

## STABILITY AND BOUNDEDNESS BEHAVIOUR OF SOLUTIONS OF A CERTAIN SECOND ORDER NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In this paper, we give sufficient conditions for the stability and ultimate boundedness of solutions to a certain second order non-autonomous differential equations with damped and forced functions. Our results improve and extend some of the stability and boundedness results in the literature which themselves are extensions of some results cited therein. We give example to illustrate the result obtained.

### 1. INTRODUCTION

In [4], the problem of stability of solutions for second order non-autonomous differential equations of the form

$$(r(t)x')' + \phi(t, x, x')x' + p(t)f(x) = 0$$

was studied. The same differential equation had been considered by [10] for the problem of boundedness of solutions where he used more than one Lyapunov functions. Furthermore, [2] studied the stability and boundedness of a related

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second order non-autonomous differential equation without damping and subsequently extended and improved their result to a damped case in [3] using a single complete Lyapunov function with some restrictions to obtain stability and boundedness results. Other results on non-autonomous nonlinear second order differential equations include [1], [6], [7], [8], [9], [11], [12] and [13].

In this paper, we consider the following second order non-autonomous differential equation of the form

$$(1.1) \quad (\alpha(t)x')' + \beta(t)f(x, x') = p(t, x, x')$$

where  $\alpha(t), \beta(t)$  are positive continuously differentiable functions,  $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$  and  $p \in C([0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $\mathbb{R}$  the real line  $-\infty < t < \infty$ . It must be noted that equation (1.1) is a special case of all the equations studied in [2], [3], [4] and [10]. Our present study is motivated by these earlier investigations. Analysis of qualitative behavior of solutions of non-autonomous nonlinear differential equations is usually complicated. This difficulty increases depending on the assumption made on the damped and forced functions and the requirement for a complete Lyapunov function. (see also [5]). Lyapunov function for non-autonomous differential equations of the form (1.1) is difficult to obtain and apply if more satisfactory results are to be obtained. The problem of such system behavior of solutions is considered in this study. By defining an explicit complete Lyapunov function, we establish sufficient conditions under which the solutions of equation (1.1) are stable and ultimately bounded. An example is included to illustrate the results obtained and provide geometric arguments on the behavior of solutions of the non-autonomous systems of differential equations. The results obtained here are different and improve the results of [2], [3], [4] and [10] and some others mentioned in the literature.

## 2. STATEMENT OF RESULT

**Theorem 2.1.** *In addition to the basic assumptions imposed on the functions  $\alpha(t), \beta(t), f$  and  $p$ , we suppose also that there exist positive constants  $\alpha_o, \beta_o, \alpha_1, \beta_1, \delta_o, c, b, L$  and  $\alpha_o < \beta_o, \delta_o > \frac{1}{2}, \beta_o > \frac{2\alpha_o}{2b-1}$  such that the following conditions are satisfied:*

- i.  $0 < \alpha_o \leq \alpha(t) \leq \alpha_1, 0 < \beta_o \leq \beta(t) \leq \beta_1, \alpha'(t) \leq \beta'(t) \leq 0$  for  $t \in I = [0, \infty)$ ;

- ii.  $\frac{f(x,0)}{x} \geq c, \frac{f(x,y)}{x} \geq \delta_o, x \neq 0; f_y(x, \theta y) \geq b;$
- iii.  $\Phi(t) = \int_{\sigma_1(t)}^{\sigma_2(t)} \left| \frac{\beta'(s)}{\beta^2(s)} \right| ds \leq \frac{1}{\beta_o^2} \int_0^t |\beta'(s)| ds \leq L < \infty,$

where  $\sigma_1(t) = \min\{x(0), x(t)\}$  and  $\sigma_2(t) = \max\{x(0), x(t)\}$ . Then, every solution of (1.1) is uniformly asymptotically stable.

For simplicity, we assume the following notation

$$\Phi(t) = \int_0^t |\xi(s)| ds, \quad \text{where } |\xi(s)| = \left| \frac{\beta'(s)}{\beta^2(s)} \right|.$$

Our proof of Theorem 2.1 rests entirely on the following lemma and the scalar function  $V = V(t, x, y)$  defined by

$$(2.1) \quad V(t, x, y) = e^{\left(\frac{\Phi(t)}{\rho}\right)} U(t, x, y),$$

where

$$U(t, x, y) = \frac{1}{2}x^2 + \frac{\alpha(t)}{\beta(t)}xy + \int_0^x f(\vartheta, 0)d\vartheta + \frac{1}{2}\frac{\alpha(t)}{\beta(t)}y^2$$

and  $\rho > 0$  is an arbitrary fixed constant which will be determined later.

**Lemma 2.1.** *Subject to the conditions of Theorem 2.1 there exist positive constants  $D_1, D_2$  depending only on  $\alpha_o, \beta_o, c, b$  such that*

$$(2.2) \quad D_1(x^2 + y^2) \leq U(t, x, y) \leq D_2(x^2 + y^2).$$

To verify (2.2) of lemma 2.1 observe first that the expression  $U$  in (2.1) may be arranged in the form

$$U(t, x, y) = \frac{1}{2} \left( x + \frac{\alpha(t)}{\beta(t)} y \right)^2 + x^2 \frac{f(x, 0)}{x} + \frac{1}{2} \frac{\alpha(t)}{\beta(t)} \left( 1 - \frac{\alpha(t)}{\beta(t)} \right) y^2,$$

using i and ii of Theorem 2.1, we have that

$$U(t, x, y) \geq \frac{1}{2} \left( x + \frac{\alpha_o}{\beta_o} y \right)^2 + cx^2 + \frac{1}{2} \frac{\alpha_o}{\beta_o} \left( 1 - \frac{\alpha_o}{\beta_o} \right) y^2.$$

So that,

$$U(t, x, y) \geq k_1(x^2 + y^2),$$

where  $k_1 = \min\{c, \frac{1}{2} \frac{\alpha_o}{\beta_o} (1 - \frac{\alpha_o}{\beta_o})\}$ .

Thus, it is evident from the terms contained in the above inequality that there exist a constant  $D_1 > 0$  small enough such that

$$U(t, x, y) \geq D_1(x^2 + y^2).$$

To prove the right side of inequality (2.2), the hypotheses i - ii and using the fact that  $2|x||y| \leq x^2 + y^2$  yields for  $U$ , term by term

$$\left| \frac{\alpha(t)}{\beta(t)} xy \right| \leq \left| \frac{\alpha(t)}{\beta(t)} \right| |x||y| \leq \frac{1}{2} \frac{\alpha_1}{\beta_1} (x^2 + y^2),$$

$$\int_0^x f(\vartheta, 0) d\vartheta \leq \frac{1}{2} cx^2.$$

It follows that

$$U(t, x, y) \leq \frac{1}{2} \left( \frac{\alpha_1}{\beta_1} + c + 1 \right) x^2 + \frac{\alpha_1}{\beta_1} y^2$$

$$\leq k_2(x^2 + y^2),$$

where  $k_2 = \max\{\frac{1}{2}(\frac{\alpha_1}{\beta_1} + c + 1), \frac{\alpha_1}{\beta_1}\}$ .

If we choose a positive constant  $D_2$ , then we have

$$U(t, x, y) \leq D_2(x^2 + y^2).$$

Thus, (2.2) of Lemma 2.1 is established when  $D_1, D_2$  are finite constants since  $e^{(\frac{\Phi(t)}{\rho})}$  is finite and non-negative in (2.1).

*Proof.* It is convenient to consider equation (1.1) in the equivalent system form

$$(2.3) \quad \begin{aligned} x' &= y, \\ y' &= -\frac{\beta(t)f(x, y)}{\alpha(t)} - \frac{\alpha'(t)}{\alpha(t)}y + \frac{p(t, x, y)}{\alpha(t)}. \end{aligned}$$

In order to prove Theorem 2.1, we consider the case where  $p = 0$  in (2.3).

Now, differentiating  $U$  in (2.1) along the system (2.3), we have

$$\begin{aligned} \frac{dU(t, x, y)}{dt} &= -y[f(x, y) - f(x, 0)] - xf(x, y) - \alpha(t) \frac{\beta'(t)}{\beta^2(t)} (xy + y^2) + xy \\ &\quad - \frac{\alpha'(t)}{\beta(t)} xy + \alpha'(t) \frac{\beta(t)}{\beta^2(t)} xy + \alpha'(t) \frac{\beta(t)}{2\beta^2(t)} y^2 + \frac{\alpha(t)}{\beta(t)} y^2. \end{aligned}$$

By the hypotheses i - ii of Theorem 2.1 and the term

$$y^2 \frac{[f(x, y) - f(x, 0)]}{y} \geq y^2 f_y(x, \theta y) \geq by^2$$

and

$$x^2 \frac{f(x, y)}{x} \geq \delta_o x^2.$$

It follows that

$$\frac{dU(t, x, y)}{dt} \leq -\delta_o x^2 - (b - \frac{\alpha_o}{\beta_o})y^2 - \alpha_o |\xi(t)|y^2 + (1 - \alpha_o |\xi(t)|)|xy|.$$

Using the fact that  $|xy| \leq \frac{1}{2}(x^2 + y^2)$ , we obtain

$$\frac{dU(t, x, y)}{dt} \leq -(\delta_o - \frac{1}{2})x^2 - (b - \frac{\alpha_o}{\beta_o} - \frac{1}{2})y^2 - \frac{\alpha_o}{2}|\xi(t)|(x^2 + 3y^2)$$

$$\frac{dU(t, x, y)}{dt} \leq -\delta_1(x^2 + y^2) - \delta_2|\xi(t)|(x^2 + y^2),$$

where  $\delta_1 = \min\{\delta_o - \frac{1}{2}, (b - \frac{\alpha_o}{\beta_o} - \frac{1}{2})\}$  and  $\delta_2 = \min\{\frac{\alpha_o}{2}, \frac{3\alpha_o}{2}\}$ . Since

$$V(t, x, y) = e^{(\frac{\Phi(t)}{\rho})}U(t, x, y) \quad \text{in (2.1),}$$

differentiating  $V$  in (2.1) and putting  $\rho = \frac{D_1}{\delta_2}$ , we have

$$\frac{dV(t, x, y)}{dt} = e^{(\frac{\delta_2\Phi(t)}{D_1})}\frac{dU(t, x, y)}{dt} + \frac{\delta_2|\xi(t)|}{D_1}e^{(\frac{\delta_2\Phi(t)}{D_1})}U$$

$$\frac{dV(t, x, y)}{dt} = e^{(\frac{\delta_2\Phi(t)}{D_1})}\left[\frac{dU(t, x, y)}{dt} + \frac{\delta_2|\xi(t)|}{D_1}U\right]$$

It follows that

$$\frac{dV(t, x, y)}{dt} \leq e^{(\frac{\delta_2\Phi(t)}{D_1})}\left[-\delta_1(x^2 + y^2) - \delta_2|\xi(t)|(x^2 + y^2) + \frac{\delta_2|\xi(t)|}{D_1}U\right].$$

In view of inequality (2.2), we have that

$$\begin{aligned} \frac{dV(t, x, y)}{dt} &\leq -\delta_1 e^{(\frac{\delta_2\Phi(t)}{D_1})}(x^2 + y^2) \\ (2.4) \quad \frac{dV(t, x, y)}{dt} &\leq -\delta_3(x^2 + y^2), \end{aligned}$$

where  $\delta_3 = \delta_1 e^{(\frac{\delta_2 L}{D_1})} > 0$ .

Thus, in view of (2.2) and (2.4), it shows that the zero solutions of equation (1.1) are uniformly asymptotically stable. □

**Theorem 2.2.** *Let all the conditions of Theorem 2.1 be satisfied and in addition we assume that there exist a finite constant  $A > 0$  such that  $p$  satisfies*

$$(2.5) \quad |p(t, x, y)| \leq A;$$

uniformly for all  $x, y$  in  $\mathbb{R}$ . Then, there exist a constant  $D_3 > 0$  such that any solutions  $(x(t), y(t))$  of system (2.3) uniformly ultimately satisfies

$$|x(t)| \leq D_3, \quad |y(t)| \leq D_3,$$

for all sufficiently large  $t$ , where the magnitude of  $D_3$  depends only on  $\alpha_o, \beta_o, c, b, \delta_o, L$  and  $A$ .

*Proof.* As in Theorem 2.1, the proof of Theorem 2.2 depends on the scalar differentiable Lyapunov function  $V(t, x, y)$  defined in (2.1).

Now, we consider the case  $p \neq 0$  in (2.1) and since  $V'_{(2.3)} \leq 0$  in (2.4) for all  $t, x, y$ , we have that

$$\frac{dV(t, x, y)}{dt} \leq -\delta_3(x^2 + y^2) + \frac{(|x| + |y|)}{\beta_o} |p(t, x, y)|.$$

By (2.5) of Theorem 2.2, we get

$$\frac{dV(t, x, y)}{dt} \leq -\delta_3(x^2 + y^2) + \beta_o^{-1}(|x| + |y|)A$$

Using the fact that  $2|x||y| \leq x^2 + y^2$ , we have

$$\frac{dV(t, x, y)}{dt} \leq -\delta_3(x^2 + y^2) + \sqrt{2}A\beta_o^{-1}(x^2 + y^2)^{\frac{1}{2}}$$

$$(2.6) \quad \frac{dV(t, x, y)}{dt} \leq -\delta_3(x^2 + y^2) + \delta_4(x^2 + y^2)^{\frac{1}{2}},$$

where  $\delta_4 = \sqrt{2}A\beta_o^{-1}$ .

If we choose

$$(x^2 + y^2)^{\frac{1}{2}} \geq \delta_5 = \delta_3^{-1}\delta_4,$$

the inequality (2.6) implies that

$$\frac{dV(t, x, y)}{dt} \leq -\delta_3(x^2 + y^2).$$

Then, there exist a  $\delta_6$  such that

$$\frac{dV(t, x, y)}{dt} \leq -\delta_6 \quad \text{provided} \quad (x^2 + y^2) \geq \delta_6\delta_3^{-1}.$$

This completes the proof. □

## 3. EXAMPLE

Consider equation (1.1) in the form

$$(3.1) \quad \left[ \left( \frac{1}{(1+4t^2)} + \frac{1}{2} \right) x' \right]' + \left( \frac{2}{(2+t^2)} + 2 \right) \left( x + \frac{x}{1+x^2} + x' + x'^2 \right) = \frac{1}{1+t^2+x^2+x'^2}$$

Or its equivalent system

$$(3.2) \quad \begin{aligned} x' &= y \\ y' &= -\frac{4(1+4t^2)[1+(2+t^2)^2]}{(2+t^2)^2(3+4t^2)} \left( x + \frac{x}{1+x^2} + y + y^2 \right) - \frac{16t}{(1+4t^2)(3+4t^2)} y \\ &\quad + \frac{2(1+4t^2)}{(3+4t^2)(1+t^2+x^2+y^2)}. \end{aligned}$$

With the earlier notations, it is easy to see that

$$\frac{1}{2} \leq \alpha(t) = \left( \frac{1}{(1+4t^2)} + \frac{1}{2} \right) \leq \frac{3}{2},$$

$$2 \leq \beta(t) = \left( \frac{2}{(2+t^2)} + 2 \right) \leq 3,$$

clearly  $\beta_o > \alpha_o$  and  $\beta'(t) \leq \alpha'(t) \leq 0, \forall t \geq 0$ .

$$f(x, y) = \left( x + \frac{x}{1+x^2} + y + y^2 \right)$$

$$\frac{f(x, 0)}{x} = \left( 1 + \frac{1}{1+x^2} \right) \geq 1 = c$$

$$\frac{f(x, y)}{x} = \left( 1 + \frac{1}{1+x^2} + \frac{y}{x} + \frac{y^2}{x} \right) \geq 1 = \delta_o, \quad x \neq 0$$

$$\frac{f(x, y) - f(x, 0)}{y} = y + 1 \geq 1 = b, \quad y \neq 0.$$

Also,  $\delta_o > \frac{1}{2}$  and  $\beta_o > \frac{2\alpha_o}{2b-1}$ .

Since

$$\Phi(t) = \int_{\sigma_1(t)}^{\sigma_2(t)} \frac{|\beta'(s)|}{\beta^2(s)} ds \leq \frac{1}{\beta_o^2} \int_0^t |\beta'(s)| ds \leq L,$$

$$\int_0^t |\beta'(s)| ds = \int_0^t \frac{4s}{(2+s^2)^2} ds = \frac{t^2}{2+t^2} \leq \frac{1}{3}, \quad t > 0.$$

Finally,

$$p(t, x, y) = \frac{1}{1+t^2+x^2+y^2},$$

$$|p(t, x, y)| \leq \frac{1}{1+t^2+x^2+y^2} \leq \frac{1}{1+t^2}$$

$$|p(t, x, y)| \leq \frac{1}{1+t^2} \leq 2, \quad t \geq 0.$$

Hence, this shows that all the conditions of Theorem 2.1 and Theorem 2.2 are satisfied. Thus, we conclude that all the solutions of (3.1) or (3.2) equivalently are uniformly asymptotically stable and ultimately bounded.

#### 4. STABILITY AND BOUNDEDNESS ANALYSIS OF SOLUTIONS OF NON-AUTONOMOUS SYSTEM (3.2)

- (1) In Figure 1, is a graph showing a linear combination  $\theta(t) = C_1x(t) + C_2y(t)$ , ( $C_1, C_2$  are any constants) of the solutions of (3.2) satisfying all the conditions of Theorem 2.1 for  $p = 0$  tends to almost zero as  $t \geq 0$  while in Figure 2, the solutions of (3.2) satisfying all the conditions of Theorem 2.1 for  $p = 0$  tends to zero as  $t \rightarrow \infty$  and  $x(t)$ ,  $y(t)$  are asymptotically stable as  $t \rightarrow \infty$ .
- (2) In Figure 3 and Figure 4, the phase-plane shows that under certain conditions in Theorem 2.1 on the damped, forced as well as the nonlinear function  $f$  the solutions of equation (3.2) are asymptotically stable as the the solution paths converge to  $(0,0)$  as  $t \rightarrow \infty$ . This clearly shows that the zero solutions of (3.1) or (3.2) equivalently are asymptotically stable as  $t \rightarrow \infty$ .
- (3) In Figure 5 and Figure 6, the ultimate boundedness behavior of solutions  $x(t)$  in (green) and  $y(t)$  in (blue) respectively in equation (3.2) for  $p \neq 0$  where  $x(t), y(t)$  are bounded by a single constant for all  $t \geq 0$ .



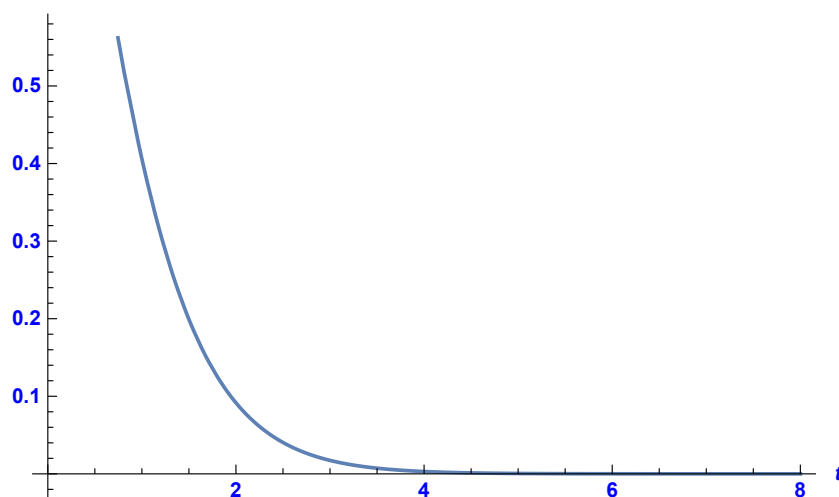


FIGURE 1. The graph of a linear combination  $\theta(t) = C_1x(t) + C_2y(t)$ , ( $C_1, C_2$  are any constants) of the solutions of (3.2) satisfying all the conditions of Theorem 2.1 for  $p = 0$  tends to zero as  $t \geq 0$ .

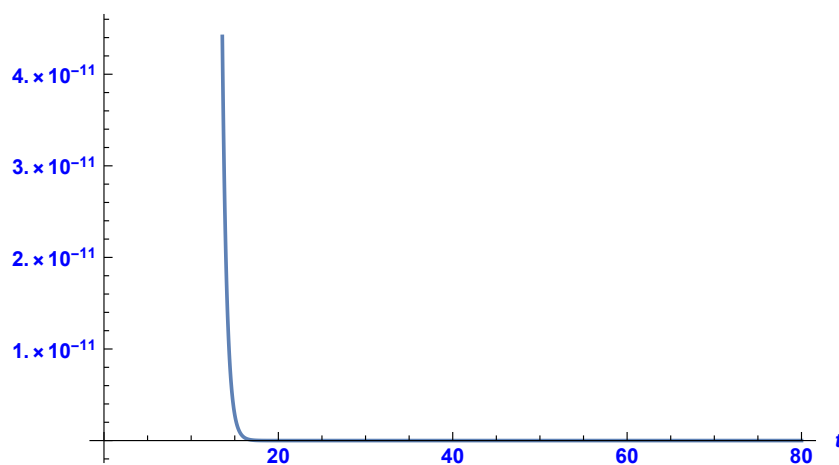


FIGURE 2. The graph of a linear combination  $\theta(t) = C_1x(t) + C_2y(t)$ , ( $C_1, C_2$  are any constants) of the solutions of (3.2) satisfying all the conditions of Theorem 2.1 for  $p = 0$  tends to zero as  $t \rightarrow \infty$ .

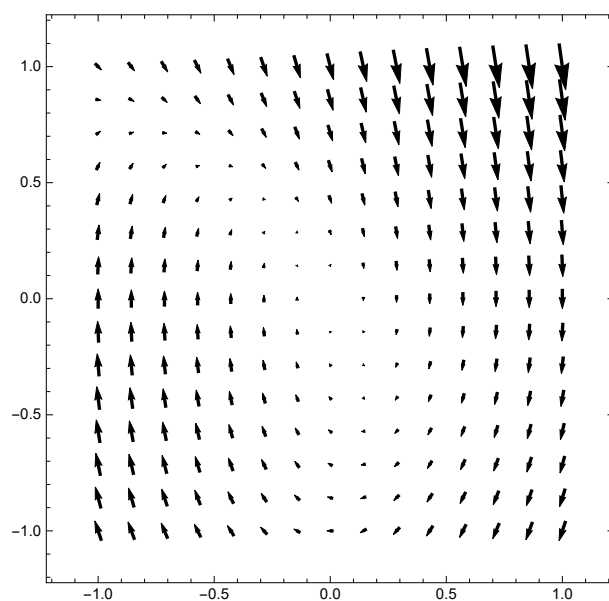


FIGURE 3. The plot showing all the solution paths satisfying the conditions of Theorem 2.1 for  $p = 0$  in (3.2) converge to  $(0, 0)$ .

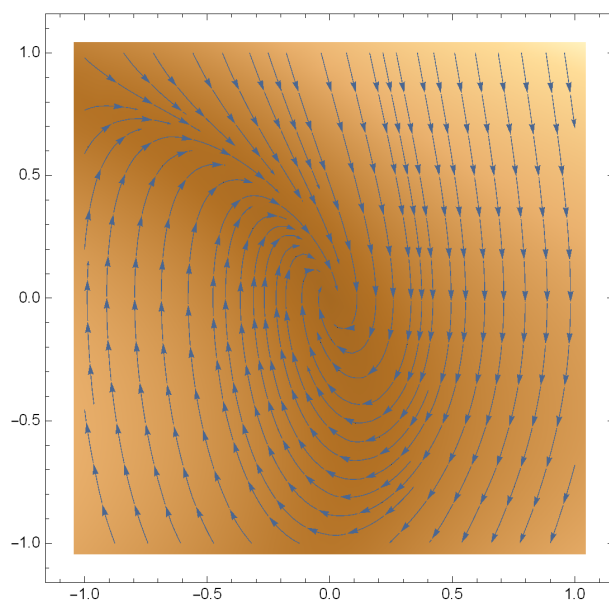


FIGURE 4. Visualizing how the solution paths satisfying the conditions of Theorem 2.1 for  $p = 0$  in (3.2) converge to  $(0, 0)$ .

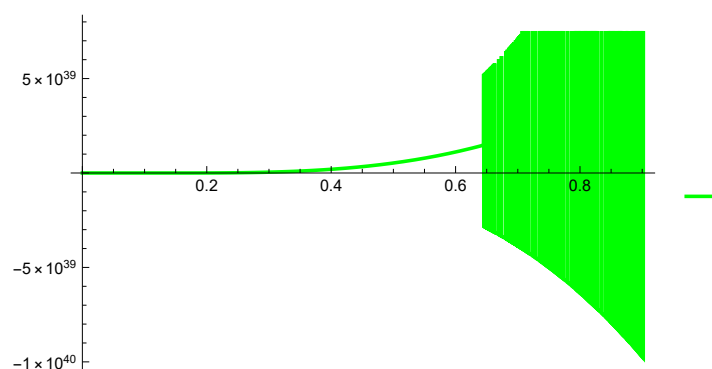


FIGURE 5. The graph of solution  $x(t)$  of (3.2) satisfying the conditions of Theorem 2.1 and Theorem 2.2 for  $p \neq 0$  is ultimately bounded by a single constant as  $t \rightarrow \infty$ .

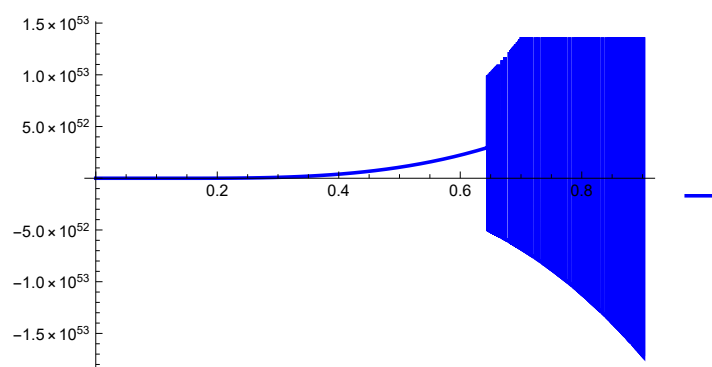


FIGURE 6. The graph of solution  $y(t)$  of (3.2) satisfying the conditions of Theorem 2.1 and Theorem 2.2 for  $p \neq 0$  is ultimately bounded by a single constant as  $t \rightarrow \infty$ .

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