

A SOLUTION TO INVERSE STURM-LIOUVILLE PROBLEMS

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ABSTRACT. In this study, we recover potential function and separable boundary conditions for the inverse Sturm-Liouville problem in normal form by using two partial subsets of the data which consist of its one spectrum and sequence of endpoints of eigenfunctions.

1. INTRODUCTION

It is possible to split the inverse Sturm-Liouville (S-L) problem in two parts:

- (i) The uniqueness theorems,
- (ii) The recovering coefficients from uniqueness data.

Although there are many studies on the uniqueness, it is observed in the literature that the studies on the recovering of the potential function and boundary coefficients do not occur too much. Some of them can be found in the references [1–8].

Consider the boundary value problems

$$(1.1) \quad -y''(x) + (\lambda + q(x))y(x) = 0,$$

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on $[0, 1]$ with the separable boundary conditions

$$(1.2) \quad y'(0) + h_0 y(0) = 0, \quad y'(1) + h_1 y(1) = 0,$$

where $q(x) \in L^2_{\mathbb{R}}[0, 1]$ and $h_0, h_1 \in \mathbb{R}$. R hrl used two spectra to obtain first the potential function $q(x)$ in [9] and then boundary coefficients h_0, h_1 along with the potential in [10]. We studied the problem for both the set of the j th elements of the infinite numbers of spectra obtained by changing boundary conditions in related problem and one spectrum with the set of the terminal velocities $\kappa_n = \log \left| \frac{g'_n(1)}{g'_n(0)} \right|$ for the Dirichlet boundary condition in order to reconstruct the potential [11]. In this paper, we recover not only potential but also boundary conditions for (1.1)-(1.2) by using another uniqueness data (see Theorem 1.1) as one spectrum and the endpoint datum

$$(1.3) \quad l_n = \log \left[\frac{(-1)^{n-1} g_n(1)}{g_n(0)} \right], \quad n \geq 1,$$

with the normalized eigenfunction g_n corresponding to λ_n .

Let $\mathbf{q} = (h_0, h_1, q) \in \mathbb{R}^2 \times L^2_{\mathbb{R}}[0, 1]$ and $(\lambda_n(\mathbf{q}), l_n(\mathbf{q}); n \geq 1)$ be given uniqueness data described as above.

Theorem 1.1. *The map*

$$\mathbf{q} \in \mathbb{R}^2 \times L^2_{\mathbb{R}}[0, 1] \rightarrow (\lambda_n(\mathbf{q}), l_n(\mathbf{q}); n \geq 1)$$

is one to one [12].

Consider (1.1) with the initial conditions

$$(1.4) \quad y_{1i}(i) = y'_{2i}(i) = 1, \quad y_{2i}(i) = y'_{1i}(i) = 0, \quad y_{3i}(i) = -1, \quad y'_{3i}(i) = h_i,$$

where $i = 0, 1$. The functions $(y_{1i}(x), y_{2i}(x))$ are fundamental solutions at the initial points $x = 0$ and $x = 1$. Thus

$$y_{3i}(x) = y_{3i}(i)y_{1i}(x) + y'_{3i}(i)y_{2i}(x) = h_i y_{2i}(x) - y_{1i}(x).$$

It is obvious that the solutions $y_{3i}(x)$ for $i = 0, 1$ satisfy the boundary conditions (1.2) respectively.

Lemma 1.1. *Let $\chi_{[i,x]} = -\chi_{[x,i]}$ for $i = 1$. Then, the following equations hold for $i = 0, 1$,*

$$(i) \quad \frac{\partial y_{3i}}{\partial q(t)}(x) = y_{3i}(t)[y_{2i}(t)y_{3i}(x) - y_{2i}(x)y_{3i}(t)]\chi_{[i,x]},$$

$$\text{ii)} \quad \frac{\partial y_{3i}}{\partial \lambda}(x) = (-1)^{i+1} \int_0^1 y_{3i}(t)[y_{2i}(t)y_{3i}(x) - y_{2i}(x)y_{3i}(t)]dt.$$

Proof. It is a corollary of the Theorem 6 in ([13], pp.21). $\square$

Lemma 1.2. Let $\{\lambda_n\}_{n \geq 1}$ be eigenvalues of boundary value problem (1.1)-(1.2). Then the functions $y_{30}(x, \lambda_n)$ and $y_{31}(x, \lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}_{n \geq 1}$ such that $y_{30}(x, \lambda_n) = \beta_n y_{31}(x, \lambda_n)$, $\beta_n \neq 0$ [14].

Lemma 1.3. For all $n \in \mathbb{N}$,

$$\begin{aligned} \text{(i)} \quad & \frac{\partial l_n}{\partial q}(t) = A_{in}(t) - [A_{in}(t)]g_n^2(t), \\ \text{(ii)} \quad & \frac{\partial l_n}{\partial h_i}(t) = (-1)^{i+1} \frac{\partial l_n}{\partial q}(i), \end{aligned}$$

where $y_{jin}(t) = y_{ji}(t, \lambda_n)$, $A_{in}(t) = y_{3in}(t)y_{2in}(t)$, and $[A_{in}(t)] = \int_0^1 A_{in}(t)dt$.

Proof. Since the functions $y_{3in}(x)$ are eigenfunctions, we get $g_n(x) = g(x, \lambda_n) = \frac{y_{3in}(x)}{\|y_{3in}(x)\|_{L^2}}$. So by virtue of (1.3)-(1.4),

$$l_n = \log \left(\frac{(-1)^{n-1} y_{3in}(1)}{y_{3in}(0)} \right) = (-1)^i \log((-1)^{(-1)^{i+n}} y_{3in}(1-i)).$$

(i) Considering $\frac{\partial \lambda_n}{\partial q(t)} = g_n^2(t)$ [13] and using Lemma 1.1, we calculate

$$\begin{aligned} & \frac{\partial l_n}{\partial q(t)} \\ &= \frac{(-1)^i}{y_{3in}(1-i)} \left\{ \frac{\partial y_{3in}(1-i)}{\partial \lambda_n} \frac{\partial \lambda_n}{\partial q(t)} + \frac{\partial}{\partial q(t)} y_{3i}(1-i, q, \lambda) \right\} \Bigg|_{\lambda=\lambda_n} \\ &= \left\{ \left(\int_i^{1-i} y_{3in}(t)[y_{3in}(t)y_{2in}(1-i) - y_{3in}(1-i)y_{2in}(t)]dt \right) g_n^2(t) \right. \\ & \quad \left. + (-1)^i (y_{3in}(t)y_{2in}(t)y_{3in}(1-i) - y_{3in}^2(t)y_{2in}(1-i)) \right\} \frac{(-1)^i}{y_{3in}(1-i)} \\ &= \left\{ y_{2in}(1-i)g_n^2(t) \int_i^{1-i} y_{3in}^2(t)dt - y_{3in}(1-i)g_n^2(t) \int_i^{1-i} y_{3in}(t)y_{2in}(t)dt \right. \\ & \quad \left. + (-1)^i y_{3in}(t)y_{2in}(t)y_{3in}(1-i) - (-1)^i y_{3in}^2(t)y_{2in}(1-i) \right\} \frac{(-1)^i}{y_{3in}(1-i)} \\ &= \left\{ y_{2in}(1-i)(-1)^i y_{3in}^2(t) - y_{3in}(1-i)(-1)^i g_n^2(t) \int_0^1 y_{3in}(t)y_{2in}(t)dt \right. \\ & \quad \left. + (-1)^i y_{3in}(t)y_{2in}(t)y_{3in}(1-i) - (-1)^i y_{3in}^2(t)y_{2in}(1-i) \right\} \frac{(-1)^i}{y_{3in}(1-i)} \\ &= y_{3in}(t)y_{2in}(t) - g_n^2(t) \int_0^1 y_{3in}(t)y_{2in}(t)dt = A_{in}(t) - [A_{in}]g_n^2(t). \end{aligned}$$

(ii) Analogously

$$\begin{aligned}
 & \frac{\partial l_n}{\partial h_i} \\
 = & \left\{ \left[y_{2in}(1-i) \int_i^{1-i} y_{3in}^2(t) dt - y_{3in}(1-i) \int_i^{1-i} y_{3in}(t) y_{2in}(t) dt \right] (-1)^{i+1} g_n^2(i) \right. \\
 & \left. + \frac{\partial}{\partial h_i} [h_i y_{2i}(1-i, \lambda) - y_{1i}(1-i, \lambda)]_{\lambda=\lambda_n} \right\} \frac{(-1)^i}{y_{3in}(1-i)} \\
 = & \frac{(-1)^i}{y_{3in}(1-i)} \left\{ (y_{2in}(1-i)(-1)^i \|y_{3in}(t)\|_{L^2}^2 \right. \\
 & \left. - y_{3in}(1-i) \int_i^{1-i} y_{3in}(t) y_{2in}(t) dt) (-1)^{i+1} g_n^2(i) + y_{2in}(1-i) \right\} \\
 = & \frac{(-1)^i}{y_{3in}(1-i)} \left\{ (-1)^{2i+1} y_{2in}(1-i) y_{3in}^2(i) - y_{3in}(1-i) (-1)^{2i+1} g_n^2(i) [A_{in}] \right. \\
 & \left. + y_{2in}(1-i) \right\} = (-1)^i [A_{in}] g_n^2(i)
 \end{aligned}$$

is obtained. On the other hand, $A_{in}(i) = 0$ since $y_{2in}(i) = 0$. Therefore

$$\begin{aligned}
 \frac{\partial l_n}{\partial h_i}(t) &= (-1)^{i+1} (-[A_{in}] g_n^2(i)) \\
 &= (-1)^{i+1} (A_{in}(i) - [A_{in}] g_n^2(i)) = (-1)^{i+1} \frac{\partial l_n}{\partial q}(i).
 \end{aligned}$$

□

Let us consider the bilinear form $\Gamma : H^1 \times H^1 \rightarrow \mathbb{R}$ with $\Gamma(f, g) = \int_0^1 [f, g] dx$ where $[\cdot, \cdot]$ is the Wronskian operator such that $[f, g] = f(x)g'(x) - f'(x)g(x)$ for differentiable functions $f, g : [0, 1] \rightarrow \mathbb{R}$. This transformation is bounded by $|\Gamma(f, g)| \leq \|f\|_{H^1} \|g\|_{H^1}$. In particular Γ is continuous on H^1 [10]. Also, it is easy to see that Γ is antisymmetric because of Wronskian. Besides that we also have some properties for Wronskian [10, 13] as below:

- (i) $[fg, FG] = fF[g, G] + gG[f, F]$ for differentiable functions f, g, F, G .
- (ii) For two arbitrary solutions f_n and f_m of the equation (1.1) with different eigenvalues λ_n and λ_m , then we have $f_n f_m = \frac{1}{\lambda_n - \lambda_m} [f_n, f_m]'$.

Since $y_{20}(x, \lambda)$ and $y_{30}(x, \lambda)$ satisfy the equation (1.1), so we obtain

$$\frac{d}{dx} [y_{20}, y_{30}] = \frac{d}{dx} (y_{20} y'_{30} - y'_{20} y_{30}) = y_{20} y_{30} (q - \lambda) - y_{20} y_{30} (q - \lambda) = 0.$$

This implies that the Wronskian is independent with respect to the variable x , thus we have

$$[y_{20}, y_{30}] = y_{20}(0)y'_{30}(0) - y'_{20}(0)y_{30}(0) = 1.$$

Lemma 1.4. *Let $\{\lambda_n\}_{n \geq 1}$ be spectrum of (1.1)-(1.2). Then the following equalities are satisfied:*

- (i) $\Gamma(g_n^2, g_m^2) = 0$,
- (ii) $\Gamma(y_{30n}y_{20n}, y_{30m}y_{20m}) = 0$,
- (iii) $\Gamma(y_{30n}y_{20n}, g_m^2) = \delta_{nm}$.

Proof. For (i), see [10]. To prove (ii) and (iii), by taking the initial condition (1.3), Lemma 1.2, and Wronskian's properties into account one can follow similar steps in ([11], Lemma 1). $\square$

2. MAIN RESULT

For our computation, we consider R  hrl's type objective functional and use another conjugate gradient algorithm which is called Fletcher-Reeves algorithm. Let $\mathbb{N}_0 = \{1, 2, \dots, k\} \subset \mathbb{N}$. For the test potential $\mathbf{Q} := (H_0, H_1, Q(x))$, objective functional and its gradient turn out to be the form as:

$$G[\mathbf{q}] = \sum_{n \in \mathbb{N}_0} ((\lambda_n^{\mathbf{q}} - \lambda_n^{\mathbf{Q}})^2 + (l_n^{\mathbf{q}} - l_n^{\mathbf{Q}})^2)$$

and

$$\nabla G[\mathbf{q}] = 2 \sum_{n \in \mathbb{N}_0} \left((\lambda_n^{\mathbf{q}} - \lambda_n^{\mathbf{Q}}) \frac{\partial \lambda_n}{\partial \mathbf{q}} + (l_n^{\mathbf{q}} - l_n^{\mathbf{Q}}) \frac{\partial l_n}{\partial \mathbf{q}} \right),$$

respectively where from [10] and Lemma 1.3

$$\frac{\partial \lambda_n}{\partial \mathbf{q}} = \begin{pmatrix} -g_n^2(0) \\ g_n^2(1) \\ g_n^2(x) \end{pmatrix} \quad \text{and} \quad \frac{\partial l_n}{\partial \mathbf{q}} = \begin{pmatrix} -\frac{\partial l_n}{\partial q}(0) \\ \frac{\partial l_n}{\partial q}(1) \\ \frac{\partial l_n}{\partial q}(x) \end{pmatrix}.$$

It is obvious that $0 = G[\mathbf{Q}] < G[\mathbf{q}]$ for $\mathbf{q} \neq \mathbf{Q} \in \mathbb{R}^2 \times L^2_{\mathbb{R}}[0, 1]$. In other words, $\mathbf{Q} \in \mathbb{R}^2 \times L^2_{\mathbb{R}}[0, 1]$ is the global minimum for $G[\mathbb{R}^2 \times \cdot]$.

Now let consider the vectors

$$V_{\lambda_m} = \begin{pmatrix} -g_m^2(0) \\ -g_m^2(1) \\ 2 \frac{d}{dx}(g_m^2(x)) \end{pmatrix} \quad \text{and} \quad V_{l_m} = \begin{pmatrix} -\frac{\partial l_m}{\partial q}(0) \\ -\frac{\partial l_m}{\partial q}(1) \\ 2 \frac{d}{dx} \left(\frac{\partial l_m}{\partial q}(x) \right) \end{pmatrix}.$$

By using integration by part, we have

$$(2.1) \quad \langle f, 2h' \rangle_{L^2} = fh|_0^1 + \Gamma(f, h),$$

where $f, h \in H^1$ – Sobolev space.

Lemma 2.1. *The following equalities are hold:*

- (i) $\langle y_{30n}y_{20n}, 2(y_{30m}y_{20m})' \rangle = y_{30n}y_{20n}y_{30m}y_{20m}|_0^1,$
- (ii) $\langle y_{30n}y_{20n}, 2(g_m^2)' \rangle = y_{30n}y_{20n}g_m^2|_0^1 + \delta_{nm},$
- (iii) $\langle g_n^2, 2(y_{30m}y_{20m})' \rangle = g_n^2y_{30m}y_{20m}|_0^1 - \delta_{nm},$
- (iv) $\langle g_n^2, 2(g_m^2)' \rangle = g_n^2g_m^2|_0^1.$

Proof. The proof of (i), (ii) and (iii) can be obtained easily from the equation (2.1), Lemma 1.4, and anti-symmetry property of the operator Γ consecutively. For the proof of (iv), see [12]. $\square$

Corollary 2.1. *For all $m, n \in \mathbb{N}$,*

- (i) $\langle \frac{\partial l_n}{\partial q}, V_{l_m} \rangle = 0,$
- (ii) $\langle \frac{\partial l_n}{\partial q}, V_{\lambda_m} \rangle = \delta_{nm},$
- (iii) $\langle \frac{\partial \lambda_n}{\partial q}, V_{l_m} \rangle = -\delta_{nm},$
- (iv) $\langle \frac{\partial \lambda_n}{\partial q}, V_{\lambda_m} \rangle = 0.$

Proof. By using Lemma 2.1, then

$$\begin{aligned}
 & (i) \\
 & \quad \langle \frac{\partial l_n}{\partial q}, V_{l_m} \rangle \\
 &= \frac{\partial l_n}{\partial q}(0) \frac{\partial l_m}{\partial q}(0) - \frac{\partial l_n}{\partial q}(1) \frac{\partial l_m}{\partial q}(1) + \langle \frac{\partial l_n}{\partial q}(x), 2 \frac{d}{dx} \left(\frac{\partial l_m}{\partial q}(x) \right) \rangle \\
 &= - \frac{\partial l_n}{\partial q} \frac{\partial l_m}{\partial q} \Big|_0^1 + \langle y_{30n}y_{20n}, 2(y_{30m}y_{20m})' \rangle - [A_{0m}] \langle y_{30n}y_{20n}, 2(g_m^2)' \rangle \\
 & \quad - [A_{0n}] \langle g_n^2, 2(y_{30m}y_{20m})' \rangle + [A_{0n}][A_{0m}] \langle g_n^2, 2(g_m^2)' \rangle \\
 &= - \frac{\partial l_n}{\partial q} \frac{\partial l_m}{\partial q} \Big|_0^1 + y_{30n}y_{20n}y_{30m}y_{20m}|_0^1 - [A_{0m}](y_{30n}y_{20n}g_m^2|_0^1 + \delta_{nm}) \\
 & \quad - [A_{0n}](g_n^2y_{30m}y_{20m}|_0^1 - \delta_{nm}) + [A_{0n}][A_{0m}]g_n^2g_m^2|_0^1 \\
 &= - \frac{\partial l_n}{\partial q} \frac{\partial l_m}{\partial q} \Big|_0^1 + \frac{\partial l_n}{\partial q} \frac{\partial l_m}{\partial q} \Big|_0^1 + \delta_{nm}([A_{0n}] - [A_{0m}]) = 0.
 \end{aligned}$$

(ii)

$$\begin{aligned}
 & \left\langle \frac{\partial l_n}{\partial \mathbf{q}}, V_{\lambda_m} \right\rangle \\
 &= -\frac{\partial l_n}{\partial q} g_m^2 \Big|_0^1 + \left\langle y_{30n} y_{20n}, 2(g_m^2)' \right\rangle - [A_{0n}] \left\langle g_n^2, 2(g_m^2)' \right\rangle \\
 &= -\frac{\partial l_n}{\partial q} g_m^2 \Big|_0^1 + \frac{\partial l_n}{\partial q} g_m^2 \Big|_0^1 + \delta_{nm} = \delta_{nm}.
 \end{aligned}$$

(iii)

$$\begin{aligned}
 & \left\langle \frac{\partial \lambda_n}{\partial \mathbf{q}}, V_{l_m} \right\rangle \\
 &= -g_n^2 \frac{\partial l_m}{\partial q} \Big|_0^1 + \left\langle g_n^2, 2(y_{30m} y_{20m})' \right\rangle - [A_{0m}] \left\langle g_n^2, 2(g_m^2)' \right\rangle \\
 &= -g_n^2 \frac{\partial l_m}{\partial q} \Big|_0^1 + g_n^2 \frac{\partial l_m}{\partial q} \Big|_0^1 - \delta_{nm} = -\delta_{nm}.
 \end{aligned}$$

(iv) See [12].

□

Theorem 2.1. *The set $\{\frac{\partial \lambda_n}{\partial \mathbf{q}} : n \in \mathbb{N}_0\} \cup \{\frac{\partial l_n}{\partial \mathbf{q}} : n \in \mathbb{N}_0\}$ is linearly independent.*

Proof. Let us suppose

$$\sum_{n \in \mathbb{N}_0} (a_n \frac{\partial \lambda_n}{\partial \mathbf{q}} + b_n \frac{\partial l_n}{\partial \mathbf{q}}) = 0,$$

where a_n, b_n are some real numbers. Since scalar product is continuous, then we find from Corollary 2.1 that

$$\begin{aligned}
 0 &= \left\langle \sum_{n \in \mathbb{N}_0} (a_n \frac{\partial \lambda_n}{\partial \mathbf{q}} + b_n \frac{\partial l_n}{\partial \mathbf{q}}), V_{\lambda_m} \right\rangle \\
 &= \sum_{n \in \mathbb{N}_0} (a_n \left\langle \frac{\partial \lambda_n}{\partial \mathbf{q}}, V_{\lambda_m} \right\rangle + b_n \left\langle \frac{\partial l_n}{\partial \mathbf{q}}, V_{\lambda_m} \right\rangle) = b_m,
 \end{aligned}$$

and

$$\begin{aligned}
 0 &= \left\langle \sum_{n \in \mathbb{N}_0} (a_n \frac{\partial \lambda_n}{\partial \mathbf{q}} + b_n \frac{\partial l_n}{\partial \mathbf{q}}), V_{l_m} \right\rangle \\
 &= \sum_{n \in \mathbb{N}_0} (a_n \left\langle \frac{\partial \lambda_n}{\partial \mathbf{q}}, V_{l_m} \right\rangle + b_n \left\langle \frac{\partial l_n}{\partial \mathbf{q}}, V_{l_m} \right\rangle) = -a_m.
 \end{aligned}$$

These complete the proof.

□

Theorem 2.2. *The functional $G(\mathbf{q})$ has no local minima at $\mathbf{q}$ with $G(\mathbf{q}) > 0$. In other words, $\nabla G(\mathbf{q}) = 0 \Leftrightarrow G(\mathbf{q}) = 0$.*

Proof. It is obvious that If $G(\mathbf{q}) = 0$, then $\nabla G(\mathbf{q}) = 0$. If $\nabla G(\mathbf{q}) = 0$, then by taking $a_n = \lambda_n^{\mathbf{q}} - \lambda_n^{\mathbf{Q}}$ and $b_n = l_n^{\mathbf{q}} - l_n^{\mathbf{Q}}$ it can be seen easily from Theorem 2.1 that $G(\mathbf{q}) = 0$. $\square$

3. NUMERICAL EXPERIMENTS

In order to present the numerical examples related with the problem, we use the test potentials $Q^1(x)$ and $Q^2(x)$ which was considered in [10, 11, 15] and the test boundary coefficient $H_0 = 1$ and $H_1 = 2$. The Fletcher-Reeves algorithm is applied as follows [16]:

Step 0. Select an initial potential as q_0 . Set $n = 0$ and $g_0 = h_0 = -\nabla G[q_0]$.

Step 1. Compute $\alpha_n > 0$ such that

$$G[q_n + \alpha_n h_n] = \min\{G[q_n + \alpha h_n] : \alpha \geq 0\}.$$

Step 2. Set $q_{n+1} = q_n + \alpha_n h_n$.

Step 3. If $G[q_{n+1}]$ is small enough, stop; else, set

$$g_{n+1} = -\nabla G[q_{n+1}],$$

$$h_{n+1} = g_{n+1} + \gamma_n h_n, \quad \text{with} \quad \gamma_n = \frac{\langle g_{n+1}, g_{n+1} \rangle}{\langle g_n, g_n \rangle},$$

set $n = n + 1$, and go to step 1.

In the numerical calculations of potentials and boundary conditions, they are taken into account the sensitivities as $G(q_n) \approx 10^{-6}$ and $G(q_n) \approx 10^{-5}$ for $Q^1(x)$ and $Q^2(x)$ respectively and graphs of Fig. 3.1 and Fig. 3.2 are sketched according to these sensitivities. In these graphs, the $\hat{q}(x)$ represents the potential which is calculated along with unknown boundary coefficients and $q(x)$ shows the potential which is obtained for known boundary coefficients $h_0 = 1$ and $h_1 = 2$ in (1.2). In order to get iterations, we start with $q_0 = 0$ and $\mathbf{q}_0 = (\hat{q}_0, h_0, h_1) = (0, 0, 0)$. While we use 10 pairs of data for $Q^2(x)$ in Fig. 3.2, besides this it is taken 5 pairs of data for $Q^1(x)$ in Fig. 3.1.

On the other hand in the calculation of potential and boundary conditions together, we reset the iterative potential q_j to zero at some iteration number j while keeping the boundary coefficients. For example, $\hat{q}_0 = \hat{q}_{60} = \hat{q}_{120} = \hat{q}_{180} = 0$

and $\hat{q}_0 = \hat{q}_{60} = 0$ are picked for $Q^1(x)$ and $Q^2(x)$ respectively. These can be seen in Fig. 3.1b for $Q^1(x)$ and Fig. 3.2b for $Q^2(x)$. The results for q and $\hat{q}(x)$ are calculated at 63 and 233 iterations for $Q^1(x)$ in Fig. 3.1a and at 53 and 148 iterations for $Q^2(x)$ in Fig. 3.2a. As a result of calculations, we find $h_0 = 0.995729, h_1 = 1.99695$ and $h_0 = 1.008437, h_1 = 2.015540$ for $Q^1(x)$ and $Q^2(x)$ respectively.

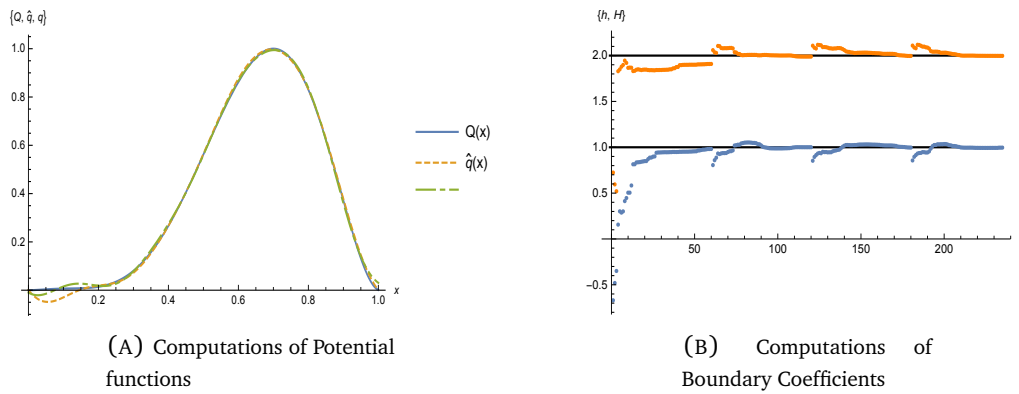


FIGURE 3.1. The graphics (a) and (b) represent the numerical results with five pairs of data for smooth $Q^1(x)$.

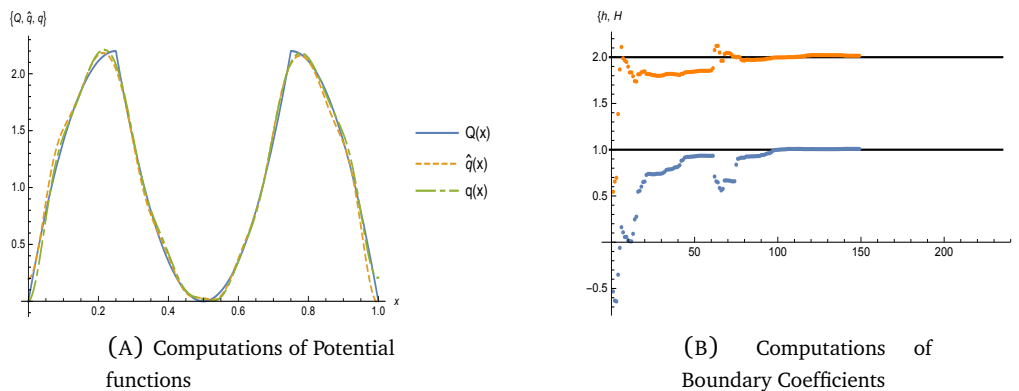


FIGURE 3.2. The graphics (a) and (b) represent the show results with ten pairs of data for non-smooth $Q^2(x)$.

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