{20}x(0) + a_{21}x(1) = -F_2'(2). \end{cases}$$

Moreover, the existence conditions for $(x(0), x(1), x'(0))$ will be defined by $D_1(p,q)$, where $D_1(p,q)$ is a determinant of the matrix

$$M_1(p,q) := \begin{pmatrix} 1 + a_{00}(1) & -1 + a_{01}(1) & 1 \\ a_{10}(2) & a_{11}(2) & 2 \\ a_{20} & a_{21} & 0 \end{pmatrix}.$$

Summarizing we have

Theorem 2.1. *Let f be a 2-periodic continuous function and $p(t)p(t+1) \neq 1$ for $t \in [0,2]$. Then*

(i) *Equation (1.2) has a unique 2-periodic solution x if and only if $D_1(p,q) \neq 0$. The 2-periodic solution x has the form (2.8), where $(x(0), x(1), x'(0))$ is the solution of (2.9).*

(ii) *If $D_1(p,q) = 0$ and $\mathbf{F} = (F_2(1), F_2(2), F_2'(2)) = (0,0,0)$, then equation (1.2) has an infinite number of 2-periodic solutions having the form*

$$x_\alpha(t) = \alpha(x(0) + x'(0)t + a_{00}(t)x(0) + a_{01}(t)x(1)) + F_2(t), \qquad t \in [0;\, 1),$$
$$x_\alpha(t) = \alpha(x(0) + x'(0)t + a_{10}(t)x(0) + a_{11}(t)x(1)) + F_2(t), \qquad t \in [1;\, 2),$$

where $(x(0), x(1), x'(0))$ is an eigenvector of $M_1(p,q)$ corresponding to 0, α is any number.

(iii) *If $D_1(p,q) = 0$ and rank$\mathbf{M}_1(p,q) <$ rank$\left(\mathbf{M}_1(p,q)|\mathbf{F}^T\right)$, where $\mathbf{F} = (F_2(1), F_2(2), F_2'(2))$, then equation (1.2) has no 2-periodic solution.*

Proof. (i) Let x be a unique 2-periodic solution of (1.2). Then x can be presented by (2.8), where $(x(0), x(1), x'(0))$ is the solution of (2.9). The linear system (2.9) has unique solution if and only if $D_1(p,q) \neq 0$. Hence $D_1(p,q) \neq 0$. Conversely, if $D_1(p,q) \neq 0$, the equation (2.9) has a unique solution $(x(0), x(1), x'(0))$. One can check that the function x having the form (2.8) is the solution of (1.2). The uniqueness of solution of (1.2) is trivial.

(ii) Let $F_2(1) = F_2(2) = F_2'(2) = 0$. Then equation (2.9) reduces to a non-homogeneous equation. This equation has non-trivial solution if and only if $D_1(p,q) = 0$. This non-trivial solution $(x(0), x(1), x'(2))$ is an eigenvector of $M_1(p,q) = 0$ corresponding to the number 0. Then the 2-periodic function

$$x_\alpha(t) = \alpha\Big(x(0) + x'(0)t + a_{00}(t)x(0) + a_{01}(t)x(1)\Big) + F_2(t) \quad \text{for} \quad t \in [0;\, 1),$$

$$x_\alpha(t) = \alpha\Big(x(0) + x'(0)t + a_{10}(t)x(0) + a_{11}(t)x(1)\Big) + F_2(t) \quad \text{for} \quad t \in [1;\, 2)$$

is a solution of (1.2), where α is any number.

(iii) Let $D_1(p,q) = 0$ and $\text{rank}\mathbf{M}_1(p,q) < \text{rank}\Big(\mathbf{M}_1(p,q)|\mathbf{F}^T\Big)$, where $\mathbf{F} = (F_2(1), F_2(2), F_2'(2))$. Then the linear system (2.9) has no any solution $(x(0), x(1), x'(2))$. Therefore, the equation (1.2) has no 2-periodic solution.

Theorem is proved. □

We remark that if $p(t) = p$, $q(t) = q$ are constants then

$$\mathbf{M}_1(p,q) = \begin{pmatrix} \frac{1}{2}\frac{2p^2-q-2}{p^2-1} & -\frac{1}{2}\frac{2p^2-pq-2}{p^2-1} & 1 \\ \frac{1}{2}\frac{q(p-3)}{p^2-1} & \frac{1}{2}\frac{q(3p-1)}{p^2-1} & 2 \\ \frac{q}{1+p} & \frac{q}{1+p} & 0 \end{pmatrix}.$$

One can check that $D_1(p,q) = \frac{4q}{1+p} \neq 0$. Hence, for this case the equation (1.2) has only unique solution.

3. Examples

In the following example are given parameters of the equation satisfying the conditions for (i) of Theorem 2.1.

Example 1. *Let* $p(t) = 2, \quad q(t) = sin\pi t, \quad f(t) = cos\pi t$. *For this case*

$$D_1(p,q) = \begin{vmatrix} 1 - \frac{1}{3\pi} & -1 - \frac{2}{3\pi} & 1 \\ -\frac{1}{3\pi} & -\frac{5}{3\pi} & 2 \\ \frac{2}{3\pi} & \frac{2}{3\pi} & 0 \end{vmatrix} = \frac{4 - 24\pi}{9\pi^2}.$$

Then the linear system (2.9) has a solution $(x(0), x(1), x'(2))$,

$$x(0) = \frac{6}{6\pi^2 - \pi}, \quad x(1) = -\frac{6}{6\pi^2 - \pi}, \quad x'(2) = -\frac{4}{6\pi^3 - \pi^2}.$$

The 2-periodic solution of the equation is

$$x(t) = \frac{cos(\pi t)(6\pi - 1) - 2sin(\pi t) - 2\pi t + \pi}{\pi^3(6\pi - 1)}, \quad t \in [0; 1),$$

$$x(t) = \frac{cos(\pi t)(6\pi - 1) + 2sin(\pi t) + 2\pi t - 3\pi}{\pi^3(6\pi - 1)}, \quad t \in [1, 2].$$

The graph of this solution is shown in the Figure 1.

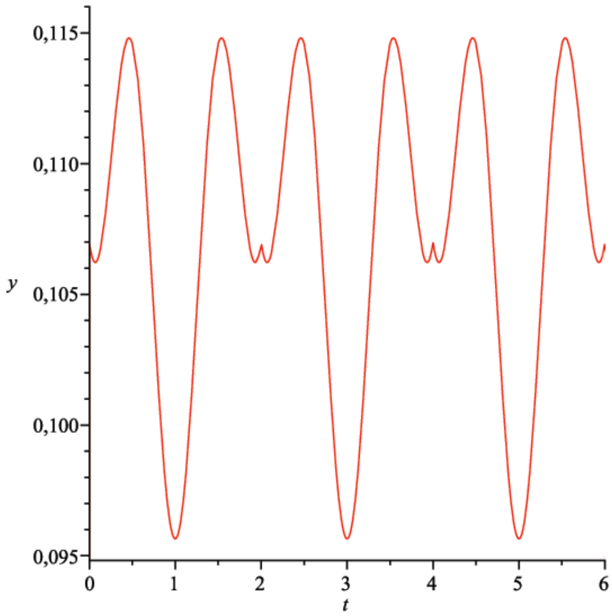

FIGURE 1. The graph of $x(t)$.

The following example has the parameters of the equation satisfying the conditions for (ii) of Theorem 2.1.

Example 2. *Let* $p(t) = \frac{1-2\pi}{2\pi}$, $q(t) = sin(\pi t)$, $f(t) = cos(2\pi t)$. *For this case*

$$M_1(p,q) = \begin{pmatrix} \frac{8\pi-1}{4\pi-1} & -\frac{8\pi-3}{4\pi-1} & 1 \\ \frac{2(8\pi-1)}{4\pi-1} & -\frac{2(8\pi-3)}{4\pi-1} & 2 \\ 4 & 4 & 0 \end{pmatrix},$$

$D_1(p,q) = 0$ *and* $F_2(t) = -\frac{1}{2}\frac{\cos(2\pi t)-1}{\pi}$, *hence* $F_2(1) = 0$, $F_2(2) = 0$ *and* $F_2'(2) = 0$. *The eigenvector of* $M_1(p,q)$ *corresponding to* 0 *is* $(-\frac{1}{4}, \frac{1}{4}, 1)$. *The 2-periodic solution of equation is*

$$x_\alpha(t) = \frac{1}{4\pi}(2sin(\pi t) + 2\pi t - \pi)\alpha + 1 - cos(2\pi t), \quad t \in [0,1),$$
$$x_\alpha(t) = -\frac{1}{4\pi}(2sin(\pi t) + 2\pi t - 3\pi)\alpha + 1 - cos(2\pi t), \quad t \in [1,2].$$

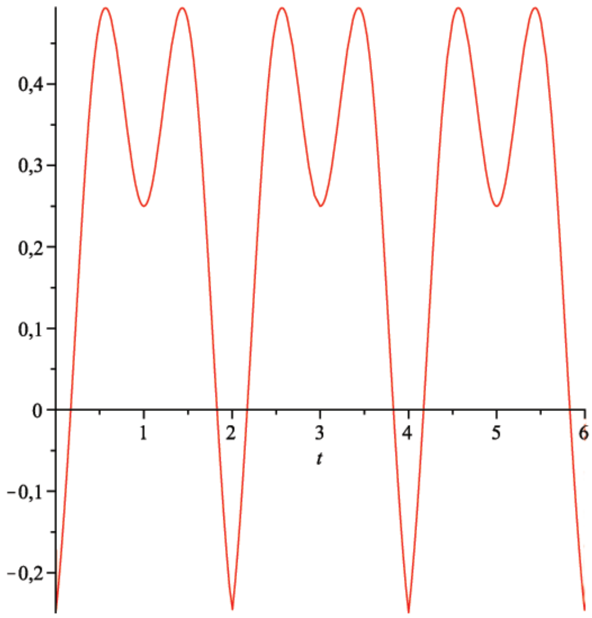

FIGURE 2. The graph of $x_\alpha(t)$ as $\alpha = 1$.

4. 3-PERIODIC SOLUTIONS

Let f, p, q be continuous 3-periodic functions and x be a 3-periodic solution of (1.2). It follows from (1.2) and 3-periodicity of $x(t)$, that

$$x''(t) + p(t)x''(t-1) = q(t)x([t]) + f(t),$$
$$x''(t+1) + p(t+1)x''(t) = q(t+1)x([t+1]) + f(t+1),$$
$$x''(t+2) + p(t+2)x''(t+1) = q(t+2)x([t+2]) + f(t+2)$$

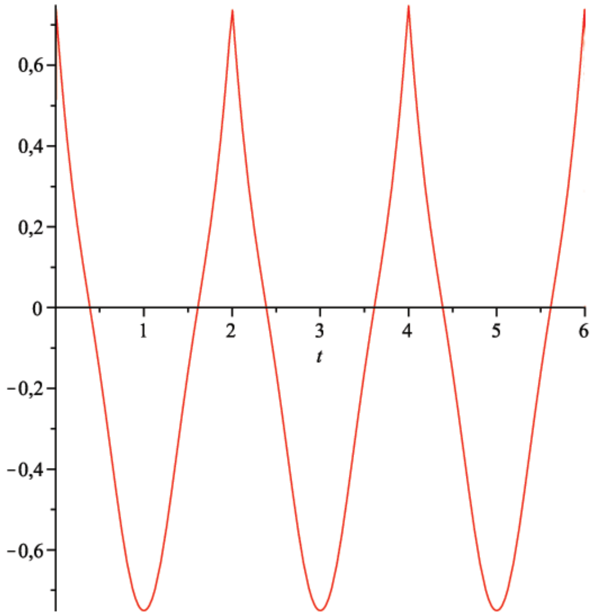

FIGURE 3. The graph of $x_\alpha(t)$ as $\alpha = -3$.

or

$$\begin{aligned} x''(t) + p(t)x''(t+2) &= q(t)x([t]) + f(t), \\ p(t+1)x''(t) + x''(t+1) &= q(t+1)x([t+1]) + f(t+1), \\ p(t+2)x''(t+1) + x''(t+2) &= q(t+2)x([t+2]) + f(t+2). \end{aligned} \tag{4.1}$$

This system of equations with respect to $x''(t), x''(t+1), x''(t+2)$ is solvable if, and only if,

$$\Delta(t) := \begin{vmatrix} 1 & 0 & p(t) \\ p(t+1) & 1 & 0 \\ 0 & p(t+2) & 1 \end{vmatrix} = 1 + p(t)p(t+1)p(t+2) \neq 0.$$

Then from (4.1) yields

$$x''(t) = \tfrac{\Delta(p,q)}{\Delta(t)}, \tag{4.2}$$

where

$$\Delta(p,q) := \begin{vmatrix} \Phi_1(t) & 0 & p(t) \\ \Phi_2(t) & 1 & 0 \\ \Phi_3(t) & p(t+2) & 1 \end{vmatrix},$$

$$\Phi_{k+1}(t) = q(t+k)x([t+k]) + f(t+k),\ k = 0, 1, 2.$$

Simple calculations give

$$
\begin{aligned}
x''(t) &= \frac{1}{\Delta(t)}\Big(q(t)x([t]) + p(t)p(t+2)q(t+1)x([t+1]) \\
&\qquad - p(t)q(t+2)x([t+2])\Big) + F(t), \\
F(t) &= \frac{p(t)p(t+2)f(t+1) - p(t)f(t+2) + f(t)}{\Delta(t)}.
\end{aligned} \tag{4.3}
$$

Integrating (4.3) we obtain

$$
\begin{aligned}
x(t) &= x(0) + x'(0)t + \int_0^t \int_0^{t_1} A(s)x([s])\,ds\,dt_1 \\
&\quad + \int_0^t \int_0^{t_1} B(s)x([s+1])\,ds\,dt_1 - \int_0^t \int_0^{t_1} C(s)x([s+2])\,ds\,dt_1 + F_3(t),
\end{aligned} \tag{4.4}
$$

where

$$
\begin{aligned}
A(s) &= \frac{q(s)}{\Delta(s)}, \quad B(s) = \frac{p(s)p(s+2)q(s+1)}{\Delta(s)}, \\
C(s) &= \frac{p(s)q(s+2)}{\Delta(s)}, \quad F_3(t) = \int_0^t \int_0^{t_1} F(s)\,ds\,dt_1.
\end{aligned}
$$

The function $x(\cdot)$ on $[0,1)$ represents as

$$
\begin{aligned}
x(t) &= x(0) + x'(0)t + \int_0^t \int_0^{t_1} A(s)\,ds\,dt_1 x(0) \\
&\quad + \int_0^t \int_0^{t_1} B(s)\,ds\,dt_1 x(1) - \int_0^t \int_0^{t_1} C(s)\,ds\,dt_1 x(2) + F_3(t)
\end{aligned}
$$

or

$$
\begin{aligned}
x(t) &= x(0) + x'(0)t + a_{00}(t)x(0) + a_{01}(t)x(1) - a_{02}(t)x(2) + F_3(t), \\
a_{00}(t) &= \int_0^t \int_0^{t_1} A(s)\,ds\,dt_1, a_{01}(t) = \int_0^t \int_0^{t_1} B(s)\,ds\,dt_1, a_{02}(t) \\
&= \int_0^t \int_0^{t_1} C(s)\,ds\,dt_1
\end{aligned} \tag{4.5}
$$

Using the properties of integral, the equation (4.4) for $t \in [1,2)$ can be written as

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \int_0^1 \int_0^{t_1} A(s)\,ds\,dt_1 x(0) + \int_1^t \int_0^1 A(s)\,ds\,dt_1 x(0) \\ &+ \int_1^t \int_1^{t_1} A(s)\,ds\,dt_1 x(1) + \int_0^1 \int_0^{t_1} B(s)\,ds\,dt_1 x(1) + \int_1^t \int_0^1 B(s)\,ds\,dt_1 x(1) \\ &+ \int_1^t \int_1^{t_1} B(s)\,ds\,dt_1 x(2) - \int_0^1 \int_0^{t_1} C(s)\,ds\,dt_1 x(2) - \int_1^t \int_0^1 C(s)\,ds\,dt_1 x(2) \\ &- \int_1^t \int_1^{t_1} C(s)\,ds\,dt_1 x(0) + F_3(t) \end{aligned}$$

or

$$x(t) = x(0) + x'(0)t + a_{10}(t)x(0) + a_{11}(t)x(1) + a_{12}(t)x(2) + F_3(t), \tag{4.6}$$

where

$$\begin{aligned} a_{10}(t) &= \int_0^1 \int_0^{t_1} A(s)\,ds\,dt_1 + \int_1^t \int_0^1 A(s)\,ds\,dt_1 - \int_1^t \int_1^{t_1} C(s)\,ds\,dt_1, \\ a_{11}(t) &= \int_0^1 \int_0^{t_1} B(s)\,ds\,dt_1 + \int_1^t \int_0^1 B(s)\,ds\,dt_1 + \int_1^t \int_1^{t_1} A(s)\,ds\,dt_1, \\ a_{12}(t) &= -\int_0^1 \int_0^{t_1} C(s)\,ds\,dt_1 - \int_1^t \int_0^1 C(s)\,ds\,dt_1 + \int_1^t \int_1^{t_1} B(s)\,ds\,dt_1. \end{aligned}$$

By similar way the equation (4.4) for $t \in [2,3)$ represents as

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \int_0^1 \int_0^{t_1} A(s)\,ds\,dt_1 x(0) + \int_1^2 \int_0^1 A(s)\,ds\,dt_1 x(0) \\ &+ \int_1^2 \int_1^{t_1} A(s)\,ds\,dt_1 x(1) + \int_2^t \int_0^1 A(s)\,ds\,dt_1 x(0) \\ &+ \int_2^t \int_1^2 A(s)\,ds\,dt_1 x(1) + \int_2^t \int_2^{t_1} A(s)\,ds\,dt_1 x(2) \\ &+ \int_0^1 \int_0^{t_1} B(s)\,ds\,dt_1 x(1) + \int_1^2 \int_0^1 B(s)\,ds\,dt_1 x(1) \\ &+ \int_1^2 \int_1^{t_1} B(s)\,ds\,dt_1 x(2) + \int_2^t \int_0^1 B(s)\,ds\,dt_1 x(1) \\ &+ \int_2^t \int_1^2 B(s)\,ds\,dt_1 x(2) + \int_2^t \int_2^{t_1} B(s)\,ds\,dt_1 x(0) \end{aligned}$$

$$- \int_0^1 \int_0^{t_1} C(s)\, ds\, dt_1 x(2) - \int_1^2 \int_0^1 C(s)\, ds\, dt_1 x(2)$$
$$- \int_1^2 \int_1^{t_1} C(s)\, ds\, dt_1 x(0) - \int_2^t \int_0^1 C(s)\, ds\, dt_1 x(2)$$
$$- \int_2^t \int_1^2 C(s)\, ds\, dt_1 x(0) - \int_2^t \int_2^{t_1} C(s)\, ds\, dt_1 x(1) + F_3(t).$$

(4.7) $$x(t) = x(0) + x'(0)t + a_{20}(t)x(0) + a_{21}(t)x(1) + a_{22}(t)x(2) + F_3(t),$$

where

$$a_{20}(t) = \int_0^1 \int_0^{t_1} A(s)\, ds\, dt_1 + \int_1^2 \int_0^1 A(s)\, ds\, dt_1 + \int_2^t \int_0^1 A(s)\, ds\, dt_1$$
$$+ \int_2^t \int_2^{t_1} B(s)\, ds\, dt_1 - \int_1^2 \int_1^{t_1} C(s)\, ds\, dt_1 - \int_2^t \int_1^2 C(s)\, ds\, dt_1,$$
$$a_{21}(t) = \int_1^2 \int_1^{t_1} A(s)\, ds\, dt_1 + \int_2^t \int_1^2 A(s)\, ds\, dt_1 + \int_0^1 \int_0^{t_1} B(s)\, ds\, dt_1$$
$$+ \int_1^2 \int_0^1 B(s)\, ds\, dt_1 + \int_2^t \int_0^1 B(s)\, ds\, dt_1 - \int_2^t \int_2^{t_1} C(s)\, ds\, dt_1,$$
$$a_{22}(t) = \int_2^t \int_2^{t_1} A(s)\, ds\, dt_1 + \int_1^2 \int_1^{t_1} B(s)\, ds\, dt_1 + \int_2^t \int_1^2 B(s)\, ds\, dt_1$$
$$- \int_0^1 \int_0^{t_1} C(s)\, ds\, dt_1 - \int_1^2 \int_0^1 C(s)\, ds\, dt_1 - \int_2^t \int_0^1 C(s)\, ds\, dt_1.$$

Note that (4.7) we can find $x'(3)$ as

(4.8) $$x'(t) = x'(0) + a'_{20}(t)x(0) + a'_{21}(t)x(1) + a'_{22}(t)x(2) + F'_3(t), \quad for \quad t \in (2, 3),$$

where

$$a'_{20}(t) = \int_0^1 A(s)ds + \int_2^t B(s)ds - \int_1^2 C(s)ds,$$
$$a'_{21}(t) = \int_1^2 A(s)dsd + \int_0^1 B(s)ds - \int_2^t C(s)ds,$$
$$a'_{22}(t) = \int_2^t A(s)ds + \int_1^2 B(s)ds - \int_0^1 C(s)ds.$$

The right hand sides of the equations (4.5-4.8) contain unknown numbers $x(0), x(1), x(2)$ and $x'(0)$, only. Using continuity and periodicity conditions $x(\cdot)$

and $x'(\cdot)$ from (4.5-4.8) we can get a system of linear equations with respect to $x(0), x(1), x(2)$ and $x'(0)$ written as

$$\left\{\begin{aligned} (1+a_{00}(1))x(0)+(-1+a_{01}(1))x(1)-a_{02}(1)x(2)+x'(0)&=-F_3(1)\\ (1+a_{10}(2))x(0)+a_{11}(2)x(1)+(-1+a_{12}(2))x(2)+2x'(0)&=-F_3(2)\\ a_{20}(3)x(0)+a_{21}(3)x(1)+a_{22}(3)x(2)+3x'(0)&=-F_3(3)\\ a_{30}x(0)+a_{31}x(1)+a_{32}x(2)&=-F_3'(3), \end{aligned}\right. \tag{4.9}$$

where

$$a_{30}=a_{20}'(3), \quad a_{31}=a_{21}'(3), \quad a_{32}=a_{22}'(3).$$

We denote by $D_3(p,q)$ a determinant of the matrix $M_3(p,q)$, where

$$M_3(p,q)=\begin{pmatrix} 1+a_{00}(1) & -1+a_{01}(1) & -a_{02}(1) & 1\\ 1+a_{10}(2) & a_{11}(2) & -1+a_{12}(2) & 2\\ a_{20}(3) & a_{21}(3) & a_{22}(3) & 3\\ a_{30} & a_{31} & a_{32} & 0 \end{pmatrix}$$

Now we are able to describe an existence conditions of the 3-periodic solutions of (1.2).

(4.10)

$$\begin{aligned} x(t)&=x(0)+x'(0)t+a_{00}(t)x(0)+a_{01}(t)x(1)-a_{02}(t)x(2)+F_3(t), \quad t\in[0,1)\\ x(t)&=x(0)+x'(0)t+a_{10}(t)x(0)+a_{11}(t)x(1)+a_{12}(t)x(2)+F_3(t), \quad t\in[1,2)\ .\\ x(t)&=x(0)+x'(0)t+a_{20}x(0)+a_{21}x(1)+a_{22}x(2)+F_3(t), \quad t\in[2,3] \end{aligned}$$

Theorem 4.1. *Let f be a 3-periodic continuous function and $p(t)p(t+1)p(t+2)\neq -1$ for $t\in[0,3]$. Then*

(i) Equation (1.2) has a unique 3-periodic solution x if and only if $D_3(p,q)\neq 0$. The 3-periodic solution x has the form (4.10), where $(x(0),x(1),x(2),x'(0))$ is the solution of (4.9).

(ii) If $D_3(p,q)=0$ and $\mathbf{F}=(F_3(1),F_3(2),F_3(3),F_3'(3))=(0,0,0,0)$, then equation (1.2) has an infinite number of 3-periodic solutions having the form

(4.11)

$$\begin{aligned} x(t)&=\alpha\Big(x(0)+x'(0)t+a_{00}(t)x(0)+a_{01}(t)x(1)-a_{02}(t)x(2)\Big)+F_3(t), \quad t\in[0,1)\\ x(t)&=\alpha\Big(x(0)+x'(0)t+a_{10}(t)x(0)+a_{11}(t)x(1)+a_{12}(t)x(2)\Big)+F_3(t), \quad t\in[1,2)\ .\\ x(t)&=\alpha\Big(x(0)+x'(0)t+a_{20}x(0)+a_{21}x(1)+a_{22}x(2)\Big)+F_3(t), \quad t\in[2,3] \end{aligned}$$

where $(x(0),x(1),x(2),x'(0))$ is an eigenvector of $M_3(p,q)$ corresponding to 0, α is any number.

(iii) If $D_3(p,q) = 0$ *and rank*$\mathbf{M}_3(p,q) <$ *rank*$\Big(\mathbf{M}_3|\mathbf{F}^t\Big)$*, where* $\mathbf{F} = (F_2(1), F_2(2), F_2'(2))$*, then equation (1.2) has no 3-periodic solution.*

The proof of this theorem is similar to the proof of Theorem 2.1.

Example 3. *Let* $p(t) = 1, \quad q(t) = sin(\frac{2\pi}{3}t), \quad f(t) = cos(\frac{2\pi}{3}t)$. *For this case*

$$D_3(p,q) = \frac{243\sqrt{3}}{32\pi^3}$$

Then the linear system (4.9) has a solution $(x(0), x(1), x(2), x'(0))$,

$$x(0) = -\frac{1}{16}\left(\frac{3(\sqrt{3}+4\pi^2-\pi)\sqrt{3}}{\pi^2}\right),$$

$$x(1) = \frac{1}{32}\left(\frac{\sqrt{3}(-8\pi^2+2\pi+3\sqrt{3})}{\pi^2}\right),$$

$$x(2) = -\frac{1}{16}\frac{(3\sqrt{3}+4\pi^2-\pi)\sqrt{3}}{\pi^2},$$

$$x(0)' = \frac{3}{256}\left(\frac{(16\pi^3-4\pi^2+15\sqrt{3}\pi-27)\sqrt{3}}{\pi^4}\right).$$

The 3*-periodic solution of the equation is*

$$x(t) = \frac{9}{32}\frac{sin(\frac{2}{3}\pi t)}{\pi^2}\left(\frac{63}{16\pi^2} - \frac{\sqrt{3}}{4\pi} + \sqrt{3}\right) + \frac{9}{32}\frac{cos(\frac{2}{3}\pi t)}{\pi^2}\left(\frac{9\sqrt{3}}{16\pi^2} - \frac{3}{4\pi} - 1\right)$$
$$- \frac{27t}{128\pi^3}\left(\frac{9\sqrt{3}}{2\pi} + 1\right) - \frac{81\sqrt{3}}{512\pi^4} + \frac{27}{128\pi^3} - \frac{9}{32\pi^2} - \frac{\sqrt{3}}{16\pi} - \frac{\sqrt{3}}{4}, \quad t \in [0,1),$$

$$x(t) = \frac{9}{32}\frac{sin(\frac{2}{3}\pi t)}{\pi^2}\left(-\frac{9}{8\pi^2} - \frac{\sqrt{3}}{4\pi} + \sqrt{3}\right) + \frac{9}{32}\frac{cos(\frac{2}{3}\pi t)}{\pi^2}\left(\frac{9\sqrt{3}}{4\pi^2} - \frac{3}{4\pi} - 1\right)$$
$$- \frac{27t}{128\pi^3}\left(\frac{3\sqrt{3}}{2\pi} + 1\right) + \frac{405\sqrt{3}}{512\pi^4} + \frac{27}{128\pi^3} - \frac{9}{32\pi^2} + \frac{\sqrt{3}}{16\pi} - \frac{\sqrt{3}}{4}, \quad t \in [1,2),$$

$$x(t) = \frac{9}{32}\frac{sin(\frac{2}{3}\pi t)}{\pi^2}\left(\frac{9}{16\pi^2} - \frac{\sqrt{3}}{4\pi} + \sqrt{3}\right) + \frac{9}{32}\frac{cos(\frac{2}{3}\pi t)}{\pi^2}\left(\frac{9\sqrt{3}}{16\pi^2} - \frac{3}{4\pi} - 1\right)$$
$$+ \frac{27t}{64\pi^3}\left(-\frac{3\sqrt{3}}{4\pi} + 1\right) + \frac{405\sqrt{3}}{512\pi^4} - \frac{135}{128\pi^3} - \frac{9}{32\pi^2} + \frac{\sqrt{3}}{16\pi} - \frac{\sqrt{3}}{4}, \quad t \in [2,3].$$

The graph of this solution is shown in the Figure 4.

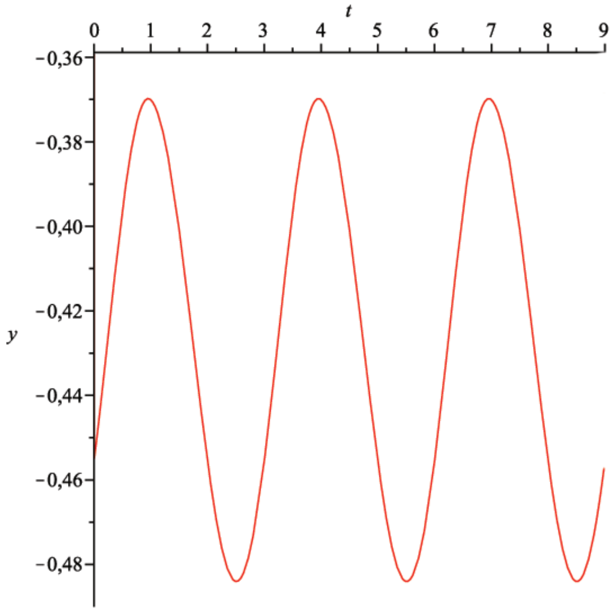

FIGURE 4. The graph of $x(t)$.

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