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COMBINED METHOD OF INTEGRAL TRANSFORMS FOR THE SPHERICALLY SYMMETRIC DROPLET HEATING PROBLEM

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ABSTRACT. In this paper, we analysed the spherically symmetric heat diffusion equation, which governs the temperature distribution inside a heated but non-evaporating droplet. The spherical droplet, with an initial uniform temperature, is assumed at rest in an unsteady gas environment. The classical Fourier sine integral transform (FSIT) and the unilateral Laplace integral transform (LIT) are successively used to solve the resulting initial-boundary value problem, first reduced in a dimensionless form. An explicit solution in the Laplace domain is obtained for the temperature inside the droplet. Then, depending on the time-varying temperature of the gas environment at the immediate vicinity of the droplet, an exact series solution and an approximate analytical solution in short time limits are derived for the droplet internal temperature. In the case of steady gas environment at constant temperature, the standard series solution obtained in the literature for the symmetrical problem of heating or cooling of a solid spherical body, is recovered. The results may be useful for time step analysis in droplets and sprays vaporization models.

1. INTRODUCTION

The goal of this paper consists in solving, by use of two classical integral transforms, the initial boundary value problem for the heating without evaporation of

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a spherical droplet of radius R_s. Assuming the spherical symmetry of the problem, the droplet internal temperature $T_l(R,t)$ to be determined is a function of radial coordinate R and time t, and the heat diffusion equation in spherical coordinates inside the droplet reads:

$$\frac{\partial T_l}{\partial t} - \alpha_l \left(\frac{\partial^2 T_l}{\partial R^2} + \frac{2}{R}\frac{\partial T_l}{\partial R} \right) = 0, \quad 0 \leq R < R_s, \quad t > 0. \tag{1.1}$$

Equation (1.1) can be equally written as:

$$\frac{\partial T_l}{\partial t} - \frac{\alpha_l}{R}\frac{\partial^2 (RT_l)}{\partial R^2} = 0, \quad 0 \leq R < R_s, \quad t > 0, \tag{1.2}$$

where the thermal diffusivity of the liquid is introduced as $\alpha_l = k_l/\rho_l c_l$. For brevity, the specific heat capacity c_l, the thermal conductivity k_l and the density ρ_l of the liquid are assumed to be constant as well as the corresponding thermophysical properties c_g, k_g and ρ_g of the gas at the immediate vicinity of the droplet. The initial and boundary conditions are written as:

$$T_l(R, t=0) = \overline{T}_0, \tag{1.3}$$

$$\frac{\partial T_l}{\partial R}(R=0, t) = 0, \tag{1.4}$$

$$\frac{\partial T_l}{\partial R}(R=R_s,\, t) = Q_s(t) = \frac{h}{k_l}(T_g(t) - T_s(t)), \tag{1.5}$$

where $\overline{T}_0$ is the initial temperature of the droplet, $T_g(t)$ is the time-evolving temperature of the surrounding gas at the immediate vicinity of the droplet, $Q_s(t)$ is the temperature gradient at the droplet surface, which is connected to the surface temperature $T_s(t)$ through the convection heat transfer coefficient $h = k_g/R_s$. Equation (1.5) specifies the energy balance condition at the droplet surface. The main assumptions for the problem are that the droplet remained spherical during the heating/cooling process and its surface temperature is uniform even if, it can vary with time [1, 2]. This latter assumption permits to separate the analysis of gas and liquid phases and then to match the solutions at the droplet surface. Engineering applications are particularly interested in the values of droplets surface temperature, which determine the rate of evaporation and break-up of droplets [2].

From the problem (1.2)-(1.5), equations with non-dimensional variables can be obtained using the following relations:

(1.6)
$$r = \frac{R}{R_s},\ \tau = \frac{t}{R_s^2/\alpha_l},\ \theta_l = \frac{c_l}{\ell}(T_l - \overline{T}_0),\ \theta_s = \frac{c_l}{\ell}(T_s - \overline{T}_0),\ \theta_g = \frac{c_l}{\ell}(T_g - \overline{T}_0),$$

where ℓ is the specific heat of evaporation of the droplet. The problem under consideration can then be recast as:

$$\frac{\partial(r\theta_l)}{\partial\tau} - \frac{\partial^2(r\theta_l)}{\partial r^2} = 0, \qquad 0 \le r < 1, \qquad \tau > 0, \tag{1.7}$$

with the following initial and boundary conditions:

$$\theta_l(r, \tau = 0) = 0, \tag{1.8}$$

$$\frac{\partial\theta_l}{\partial r}(r = 0, \tau) = 0, \tag{1.9}$$

$$\frac{\partial\theta_l}{\partial r}(r = 1, \tau) = K(\theta_g(\tau) - \theta_s(\tau)) = q_s(\tau), \tag{1.10}$$

where $K = k_g/k_l$. The function $\theta_l \equiv \theta_l(r, \tau)$ to be determined is the droplet inside temperature distribution at dimensionless time τ and distance r from the droplet centre. The initial condition for the function θ_l is simply brought to $\theta_l(r, \tau = 0) = 0$ and the zero temperature gradient at the droplet centre assures the spherical symmetry of the problem. The expressions of the dimensionless temperature gradient $q_s(\tau)$ and of the surface temperature $\theta_s(\tau)$ are related to each other. The resulting Robin boundary condition can be considered equivalent to the Dirichlet boundary condition $\theta_l(r = 1, \tau) = \theta_s(\tau)$. However, $q_s(\tau)$ and $\theta_s(\tau)$ are not known beforehand, and must be determined as part of the solution of equation (1.7).

Many authors have used various methods to treat the droplet heating/cooling problem. In Computational Fluid Dynamics (CFD) codes for example, the droplet transient heating and vaporization are treated as a time step analysis, where the the droplet radius is assumed fixed during the time step, but the time-evolving temperature of the surrounding gas-phase mixture can be estimated at each time step [3]. Likewise, a number of approximate analytical models formulated through physical considerations, as the power law, the polynomial approximations and the heat balance integral methods, have being used in consideration

to the important number of droplets involved in liquid fuel combustion mechanisms [4]. However, the exact analytical solution of the symmetrical problem of heating or cooling of a solid spherical body has been addressed in several books [5–8], by using the standard method of separation of variables. In the reference [5] for example, the method of separation of variables and that of the Laplace integral transform have been used for deriving exact series solutions for the spherical body heating/cooling problem with prescribed expressions of the ambient gas temperature. In the same book [5], a series of similar heating or cooling problems with spherical symmetry and source terms are solved by using finite integral transforms, while the classical Fourier sine integral applied to space coordinates has being validated only for infinite and semi-infinite solids.

In the present paper, by using the classical Fourier sine integral transform (FSIT) in combination with the Laplace integral transform (LIT), explicit solutions in the Laplace and the time domains are obtained for the spherically symmetric heat diffusion equation inside a droplet suspended in an unsteady gas environment. In the next section, some basic mathematical definitions and properties used in the paper are recalled. In sections 3 and 4, The FSIT and the LIT methods are introduced in the analysis and explicit solutions in the Laplace domain are derived for the droplet temperature field. In section 5, an exact series solution and an approximate analytical solution in short time limits are obtained for the droplet internal temperature in function of the time-varying temperature of the gas environment. In the case of steady gas environment at constant temperature, the standard series solution in the time domain for the symmetrical problem of heating or cooling of a solid spherical body is recovered (see reference [8] for example). Finally, section 6 outlines the conclusion.

2. Basic definitions and theorems

Some basic definitions and properties of the classical Fourier sine and the Laplace integral transforms which are recalled in order to be used further in this paper. We denoted the exponential function by e instead of $\exp$.

Definition 2.1. *(see [9]) For any real-valued function $\phi(x)$, absolutely integrable on $[0,+\infty[$ i.e. $\displaystyle\int_0^\infty |\phi(x)|dx < \infty$, its Fourier sine integral transform (FSIT) in*

terms of $\lambda \in [0, +\infty[$ *is:*

$$\Phi_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \phi(x) \sin(\lambda x) dx,$$

and the inversion formula reads:

$$\phi(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \Phi_s(\lambda) \sin(x\lambda) d\lambda.$$

Remark 2.1. *The integral* Φ *and its derivative converge uniformly with respect to* λ *and* Φ_s *tends to* 0 *as* λ *goes to* $+\infty$*, since it's the imaginary part of the classical Fourier integral transform for which those properties hold.*

Definition 2.2. *(see [9]) If* $f(\tau)$ *is a function defined in* $\tau \geq 0$*, then its unilateral Laplace integral transform (LIT) is given in the complex* p*-plane by:*

$$F(p) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) \mathrm{e}^{-pt} dt,$$

provided that $f(t)$ *be of exponential order, that is, there are constants* C *and* σ *so that* $|f(t)| < C\mathrm{e}^{\sigma t}$*, when t is sufficiently large. The inversion, from the Laplace domain* p *to the time domain* t *is given by the complex integral:*

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(p) \mathrm{e}^{pt} dp,$$

where $\gamma > \sigma$ *is chosen so that* $F(p)$ *converges absolutely on the real part of* p *line* $\Re(p) = \gamma$*, and* $F(p)$ *is analytic to the right of this line.*

Theorem 2.1. *(The Initial Value Theorem) If* $f(t)$ *is a function defined in* $t \geq 0$ *and* $\mathcal{L}\{f(t)\} = F(p)$ *exists, then*

$$\lim_{p \to \infty} pF(p) = \lim_{t \to 0} f(t) = f(0).$$

Proof. (see [10]). □

Theorem 2.2. *(The Convolution Theorem) Let* $f(t)$ *and* $g(t)$ *be functions defined in* $t \geq 0$*. If* $\mathcal{L}\{f(t)\} = F(p)$ *and* $\mathcal{L}\{g(t)\} = G(p)$*, then*

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(p)G(p).$$

Or, equivalently,

$$\mathcal{L}^{-1}\{F(p)G(p)\} = f(t) * g(t),$$

where $f(t) * g(t)$ is called the convolution of $f(t)$ and $g(t)$ and is defined by the integral

$$f(t) * g(t) = \int_0^t f(t-\eta)g(\eta)d\eta.$$

Proof. (see [10]). □

Theorem 2.3. *(Heaviside's Expansion Theorem) If $F(p) = \mathcal{L}\{f(t)\} = \dfrac{N(p)}{D(p)}$, where $N(p)$ and $D(p)$ are polynomials in p and the degree of $D(p)$ is higher than that of $N(p)$, then*

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \sum_{k\geq 1} \frac{N(p_k)}{D'(p_k)} \mathrm{e}^{p_k t},$$

where D' denotes the derivative of D and p_k are the distinct roots of the equation $D(p) = 0$.

Proof. (see [10]). □

Remark 2.2. *The Heaviside Expansion Theorem can be applied even if $N(p)$ or $D(p)$ are generalized polynomials with respect to p i.e. some infinite convergent power series, the exponents of which are positive integers. (see [5])*

Lemma 2.1. *(Watson's Lemma) If (i) $f(t) = O(\mathrm{e}^{at})$ as $t \to \infty$, that is, $\mid f(t) \mid \leq Kexp(at)$ for $t > T$ where K and T are constants, and (ii) $f(t)$ has the expansion*

$$f(t) = t^{\alpha}\left[\sum_{r=0}^{n} a_r t^r + R_{n+1}(t)\right] \text{ for } 0 < t < T \text{ and } \alpha > -1,$$

where $\mid R_{n+1}(t) \mid < At^{n+1}$ for $0 < t < T$ and A is a constant, then the Laplace transform $F(p) = \mathcal{L}\{f(t)\}$ has the asymptotic expansion:

$$F(p) = \sum_{r=0}^{n} a_r \frac{\Gamma(\alpha + r + 1)}{p^{\alpha+r+1}} + O\left(\frac{1}{p^{\alpha+n+2}}\right) \text{ for } p \to \infty,$$

where the gamma function $\Gamma(p)$ is defined for the real part of p greater than 0 ($\Re(p) > 0$) by:

$$\Gamma(p) = \int_0^{\infty} \mathrm{e}^{-t} t^{p-1} dt.$$

Proof. (see [10]). □

Lemma 2.2. *(Converse to Watson's Lemma) Let $f(t)$ be a continuous function in $[0,+\infty[$, $f(t)=0$ for $t<0$, and it exists $c>0$ such that $\mathrm{e}^{ct}f(t)\in L^1[0,+\infty[$ that is $\mathrm{e}^{ct}f(t)$ is absolutely integrable on $[0,+\infty[$. Let $F(p)$ be the unilateral Laplace integral transform of $f(t)$ i.e. $F(p)=\mathcal{L}\{f(t)\}$. If*

$$F(p)=\sum_{k=0}^{n}a_k\frac{\Gamma(\lambda_k)}{p^{\lambda_k}}+O\left(\frac{1}{p^{\lambda_{n+1}}}\right)\sim\sum_{n=0}^{\infty}a_n\Gamma(\lambda_n)p^{-\lambda_n}$$

as $p\to\infty$ uniformly in $\mid\arg(z-c)\mid<\frac{\pi}{2}$ where $\Re(\lambda_n)>0$ and $\Re(\lambda_n)\to\infty$ as $n\to\infty$, then as $t\to 0^+$

$$f(t)=\sum_{k=0}^{n}a_kt^{\lambda_k-1}+O(t^{\lambda_{n+1}-1})\sim\sum_{n=0}^{\infty}a_nt^{\lambda_n-1}.$$

Proof. (see [11]). □

3. The Fourier sine integral transform method

In this section, we apply the classical Fourier sine integral transform to the problem (1.7)-(1.10). We seek for a solution $\theta_l=\theta_l(r,\tau)$ in the form of a continuously twice differentiable function in the domain $\tau>0$ and $r\in[0,1]$. The surrounding gas temperature $\theta_g(\tau)$ is assumed to be bounded and continuous with time.

Lemma 3.1. *Assume that $\theta_l=\theta_l(r,\tau)$ be a solution for the problem (1.7)-(1.10). The Fourier Sine Integral Transform (FSIT) $V_s(\lambda,\tau)$ of the solution $\theta_l=\theta_l(r,\tau)$, defined as:*

$$V_s(\lambda,\tau)=\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\theta_l\sin(\lambda r)dr=\sqrt{\frac{2}{\pi}}\int_0^{1}\theta_l\sin(\lambda r)dr$$

is solution of the differential equation:

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\tau}\frac{\partial V_s(\lambda,\tau)}{\partial\lambda}+\lambda^2\frac{\partial V_s(\lambda,\tau)}{\partial\lambda}\right)\\&=\sqrt{\frac{2}{\pi}}(\lambda\theta_s(\tau)\cos\lambda-q_s(\tau)\sin\lambda),\qquad\lambda\geq 0,\qquad\tau>0.\end{aligned}\tag{3.1}$$

Proof. We are interested in the dimensionless temperature $\theta_l(r,\tau)$ inside the droplet. Thus, θ_l can be considered null outside the interval $[0,1]$ without lost

of generality. Assuming now the existence of a solution for the problem (1.7)-(1.10), θ_l should be continuously differentiable for $0 \leq r \leq 1$ and for $\tau > 0$ according to the same equations (1.7)-(1.10). So, the functions θ_l, $\frac{\partial \theta_l}{\partial r}$, $\frac{\partial^2 \theta_l}{\partial r^2}$ are absolutely integrable in respect to r on $[0,1] \subset [0,+\infty[$ and the FSIT of the temperature function $\theta_l \equiv \theta_l(r,\tau)$ is written as:

$$V_s(\lambda,\tau) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \theta_l \sin(\lambda r) dr = \sqrt{\frac{2}{\pi}} \int_0^1 \theta_l \sin(\lambda r) dr.$$

Likewise, the terms mentioned in the dimensionless equation (1.7) are absolutely integrable in respect to r on $[0,+\infty[$. The FSIT will now be applied to these terms. For convenience, equation (1.7) is first multiplied by r and reads:

$$\frac{r^2 \partial \theta_l}{\partial \tau} - r \frac{\partial^2 (r\theta_l)}{\partial r^2} = 0. \tag{3.2}$$

Applying the FSIT (denoted by F_s) to the first term of equation (3.2), we have:

$$A(\lambda) = F_s \left[\frac{r^2 \partial \theta_l}{\partial \tau} \right] = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial \tau} \int_0^1 \theta_l r^2 \sin(\lambda r) dr,$$

which, as $\frac{\partial^2}{\partial \lambda^2} \sin(\lambda r) = -r^2 \sin(\lambda r)$, can be equally written as:

$$A(\lambda) = -\sqrt{\frac{2}{\pi}} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \lambda} \int_0^1 r\theta_l \cos(\lambda r) dr = -\frac{\partial}{\partial \tau} \frac{\partial^2}{\partial \lambda^2} V_s(\lambda,\tau). \tag{3.3}$$

The FSIT is also applied to the diffusion term of equation (3.2) and gives:

$$C(\lambda) = F_s \left[-r \frac{\partial^2 (r\theta_l)}{\partial r^2} \right] = -\sqrt{\frac{2}{\pi}} \int_0^1 r \frac{\partial^2 (r\theta_l)}{\partial r^2} \sin(\lambda r) dr. \tag{3.4}$$

In order to perform a first integration by parts, we write:

$$u_1(r) = r \sin(\lambda r) \Rightarrow u_1'(r) = \sin(\lambda r) + r\lambda \cos(\lambda r)$$

and

$$v_1'(r) = \frac{\partial^2 (r\theta_l)}{\partial r^2} \Rightarrow v_1(r) = \frac{\partial (r\theta_l)}{\partial r}.$$

Equation (3.4) then becomes:

$$\begin{aligned} -\sqrt{\frac{\pi}{2}} C(\lambda) &= \int_0^1 r \frac{\partial^2 (r\theta_l)}{\partial r^2} \sin(\lambda r) dr = [u_1 v_1]_0^1 - \int_0^1 u_1' v_1 dr \\ &= \left[r \sin(\lambda r) \frac{\partial (r\theta_l)}{\partial r} \right]_0^1 - \int_0^1 \frac{\partial (r\theta_l)}{\partial r} \sin(\lambda r) dr \\ &- \int_0^1 \frac{\partial (r\theta_l)}{\partial r} r\lambda \cos(\lambda r) dr = C_1 - C_2 - C_3. \end{aligned} \tag{3.5}$$

The term C_1 in equation (3.5) is calculated as:

$$C_1 = \left[r\sin(\lambda r)\frac{\partial(r\theta_l)}{\partial r} \right]_0^1 = \left[r\sin(\lambda r)(\theta_l + r\frac{\partial \theta_l}{\partial r}) \right]_0^1 \\ = \sin(\lambda)\theta_s(\tau) + q_s(\tau)\sin(\lambda). \tag{3.6}$$

The term C_2 in equation (3.5) is calculated by integrating by parts as follows:

$$u_2' = \frac{\partial(r\theta_l)}{\partial r} \Rightarrow u_2 = r\theta_l; \quad v_2 = \sin(\lambda r) \Rightarrow v_2' = \lambda\cos(\lambda r)$$

and:

$$C_2 = \int_0^1 \frac{\partial(r\theta_l)}{\partial r}\sin(\lambda r)dr = [u_2 v_2]_0^1 - \int_0^1 u_2 v_2' dr \\ = [r\theta_l \sin(\lambda r)]_0^1 - \int_0^1 r\theta_l \lambda\cos(\lambda r)dr \\ = \sin(\lambda)\theta_s(\tau) - \int_0^1 r\theta_l\lambda\cos(\lambda r)dr \\ = \sin(\lambda)\theta_s(\tau) - \sqrt{\frac{\pi}{2}}\lambda\frac{\partial V_s(\lambda,\tau)}{\partial\lambda}. \tag{3.7}$$

The third term C_3 in equation (3.5) can also be calculated by integration by parts:

$$u_3 = r\cos(\lambda r) \Rightarrow u_3' = \cos(\lambda r) - \lambda r\sin(\lambda r);$$

$$v_3' = \frac{\partial(r\theta_l)}{\partial r} \Rightarrow v_3 = r\theta_l$$

and

$$C_3 = \int_0^1 \frac{\partial(r\theta_l)}{\partial r} r\lambda\cos(\lambda r)dr = \lambda\int_0^1 \frac{\partial(r\theta_l)}{\partial r} r\cos(\lambda r)dr \\ = \lambda\left([u_3 v_3]_0^1 - \int_0^1 u_3' v_3 dr \right) = \lambda[r^2\theta_l\cos(\lambda r)]_0^1 \\ +\lambda\left(-\int_0^1 r\theta_l\cos(\lambda r)dr + \lambda\int_0^1 r^2\theta_l\sin(\lambda r)dr \right) \\ = \lambda\left(\cos(\lambda)\theta_s - \sqrt{\frac{\pi}{2}}\frac{\partial V_s(\lambda,\tau)}{\partial\lambda} - \sqrt{\frac{\pi}{2}}\lambda\frac{\partial^2 V_s(\lambda,\tau)}{\partial\lambda^2} \right). \tag{3.8}$$

So, by successive integration by parts over the droplet radius with consideration to the boundary conditions (1.9)-(1.10), equation (3.4) is transformed by combination of equations (3.5)-(3.8) into:

$$C(\lambda) = -\sqrt{\frac{2}{\pi}}(q_s\sin\lambda - \lambda\theta_s\cos\lambda) - \frac{\partial}{\partial\lambda}\left(\lambda^2\frac{\partial V_s(\lambda,\tau)}{\partial\lambda} \right). \tag{3.9}$$

Using expressions (3.3) and (3.9), the system (1.7)-(1.10) is finally transformed into equation (3.1). This completes the proof of the lemma. □

Proposition 3.1. *The temperature function $\theta_l(r,\tau)$, solution of the initial-boundary value problem (1.7)-(1.10) can be written as:*

$$\begin{aligned}\theta_l(r,\tau) &= \tfrac{1}{r\sqrt{\pi}}\int_0^\tau \theta_c(\tau-\eta)\frac{(1-\mathrm{e}^{-\frac{r^2}{4\eta}})}{\sqrt{\eta}}d\eta \\ &+\tfrac{1}{r\sqrt{\pi}}\int_0^\tau \theta_s(\tau-\eta)\frac{(-\mathrm{e}^{-\frac{1}{4\eta}}+\frac{(r+1)}{2}\mathrm{e}^{-\frac{(r+1)^2}{4\eta}}+\frac{(1-r)}{2}\mathrm{e}^{-\frac{(1-r)^2}{4\eta}})}{2\eta^{\frac{3}{2}}}d\eta \\ &+\tfrac{1}{r\sqrt{\pi}}\int_0^\tau P(\tau-\eta)\frac{(-\mathrm{e}^{-\frac{1}{4\eta}}+\frac{1}{2}\mathrm{e}^{-\frac{(r+1)^2}{4\eta}}+\frac{1}{2}\mathrm{e}^{-\frac{(r-1)^2}{4\eta}})}{\sqrt{\eta}}d\eta,\end{aligned} \tag{3.10}$$

where $P = q_s + \theta_s$, q_s is the temperature gradient at the droplet surface, θ_c and θ_s are respectively the temperatures at the centre and at the surface of the droplet.

Proof. In order to derive the expression of the transformed function $V_s(\lambda,\tau)$, it's necessary to solve the partial differential equation (3.1). A first integration over λ gives:

$$\begin{aligned}&\frac{\partial^2 V_s(\lambda,\tau)}{\partial\tau\partial\lambda}+\lambda^2\frac{\partial V_s(\lambda,\tau)}{\partial\lambda} \\ &=\sqrt{\frac{2}{\pi}}(q_s(\tau)\cos\lambda+(\lambda\sin\lambda+\cos\lambda)\theta_s(\tau))+a_1(\tau),\end{aligned} \tag{3.11}$$

with $a_1(\tau)$ a function to be determined. By tending λ to 0 in equation (3.11), we find:

$$\left.\frac{\partial^2 V_s(\lambda,\tau)}{\partial\tau\partial\lambda}\right|_{\lambda=0} = \sqrt{\frac{2}{\pi}}\left(q_s(\tau)+\theta_s(\tau)\right)+a_1(\tau), \tag{3.12}$$

since $\lambda^2\left.\frac{\partial V_s(\lambda,\tau)}{\partial\lambda}\right|_{\lambda=0} = 0$, as the quantity $\left.\frac{\partial V_s(\lambda,\tau)}{\partial\lambda}\right|_{\lambda=0} = \int_0^1 r\theta_l dr$ is finite. Now, according to equation (1.7), the following equality holds:

$$\left.\frac{\partial^2 V_s(\lambda,\tau)}{\partial\tau\partial\lambda}\right|_{\lambda=0} = \sqrt{\frac{2}{\pi}}\int_0^1\frac{\partial(r\theta_l)}{\partial\tau}dr = \sqrt{\frac{2}{\pi}}\int_0^1\frac{\partial^2(r\theta_l)}{\partial r^2}dr. \tag{3.13}$$

Then, from equations (3.12) and (3.13) one has,

$$a_1(\tau) = \sqrt{\frac{2}{\pi}}\left[\frac{\partial(r\theta_l)}{\partial r}\right]_0^1 - \sqrt{\frac{2}{\pi}}\left(q_s(\tau)+\theta_s(\tau)\right),$$

and therefore

$$a_1(\tau) = -\sqrt{\frac{2}{\pi}}\theta_c(\tau), \tag{3.14}$$

where $\theta_c(\tau)$ is the time-varying temperature at the droplet centre. Equation (3.11) becomes:

$$\begin{aligned}&\frac{\partial^2 V_s(\lambda,\tau)}{\partial\tau\partial\lambda} + \lambda^2\frac{\partial V_s(\lambda,\tau)}{\partial\lambda}\\ &= \sqrt{\frac{2}{\pi}}(q_s(\tau)\cos\lambda + (\lambda\sin\lambda + \cos\lambda)\theta_s(\tau) - \theta_c(\tau)).\end{aligned} \tag{3.15}$$

Setting $W_s(\lambda,\tau) = \dfrac{\partial V_s(\lambda,\tau)}{\partial\lambda}$ and multiplying the equation (3.15) by $\mathrm{e}^{\lambda^2\tau}$, we write:

$$\begin{aligned}&\mathrm{e}^{\lambda^2\tau}\frac{\partial W_s(\lambda,\tau)}{\partial\tau} + \lambda^2\mathrm{e}^{\lambda^2\tau}W_s(\lambda,\tau)\\ &= \sqrt{\frac{2}{\pi}}\left[q_s(\tau)\cos\lambda + (\lambda\sin\lambda + \cos\lambda)\theta_s(\tau) - \theta_c(\tau)\right]\mathrm{e}^{\lambda^2\tau},\end{aligned}$$

which is equivalent to:

$$\begin{aligned}&\frac{\partial(\mathrm{e}^{\lambda^2\tau}W_s(\lambda,\tau))}{\partial\tau}\\ &= \sqrt{\frac{2}{\pi}}\left[q_s(\tau)\cos\lambda + (\lambda\sin\lambda + \cos\lambda)\theta_s(\tau) - \theta_c(\tau)\right]\mathrm{e}^{\lambda^2\tau}.\end{aligned} \tag{3.16}$$

The integration of equation (3.16), from 0 to τ with respect to the time variable, leads to:

$$\begin{aligned}&\mathrm{e}^{\lambda^2\tau}W_s(\lambda,\tau) - W_s(\lambda,\tau=0)\\ &= \sqrt{\frac{2}{\pi}}\int_0^\tau [q_s\cos\lambda + (\lambda\sin\lambda + \cos\lambda)\theta_s - \theta_c]\mathrm{e}^{\lambda^2\eta}d\eta.\end{aligned}$$

Now,

$$W_s(\lambda,\tau=0) = \frac{\partial V_s(\lambda,\tau=0)}{\partial\lambda} = 0$$

according to the initial condition (1.8) which reads $\theta_l(r,\tau=0) = 0$. Equation (3.16) then becomes:

$$\begin{aligned}&W_s(\lambda,\tau) = \frac{\partial V_s(\lambda,\tau)}{\partial\lambda}\\ &= \sqrt{\frac{2}{\pi}}\mathrm{e}^{-\lambda^2\tau}\int_0^\tau [q_s\cos\lambda + (\lambda\sin\lambda + \cos\lambda)\theta_s - \theta_c]\mathrm{e}^{\lambda^2\eta}d\eta.\end{aligned} \tag{3.17}$$

Since $V_s(\lambda,\tau)$ is a FSIT function and therefore cancels when λ tends to $+\infty$, it can be written that:

$$V_s(\lambda,\tau) = -\int_\lambda^\infty \frac{\partial V_s(x,\tau)}{\partial x} dx,$$

and equation (3.17) will be integrated over λ, the integration variable x going from λ to $+\infty$. Then, reversing the order of integration (that is allowed due to the uniform convergence of $V_s(\lambda,\tau)$ and of its derivative with respect to λ, as specified by the Remark 2.1), equation (3.17) leads to:

$$\begin{aligned} V_s(\lambda,\tau) &= -\sqrt{\frac{2}{\pi}} \int_0^\tau d\eta \left[\theta_c(\eta)\left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} dx\right)\right] \\ &+\sqrt{\frac{2}{\pi}} \int_0^\tau d\eta \left[\theta_s(\eta)\left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} x\sin(x) dx\right)\right] \\ &+\sqrt{\frac{2}{\pi}} \int_0^\tau d\eta \left[(q_s(\eta)+\theta_s(\eta))\left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} \cos(x) dx\right)\right]. \end{aligned} \tag{3.18}$$

The FSIT inversion formula to obtain $\theta_l(r,\tau)$ from $V_s(\lambda,\tau)$, can now be applied to each term of equation (3.18). We first calculate the following inverse integrals:

$$I_a = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} \cos(x) dx\right) \sin(\lambda r) d\lambda,$$

$$I_b = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} x\sin(x) dx\right) \sin(\lambda r) d\lambda,$$

and

$$I_c = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)}\right) \sin(\lambda r) d\lambda.$$

The quantity I_a is calculated by integrating by parts as follows:

$$u_4 = -\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} \cos(x) dx \Rightarrow u_4' = \mathrm{e}^{\lambda^2(\eta-\tau)} \cos(\lambda),$$

$$v_4' = \sin(\lambda r) \Rightarrow v_4 = -\frac{\cos(\lambda r)}{r},$$

and then

$$\begin{aligned} I_a &= -\frac{\sqrt{2}}{r\sqrt{\pi}} \int_0^\infty \mathrm{e}^{x^2(\eta-\tau)} \cos(x) dx + \frac{\sqrt{2}}{r\sqrt{\pi}} \int_0^\infty \mathrm{e}^{\lambda^2(\eta-\tau)} \cos(\lambda r)\cos(\lambda) d\lambda \\ &= \frac{1}{r\sqrt{2(\tau-\eta)}} \left(-\mathrm{e}^{-\frac{1}{4(\tau-\eta)}} + \tfrac{1}{2}\mathrm{e}^{-\frac{(1+r)^2}{4(\tau-\eta)}} + \tfrac{1}{2}\mathrm{e}^{-\frac{(r-1)^2}{4(\tau-\eta)}}\right). \end{aligned}$$

Likewise, for I_b an integration by parts:

$$u_5 = -\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)} x\sin(x)dx \Rightarrow u_5' = \mathrm{e}^{\lambda^2(\eta-\tau)}\lambda\sin(\lambda),$$

$$v_5' = \sin(\lambda r) \Rightarrow v_5 = -\frac{\cos(\lambda r)}{r},$$

leads to:

$$\begin{aligned} I_b &= -\frac{\sqrt{2}}{r\sqrt{\pi}}\int_0^\infty \mathrm{e}^{x^2(\eta-\tau)} x\sin(x)dx + \frac{\sqrt{2}}{r\sqrt{\pi}}\int_0^\infty \lambda\mathrm{e}^{\lambda^2(\eta-\tau)}\cos(\lambda r)\sin(\lambda)d\lambda \\ &= \frac{\sqrt{2}}{4r(\tau-\eta)^{\frac{3}{2}}}\left(-\mathrm{e}^{-\frac{1}{4(\tau-\eta)}} + \frac{(1+r)}{2}\mathrm{e}^{-\frac{(1+r)^2}{4(\tau-\eta)}} + \frac{(1-r)}{2}\mathrm{e}^{-\frac{(1-r)^2}{4(\tau-\eta)}}\right) \end{aligned}$$

And by the alike technique of integration by parts, I_c is obtained as:

$$I_c = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta-\tau)}\right)\sin(\lambda r)d\lambda = \frac{1}{r\sqrt{2(\tau-\eta)}}\left(-1+\mathrm{e}^{-\frac{r^2}{4(\tau-\eta)}}\right).$$

The temperature function $\theta_l(r,\tau)$ is derived through the above expressions of I_a, I_b and I_c, by inverting equation (3.18). Finally, by changing the integration variable from η to $\eta' = \tau - \eta$, the proposition 3.1 holds. □

Remark 3.1. *In brief, an integral expression of the solution for the spherically symmetric droplet heating problem is obtained by using the FSIT method. Unfortunately, the above solution expressed by equation (3.10) doesn't permit to derive easily the expressions of the droplet surface temperature $\theta_s(\tau)$ or that of its centre temperature $\theta_c(\tau)$. In addition, the temperature gradient $q_s(\tau)$ and the surface temperature $\theta_s(\tau)$ are related as mentioned in the introduction. We introduce the Laplace integral transform (LIT) in order to express more simply the dependencies between the time-varying functions mentioned in equation (3.10).*

4. Explicit solutions in the Laplace domain

The Laplace integral transform (LIT) of the surface temperature gradient $q_s(\tau)$ and of the surface temperature $\theta_s(\tau)$ are respectively denoted by $\mathcal{L}q_s(p)$ and $\mathcal{L}\theta_s(p)$. The droplet centre temperature $\theta_c(\tau)$ (see equation (3.14)) is transformed into $\mathcal{L}\theta_c(p)$, and the temperature distribution $\theta_l(r,\tau)$ is transformed into

$\mathcal{L}\theta_l(r,p)$. Thus, the initial boundary value problem (1.7)-(1.10) can be written in the Laplace domain, under the following form:

$$p\,r\mathcal{L}\theta_l(r,p) - \frac{d^2(r\mathcal{L}\theta_l(r,p))}{dr^2} = 0, \tag{4.1}$$

and

$$\begin{cases} p\mathcal{L}\theta_l(r,p)|_{r,p=\infty} = 0 \\ \left.\dfrac{d\mathcal{L}\theta_l(r,p)}{dr}\right|_{r=0,p} = 0 \\ \left.\dfrac{d\mathcal{L}\theta_l(r,p)}{dr}\right|_{r=1,p} = \mathcal{L}q_s(p) \end{cases} . \tag{4.2}$$

The initial condition in the Laplace domain, as expressed by the first equation of the conditions (4.2), results from Theorem 2.1.

Proposition 4.1. *The temperature gradient at the droplet surface and the droplet internal temperature field can be respectively expressed in the Laplace domain as:*

$$\mathcal{L}q_s(p) = \frac{K\,(\mathrm{e}^{-2\sqrt{p}}\sqrt{p} + \mathrm{e}^{-2\sqrt{p}} + \sqrt{p} - 1)\mathcal{L}\theta_g(p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p} - K\,\mathrm{e}^{-2\sqrt{p}} + \mathrm{e}^{-2\sqrt{p}} + \sqrt{p} + K - 1} \tag{4.3}$$

and

$$\mathcal{L}\theta_l(r,p) = \frac{K\,\mathrm{e}^{-\sqrt{p}}(\mathrm{e}^{r\sqrt{p}} - \mathrm{e}^{-r\sqrt{p}})\mathcal{L}\theta_g(p)}{r(\mathrm{e}^{-2\sqrt{p}}\sqrt{p} - K\,\mathrm{e}^{-2\sqrt{p}} + \mathrm{e}^{-2\sqrt{p}} + \sqrt{p} + K - 1)}, \tag{4.4}$$

where $K = k_g/k_l$ is the ratio of conductivities and $\mathcal{L}\theta_g(p)$ is the Laplace transform of the dimensionless gas temperature $\theta_g(\tau)$.

Proof. According to the Convolution Theorem (see Theorem 2.2), equation (3.10) is transformed by the LIT into:

$$\begin{aligned} \mathcal{L}\theta_l(r,p) = {} & \frac{1}{r\sqrt{p}}\left(-\mathrm{e}^{-\sqrt{p}} + \frac{1}{2}\mathrm{e}^{-(1+r)\sqrt{p}} + \frac{1}{2}\mathrm{e}^{-(1-r)\sqrt{p}}\right) \\ & \times[\mathcal{L}q_s(p) + \mathcal{L}\theta_s(p) + \sqrt{p}\mathcal{L}\theta_s(p)] + \frac{1}{r\sqrt{p}}\left(1 - \mathrm{e}^{-r\sqrt{p}}\right)\mathcal{L}\theta_c(p). \end{aligned} \tag{4.5}$$

As already mentioned in the introduction, the condition of the temperature gradient is satisfied together with the condition at the droplet surface ($\mathcal{L}\theta_l(r,p)|_{r=1,p} = \mathcal{L}\theta_s(p)$). By using this latter condition and substituting the expression of

$\mathcal{L}\theta_l(r,p)$ given by equation (4.5) into equation (4.1), the droplet surface and centre temperatures are respectively derived in the Laplace domain as:

$$\mathcal{L}\theta_s(p) = -\frac{(\mathrm{e}^{-2\sqrt{p}}-1)\mathcal{L}q_s(p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}-1} \tag{4.6}$$

and

$$\mathcal{L}\theta_c(p) = \frac{2\mathrm{e}^{-\sqrt{p}}\sqrt{p}\mathcal{L}q_s(p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}-1}. \tag{4.7}$$

Combining equations (4.5)-(4.7), the exact operational or Laplace domain solution of the initial boundary value problem (1.7)-(1.10) is expressed in function of $\mathcal{L}q_s(p)$ by:

$$\mathcal{L}\theta_l(r,p) = \frac{\mathrm{e}^{-\sqrt{p}}(\mathrm{e}^{r\sqrt{p}}-\mathrm{e}^{-r\sqrt{p}})\mathcal{L}q_s(p)}{r(\mathrm{e}^{-2\sqrt{p}}\sqrt{p}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}-1)}. \tag{4.8}$$

The condition at the droplet surface, written in terms of dimensionless variables in equation (1.10), becomes in the Laplace domain:

$$\mathcal{L}q_s(p) = K\left(\mathcal{L}\theta_g(p) - \mathcal{L}\theta_s(p)\right), \tag{4.9}$$

where $K = k_g/k_l$ and $\mathcal{L}\theta_g(p)$ is the Laplace transform of the dimensionless gas temperature $\theta_g(\tau)$. Substituting $r = 1$ in equation (4.8) and using the above expression of $\mathcal{L}q_s(p)$, the surface temperature is deduced in the Laplace domain as:

$$\mathcal{L}\theta_s(p) = -\frac{K\,(\mathrm{e}^{-2\sqrt{p}}-1)\mathcal{L}\theta_g(p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1}. \tag{4.10}$$

From equations (4.9) and (4.10), an explicit expression of the temperature gradient at the droplet surface is obtained in the Laplace domain by the equation (4.3) as expressed in the proposition 4.1. Now, by tending r to 0 in equation (4.8) and using equation (4.3), the temperature at the droplet centre is expressed in the Laplace domain as follows:

$$\mathcal{L}\theta_c(p) = \frac{2K\sqrt{p}\,\mathrm{e}^{-\sqrt{p}}\mathcal{L}\theta_g(p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1}. \tag{4.11}$$

Finally, combining equations (4.8) and (4.3), an explicit solution $\mathcal{L}\theta_l(r,p)$ of the temperature inside the droplet can be obtained in the Laplace domain by equation (4.4) as expressed in the proposition 4.1. □

Remark 4.1. *Regardless of the complexity of the functions involved, inverse Laplace transformations can always be accomplished numerically [12, 13]. Nevertheless, analytical transformations in form of exact series solutions and asymptotic approximations in short time limits may be sought by means of inversion theorems from the Laplace domain into the time domain. The original function $\theta_l(r,\tau)$ is the inverse Laplace transform of the complex valued function $\mathcal{L}\theta_l(r,p)$, which depends on the parameter r and is given by equation (4.4).*

5. Analytical solutions in the time domain

An exact series solution to the initial-boundary value problem (1.7)-(1.10) may be derived by using the inversion technique due to the Heaviside expansion theorem (see Theorem 2.3).

Proposition 5.1. *An exact series solution to the initial-boundary value problem (1.7)-(1.10) can be written as:*

$$\theta_l(r,\tau) = \sum_{k\geq 1} A_k \frac{\sin(r\lambda_k)}{r} \int_0^\tau \theta_g(\tau-\eta)\mathrm{e}^{-\lambda_k^2\eta}d\eta, \tag{5.1}$$

where

$$A_k = \frac{2\lambda_k[\lambda_k\cos(\lambda_k)-\sin(\lambda_k)]}{\cos(\lambda_k)\sin(\lambda_k)-\lambda_k} = (-1)^{k+1}\frac{2K[\lambda_k^2+(K-1)^2]^{1/2}}{\lambda_k^2+K^2-K}, \tag{5.2}$$

and the terms of the sequence (λ_k) are the roots of the characteristic equation:

$$\tan\lambda = -\frac{\lambda}{K-1}, \tag{5.3}$$

numbered in ascending order $0<\lambda_1<\lambda_2<...<\lambda_k<...$ and tending to $+\infty$.

Proof. Equation (4.4) implies:

$$r\,\mathcal{L}\theta_l(r,p) = \frac{K\sinh(r\sqrt{p})\,\mathcal{L}\theta_g(p)}{(K-1)\sinh(\sqrt{p})+\sqrt{p}\cosh(\sqrt{p})}. \tag{5.4}$$

Using $\mathcal{L}^{-1}$ to denote the inverse operator of the LIT operator $\mathcal{L}$, we have

$$r\theta_l(r,\tau) = \mathcal{L}^{-1}\{\mathcal{L}\theta_g(p)\times F(r,p)\},$$

where $F(r,p)$ can be written in the form of:

$$F(r,p) = \frac{N(r,p)}{D(p)}, \tag{5.5}$$

where $N(r,p) = K\sinh(r\sqrt{p})$ and $D(p) = (K-1)\sinh(\sqrt{p}) + \sqrt{p}\cosh(\sqrt{p})$, $\sinh$ and $\cosh$ being respectively the sine and cosine hyperbolic functions. These latter functions are not generalized polynomials, but they can be so reduced by multiplying or dividing both by $\sqrt{p}$ if $p \neq 0$. Since $\sinh(\sqrt{p}) = -i\sin(i\sqrt{p})$ and $\cosh(\sqrt{p}) = \cos(i\sqrt{p})$, the denominator is equally written $D(p) = -i(K-1)\sin(i\sqrt{p}) + \sqrt{p}\cos(i\sqrt{p})$. We first determine the roots p_k of $D(p)$ in order to calculate the inverse $\mathcal{L}^{-1}\{F(r,p)\}$. Note that the function $F(r,p)$ is continuous by extension for the zero root $p_0 = 0$. The remaining roots p_k for which $D(p)$ is equated to zero are then determined by the characteristic equation (5.3):

$$\tan\lambda = -\frac{\lambda}{K-1},$$

where $\lambda = i\sqrt{p}$. So, $p_k = -\lambda_k^2$ for $k \geq 1$. From equation (5.3), it can be deduced that the λ_k are abscissa of the points of intersection of the tangent curve with the straight line of slope -1/(K-1) with $K = k_g/k_l < 1$. The solution to equation (5.3) is formed by an infinite set of real positive eigenvalues λ_k numbered in ascending order $0 < \lambda_1 < \lambda_2 < ... < \lambda_k < ...$ and tending to $+\infty$. Since the not null roots λ_k, $k \geq 1$ are distinct, the inversion technique from Heaviside's expansion theorem can be applied to generalized polynomials (confer Remark 2.2). This will lead to:

$$\mathcal{L}^{-1}\{F(r,p)\} = \sum_{k\geq 1} \frac{N(r,p_k)}{D'(p_k)} \mathrm{e}^{p_k\tau}, \tag{5.6}$$

where the prime sign stands for the derivative in respect to p. The calculation for these simple and not null roots leads to:

$$D'(p_k) = \frac{i}{2\lambda_k\sin(\lambda_k)}[\cos(\lambda_k)\sin(\lambda_k) - \lambda_k]$$

and

$$N(r,p_k) = \frac{i}{\sin(\lambda_k)}[\lambda_k\cos(\lambda_k) - \sin(\lambda_k)]\sin(r\lambda_k).$$

Hence, equation (5.6) is equally written as:

$$\mathcal{L}^{-1}\{F(r,p)\} = \sum_{k\geq 1} A_k \sin(r\lambda_k)\mathrm{e}^{-\lambda_k^2\tau}, \tag{5.7}$$

where A_k is given by the formula (5.2). Thus, the original of $N(r,p)/D(p)$ is the uniformly convergent series expressed by equation (5.7) and that of $\mathcal{L}\theta_g(p)$ is the dimensionless gas phase time-varying temperature $\theta_g(\tau)$. According to

the convolution theorem (Confer Theorem 2.2), an exact series solution of the initial-boundary value problem (1.7)-(1.10) can be found in the form of equation (5.1). This completes the proof of the proposition. □

Corollary 5.1. *If the temperature of the surrounding gas is constant ($\theta_g(\tau) = \overline{\theta}_g$), the solution (5.1) is reduced to:*

$$\theta_l(r,\tau) = \overline{\theta}_g \left(1 - \sum_{k\geq 1} B_k \frac{\sin(r\lambda_k)}{r} \mathrm{e}^{-\lambda_k^2 \tau}\right), \tag{5.8}$$

where

$$B_k = \frac{A_k}{\lambda_k^2} = \frac{2[\lambda_k \cos(\lambda_k) - \sin(\lambda_k)]}{\lambda_k(\cos(\lambda_k)\sin(\lambda_k) - \lambda_k)}. \tag{5.9}$$

Proof. In the case of steady gas phase environment, the temperature θ_g of the gas does not depend on time τ, so that

$$\theta_g(\tau) = \overline{\theta}_g = \text{const.}$$

The integral in solution (5.1) can be calculated and will explicitly yield:

$$\theta_l(r,\tau) = \overline{\theta}_g \sum_{k\geq 1} B_k \frac{\sin(r\lambda_k)}{r} - \overline{\theta}_g \sum_{k\geq 1} B_k \frac{\sin(r\lambda_k)}{r} \mathrm{e}^{-\lambda_k^2 \tau}, \tag{5.10}$$

since

$$\int_0^\tau \mathrm{e}^{-\lambda_k^2 \eta} d\eta = \frac{(1 - \mathrm{e}^{-\lambda_k^2 \tau})}{\lambda_k^2}.$$

In this case, the solution (5.8) may also be obtained directly from equation (5.5) by replacing the Laplace transform of the constant temperature $\overline{\theta}_g$ by its value $\mathcal{L}\theta_g(p) = \mathcal{L}\overline{\theta}_g = \overline{\theta}_g/p$, before the calculation of the residues. Then, by using the generalized polynomials $\sqrt{p}N(r,p)$ and $\sqrt{p}D(p)$, it may be verified that the residue of $F(r,p)$ at $p_0 = 0$ is $r\overline{\theta}_g$. This will leads to equation (5.8), which is the standard series solution of the symmetrical problem of heating or cooling of a solid spherical body as expressed in [5–8]. Now, equating solutions (5.8) and (5.10), it may be admitted that:

$$\sum_{k\geq 1} B_k \frac{\sin(r\lambda_k)}{r} = 1,$$

or equivalently:

$$\sum_{k\geq 1} B_k \sin(r\lambda_k) = r. \tag{5.11}$$

This equality (5.11) is proven as follows. The coefficients of the generalized Fourier series expansion of the function $g(r) = r$ defined on $[0, 1]$ can be calculated in the complete basis formed by the orthogonal and infinite family of functions $\{\sin(r\lambda_k), k \geq 1\}$ according to the inner product

$$< g_1, g_2 >= \int_0^1 g_1(x)\overline{g_2(x)}dx.$$

In fact, the set $\{\lambda_k,\ k \geq 1\}$, where λ_k is the kth root of the characteristic equation (5.3), forms the set of eigenvalues for the following homogeneous regular Sturm-Liouville problem:

$$\left\{ \begin{array}{l} \dfrac{d^2v}{dr^2} + \lambda v = 0 \\ \left.\dfrac{dv}{dr}\right|_{r=0} = 0 \\ \left.\left(\dfrac{dv}{dr} + (K-1)v\right)\right|_{r=1} = 0 \end{array} \right. . \tag{5.12}$$

with the related eigenfunctions $v_k = \sin(r\lambda_k)$ forming the full set of non-trivial solutions of this problem (5.12). It is well-known that the family of solutions of regular Sturm-Liouville problems such as problem (5.12), that is $\{\sin(r\lambda_k), k \geq 1, r \in [0, 1]\}$, forms a complete orthogonal set of linearly independent functions on $[0, 1]$ (see [14] and [15]). Thus, the function $g(r)$, defined on $[0, 1]$, satisfies the so-called Dirichlet conditions and can be expressed in a unique way as a series (called a generalized Fourier series) of the eigenfunctions of the problem (5.12). According to the calculations, the coefficients of the generalized Fourier series expansion of the function $g(r) = r$ are identical to B_k. Hence, equation (5.11) is justified and the proof of the corollary is completed. □

Corollary 5.2. *The exact series solutions (5.1) and (5.8) of the original initial-boundary value problem (1.2)-(1.5) can be respectively written as:*

$$T_l(R,t) = \overline{T}_0 + \sum_{k\geq 1} A_k \frac{\alpha_l}{R\,R_s} \sin\left(\frac{R}{R_s}\lambda_k\right) \int_0^t (T_g(t-\eta) - \overline{T}_0)\mathrm{e}^{-\kappa\lambda_k^2\eta}d\eta \tag{5.13}$$

and

$$T_l(R,t) = \overline{T}_0 + (\overline{T}_g - \overline{T}_0)\left(1 - \sum_{k\geq 1} B_k \frac{R_s}{R} \sin\left(\frac{R}{R_s}\lambda_k\right) \mathrm{e}^{-\kappa\lambda_k^2 t}\right), \tag{5.14}$$

where $\kappa = \alpha_l/R_s^2$ and A_k, B_k are respectively given by formulae (5.2) and (5.9).

Proof. Formulae (5.13) and (5.14) are obtained by returning to the original variables (see equations (1.2)-(1.5)) through the nondimensionalized equations (1.6) stated in the introduction. □

It may be interesting to find an approximate analytical solution in short time limits for the droplet internal temperature field $\theta_l(r,\tau)$ (and particularly for the dimensional droplet surface temperature). This analytical solution will be valid only for small values of a time step $\Delta\tau$ and can be useful in the so-called time step models of vaporizing droplets, as practised in Computational Fluid Dynamics (CFD) spray modelling.

Proposition 5.2. *A truncated expansion of at least first order of $r\theta_l(r,\tau)$ during a short time step $\Delta\tau$ ($\tau \in [0,\ \Delta\tau]$) can be expressed as:*

$$
\begin{aligned}
r\theta_l(r,\tau) = K\int_0^\tau \theta_g(\tau-\eta)\mathrm{e}^{\frac{-(1-r)^2}{4\eta}}\left(\frac{1}{\sqrt{\pi\eta}}+2(1-K)^2\sqrt{\frac{\eta}{\pi}}\right)d\eta \\
+K(1-K)(r+K-Kr)\int_0^\tau \theta_g(\tau-\eta)\mathrm{erfc}\left(\frac{1-r}{2\sqrt{\eta}}\right)d\eta+O(\tau),
\end{aligned}
\tag{5.15}
$$

where the big $O()$ is the asymptotic notation, erfc *is the complementary error function defined as* $\mathrm{erfc}(x)=1-\mathrm{erf}(x)$*, and*

$$
\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{z^2}dz.
$$

Proof. The limiting case of short time duration (τ tending to 0) corresponds to a very large Laplace domain variable (p tending to $+\infty$). Since the Laplace transform $\mathcal{L}\theta_g(p)$ goes to zero as p tends to $+\infty$, the truncated asymptotic expansion of second order for the droplet internal temperature $\theta_l(r,\tau)$, can be derived from the Laplace domain solution (4.4) as follows:

$$
\begin{aligned}
r\mathcal{L}\theta_l(r,p) = K\mathcal{L}\theta_g(p)\left(\frac{1}{\sqrt{p}}+\frac{(1-K)}{p}+\frac{(1-K)^2}{p^{3/2}}\right)\mathrm{e}^{-(1-r)\sqrt{p}} \\
+O\left(\frac{1}{p^2}\right)\mathrm{e}^{-(1-r)\sqrt{p}}.
\end{aligned}
\tag{5.16}
$$

The inversion of this result in the time domain is possible by using inverse Laplace integral transform (LIT) tables as in [16]. Thus, analytical approximations of the droplet internal temperature at the earliest time of the process or after any short time step $\Delta\tau$ ($t \in [0,\ \Delta\tau]$) can be found by using the convolution theorem of the Laplace integral transform (see Theorem 2.2) and the converse

to Watson's Lemma (see Lemma 2.2). This leads to the formula (5.15) of the time-varying droplet internal temperature for $\tau \in [0, \Delta\tau]$ or during any small time step $\Delta\tau$. Note that if $r = 0$, that is at the droplet centre, the approximation (5.15) is valid for an arbitrary order $n \geq 1$ of the related truncated expansion, while for $r = 1$ (droplet surface), the formula (5.15) is valid only for the first order $n = 1$. □

Corollary 5.3. *Returning to the original variables, the analytical approximation for the droplet surface temperature $T_s(t)$ during any sufficiently small dimensional time step $\Delta t = R_s^2 \Delta\tau/\alpha_l$, ($t \in [0, \Delta t]$), can be written as:*

$$\begin{aligned} &T_s(t) = \\ &K \int_0^t (T_g(t-\eta) - \overline{T}_0) \left[\frac{\sqrt{\alpha_l}}{R_s\sqrt{\pi\eta}} + \frac{\alpha_l(1-K)}{R_s^2} + 2\frac{{\alpha_l}^{3/2}(1-K)^2}{R_s^3}\sqrt{\frac{\eta}{\pi}} \right] d\eta \\ &+ \overline{T}_0 + O(t). \end{aligned} \tag{5.17}$$

In the case of constant temperature $T_g(t) = \overline{T}_g$ of the surrounding gas phase during the time step Δt, $T_s(t)$ is reduced to:

$$T_s(t) = \overline{T}_0 + 2K(\overline{T}_g - \overline{T}_0)\frac{\sqrt{\alpha_l t}}{R_s\sqrt{\pi}} + O(t). \tag{5.18}$$

Proof. From the general asymptotic formula (5.16), that of the dimensionless droplet surface temperature $\theta_s(p)$ is obtained by substituting $r = 1$, which leads to:

$$\mathcal{L}\theta_s(p) = K\mathcal{L}\theta_g(p)\left(\frac{1}{\sqrt{p}} + \frac{(1-K)}{p} + \frac{(1-K)^2}{p^{3/2}}\right) + O\left(\frac{1}{p^2}\right). \tag{5.19}$$

Using again the nondimensionalized equations (1.6), one has: $\Delta\tau = \alpha_l \Delta t/R_s^2$ and then $\Delta t = R_s^2\Delta\tau/\alpha_l$. By using the scaling property of the Laplace integral transform, equation (5.19) will read in its dimensional form:

$$\begin{aligned} &\mathcal{L}T_s(q) = \\ &K(\mathcal{L}T_g(q) - \overline{T}_0/q)\left[\frac{1}{R_s}\left(\frac{\alpha_l}{q}\right)^{1/2} + \frac{(1-K)}{R_s^2}\frac{\alpha_l}{q} + \frac{(1-K)^2}{R_s^3}\left(\frac{\alpha_l}{q}\right)^{3/2}\right] \\ &+ \frac{\overline{T}_0}{q} + O\left(\frac{1}{q^2}\right), \end{aligned} \tag{5.20}$$

where q instead of p stands for the new Laplace domain variable. The transformation into the time domain of the formula (5.20) gives the time-varying

droplet surface temperature $T_s(t)$ during any small time step Δt, $t \in [0, \Delta t]$, as in formula (5.17). Now, if the surrounding gas phase is at constant temperature $T_g(t) = \overline{T}_g$ during the time step Δt, then $\mathcal{L}T_g(q) = \overline{T}_g/q$ and the formula (5.17) of the dimensional surface temperature $T_s(t)$ is reduced to the analytical approximation (5.18). This completes the proof of the corollary. □

Remark 5.1. *As already mentioned, the results may be useful in the modelling of transient heating and evaporation of droplets and sprays, as pertaining to CFD codes. Indeed, in CFD calculations for sprays, the droplet surface and volume-average temperature values are sufficient, instead of the complete droplet interior temperature field, to permit the estimation of the related heat and mass transfer quantities during each time step [17]. As performed above for the dimensional droplet surface temperature $T_s(t)$, the volume-average temperature during a short time step can also be derived from the analytical approximations (5.15) and (5.18). Likewise, the droplet surface, centre and volume average temperatures during each time step are derivable from the exact series solutions as expressed by equations (5.13) and (5.14).*

6. Conclusion

By combining two classical non-finite integral transform methods, this study has derived the explicit solution in the Laplace domain, for the spherically symmetric heat diffusion equation inside a non-evaporating droplet suspended in an unsteady gas environment. It was possible to obtain an exact series solution in time domain and an analytical approximate expansion in a short time duration (time step) for the droplet surface and internal temperatures distributions. These latter can be useful for applications in CFD spray modelling. The early time behaviour of the heat diffusion process at the droplet surface is of great interest in combustion engineering. The combined integral transform method presented in the above study can also be regarded as an alternative to the classical method of separation of variables for solving parabolic linear differential equations. It is promising for many engineering problems involving one-dimensional transient heat or mass diffusion with various thermal boundary conditions.

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