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SOME PROPERTIES OF EQUIDISTRIBUTED AND WELL DISTRIBUTED SEQUENCES

Leila Miller-Van Wieren

ABSTRACT. Many authors studied properties related to distribution and summability of sequences of real numbers. In these studies, different types of limit points of a sequence were introduced and studied including statistical and uniform statistical cluster points of a sequence. In this paper, we aim to prove some new results about the nature of different types of limit points, this time connected to equidistributed and well distributed sequences.

1. INTRODUCTION

In recent years, many mathematicians studied properties of real valued sequences and different types of summability of a sequence such as statistical convergence, uniform statistical convergence and ideal convergence. Related to these types of convergence various types of limit points of a sequence were introduced, connected to the density of the subsequence with the given limit. Buck [4] , Agnew [1], Buck [5], Buck and Pollard [6], Dawson [7], Miller [12], Miller and Orhan [14], Zeager [22] have studied different types of summability and the relation between the summability of a sequence and its subsequences. Later on, in [2], [9], [10], [15], [16], [20], [21] , [3], [11], [18] different types

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of convergence of a sequence and the related summability of its subsequences were further studied. More recently, in one of his last papers [13], Miller specifically studied equidistributed sequences and their subsequences, coming up with some new and interesting results. In our recent paper [17], we tried to further extend Millers results by focusing on well distributed sequences, a more strict class of sequences. In this paper, we wish to further elaborate on some properties of equidistributed sequences and well distributed sequences related to different types of limit points.

2. Preliminaries

Now let us recall some known notions. Let $K \subseteq \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. If $m, n \in \mathbb{N}$, by $K(m,n)$ we denote the cardinality of the set of numbers i in K such that $m \leq i \leq n$. The numbers

$$\underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}, \ \bar{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set K, respectively. If $\underline{d}(K) = \bar{d}(K)$ then it is said that $d(K) = \underline{d}(K) = \bar{d}(K)$ is the asymptotic density of K. The uniform density of $K \subseteq \mathbb{N}$ has been defined as follows (see [2], [20]):

$$\underline{u}(K) = \lim_{n\to\infty} \frac{\min\limits_{i\geq 0} K\left(i+1, i+n\right)}{n}, \bar{u}(K) = \lim_{n\to\infty} \frac{\max\limits_{i\geq 0} K\left(i+1, i+n\right)}{n}$$

are respectively called the lower and the upper uniform density of the set K (the existence of these bounds is also mentioned in [2]). If $\underline{u}(K) = \bar{u}(K)$, then $u(K) = \underline{u}(K)$ is called the uniform density of K. It is clear that for each $K \subseteq \mathbb{N}$ we have

$$\underline{u}(K) \leq \underline{d}(K) \leq \bar{d}(K) \leq \bar{u}(K).$$

The concept of statistical convergence has been introduced by Fast [8] as follows: Let $x = \{x_n\}$ be a sequence of real numbers. The sequence x is said to be statistically convergent to a real number l provided that for every $\epsilon > 0$ we have $d(K_\epsilon) = 0$, where $K_\epsilon = \{n \in \mathbb{N} : |x_n - l| \geq \epsilon\}$. Now let us recall the concept of uniform statistical convergence related to uniform density. A sequence of real numbers $x = \{x_n\}$ is said to be uniformly statistically convergent to a real number l provided that for every $\epsilon > 0$ we have $u(K_\epsilon) = 0$, where K_ϵ is as just mentioned.

Related to these types of convergence, we can introduce different kinds of limit points of a sequence $x = \{x_n\}$ that will the topic of discussion of this paper (see [15], [16], [20], [21]).

Definition 2.1. *Given $x = \{x_n\}$, l is called a statistical cluster point of x if for every $\epsilon > 0$, the set $\{n : |x_n - l| < \epsilon\}$ does not have asymptotic density 0. l is called a uniform statistical cluster point of $x = \{x_n\}$ if for every $\epsilon > 0$, the set $\{n : |x_n - l| < \epsilon\}$ does not have uniform density 0.*

From the inequalities connecting asymptotic and uniform upper and lower density it is clear that any statistical cluster point of a sequence is also a uniform statistical cluster point of the sequence. Next we have some more strictly defined classes of limit points, namely statistical limit points and uniform statistical limit points.

Definition 2.2. *Given $x = \{x_n\}$, l is called a statistical limit point of x if there exists a sequence $\{n_k\}$, with $\bar{d}(\{n_k : k \in \mathbb{N}\}) > 0$ and $\lim_{k\to\infty} x_{n_k} = l$. l is called a uniform statistical limit point of x if there exists a sequence $\{n_k\}$, with $\bar{u}(\{n_k : k \in \mathbb{N}\}) > 0$ and $\lim_{k\to\infty} x_{n_k} = l$.*

From the inequalities connecting asymptotic and uniform upper and lower density it is clear that any statistical limit point of a sequence is also a uniform statistical limit point of the sequence.

It is also useful to mention that subsequences of a sequence x can be naturally identified with numbers $t \in (0, 1]$ written by a binary expansion with infinitely many 1's. Thus we can denote by $\{x(t)\}$ the subsequence of x corresponding to t.

3. EQUIDISTRIBUTED SEQUENCES

Equidistributed sequences, were first introduced by Herman Weyl, 100 years ago (see [19]). Recall the following definition.

Definition 3.1. *A sequence $x = \{x_n\}$ contained in $(0, 1]$ is said to be equally distributed if for every $[a, b]$, subinterval of $(0, 1]$,*

$$lim_{n\to\infty} \frac{|\{1 \le i \le n, x_i \in [a, b]\}|}{n} = m([a, b]).$$

(Trivially closed intervals in $(0, 1]$ can be replaced with all intervals in $(0, 1]$.)

Example 1. *Let $x = \{x_n\}$ where $x_n = [cn]$, where c is irrational and [r] is the fractional part of r. It is well known that this sequence is equidistributed.*

We next state a simple lemma that gives another way of stating the definition of equidistributed sequences. It follows clearly from the definition of asymptotic density so the proof is omitted.

Lemma 3.1. *A sequence $x = \{x_n\}$ contained in $(0, 1]$ is equally distributed if for every $[a, b]$, subinterval of $(0, 1]$, $d(\{n : x_n \in [a, b]\}) = m([a, b])\}$. Closed intervals in $(0, 1]$ can be replaced with all intervals in $(0, 1]$.*

We prove the following result regarding statistical and uniform statistical cluster points of an equidistributed sequence.

Theorem 3.1. *Suppose $x = \{x_n\}$ is a equidistributed sequence of reals in $(0, 1]$. Then $[0, 1]$ is the set of statistical cluster points of x as well as the set of uniform statistical cluster points of x.*

Proof. Let $l \in [0, 1]$ be arbitrary, $\epsilon > 0$ arbitrary. Since $\{x_n\}$ is equidistributed, by Lemma 3.1, $d(\{n : |x_n - l| < \epsilon\}) = m([0, 1] \bigcap (l - \epsilon, l + \epsilon)) > 0$. Hence l is a statistical cluster point of $\{x_n\}$. Since $[0, 1]$ is the set of all limit points of $\{x_n\}$ it follows that $[0, 1]$ is the set of statistical cluster points of $\{x_n\}$. As earlier mentioned, the set of uniform statistical cluster points of $\{x_n\}$ contains the set of statistical cluster points of $\{x_n\}$, so the theorem is proved. □

However, next we show that the situation regarding statistical limit points of an equidistributed sequence is the opposite.

Theorem 3.2. *Suppose $x = \{x_n\}$ is a equidistributed sequence of reals in $(0, 1]$. Then x has no statistical limit points.*

Proof. Since $[0, 1]$ is the set of (all) limit points of x, let $l \in [0, 1]$ be arbitrary fixed. Suppose, contrary to what we want to show, that l is a statistical limit point of x. Then there exists a subsequence of $\{x_n\}$, $\{x_{n_k}\}$, such that $x_{n_k} \to l$ and $\bar{d}(\{n_k : k \in \mathbb{N}\}) = \epsilon$, for some $\epsilon > 0$. Let $I = [0, 1] \bigcap (l - \frac{\epsilon}{4}, l + \frac{\epsilon}{4})$. Then $m(I) \leq \frac{\epsilon}{2}$. Since $x_{n_k} \to l$, all but finitely many $x_{n_k} \in I$. From $\bar{d}(\{n_k : k \in \mathbb{N}\}) = \epsilon$ we conclude that $\bar{d}(\{n_k : x_{n_k} \in I\}) = \epsilon > m(I)$, and consequently that $\bar{d}(\{n : x_n \in I\}) \geq \bar{d}(\{n_k : x_{n_k} \in I\}) > m(I)$, a contradiction since x is equidistributed and therefore by Lemma 3.1, $d(\{n : x_n \in I\}) = m(I)$. This completes the proof. □

4. Results on well distributed sequences

In summability the concept of uniform statistical density and convergence are introduced as more strict than asymptotic density and statistical convergence. Parallel with these notions, from the concept of equidistributed sequences, we move to the more strict notion of well distributed sequences.

Definition 4.1. *A sequence $x = \{x_n\}$ contained in $(0,1]$ is said to be well distributed if for every $[a,b]$, subinterval of $(0,1]$,*

$$lim_{n\to\infty}\frac{|\{m+1 \leq i \leq m+n, x_i \in [a,b]\}|}{n} = m([a,b]).$$

uniformly in m. (Trivially closed intervals in $(0,1]$ can be replaced with all intervals in $(0,1]$.)

Clearly, well distributed sequences are equidistributed as well.

Remark 4.1. *The sequence described in Example 1 is well distributed (not just equidistributed). Of course there are examples of sequences that are equidistributed but not well distributed. We will give such an example at the end of this section.*

Once again the definition of well distributed sequences can be restated in the following lemma. It follows from the definition of uniform density so the proof is omitted.

Lemma 4.1. *A sequence $x = \{x_n\}$ contained in $(0,1]$ is well distributed if for every $[a,b]$, subinterval of $(0,1]$, $u(\{n : x_n \in [a,b]\}) = m([a,b])$. Closed intervals in $(0,1]$ can be replaced with all intervals in $(0,1]$.*

Now we can formulate some results concerning well distributed sequences, analogous to the results in the previous section.

Theorem 4.1. *Suppose $x = \{x_n\}$ is a well distributed sequence of reals in $(0,1]$. Then $[0,1]$ is the set of statistical cluster points of x as well as the set of uniform statistical cluster points of x.*

Proof. This theorem is a direct corollary of Theorem 3.1. Namely since x is well distributed in $(0,1]$, it is consequently equidistributed in $(0,1]$, so the conclusion follows. □

Next we have the following analogue of Theorem 3.2, and this time the conclusion is analogous but different.

Theorem 4.2. *Suppose $x = \{x_n\}$ is a well distributed sequence of reals in $(0, 1]$. Then x has no uniform statistical limit points, and consequently no statistical limit points.*

Proof. Since $[0, 1]$ is the set of (all) limit points of x, let $l \in [0, 1]$ be arbitrary fixed. Suppose, contrary to what we want to show, that l is a uniform statistical limit point of x. Then there exists a subsequence of $\{x_n\}$, $\{x_{n_k}\}$, such that $x_{n_k} \to l$ and $\bar{u}(\{n_k : k \in \mathbb{N}\}) = \epsilon$, for some $\epsilon > 0$. Let $I = [0, 1] \bigcap (l - \frac{\epsilon}{4}, l + \frac{\epsilon}{4})$. Then $m(I) \leq \frac{\epsilon}{2}$. Since $x_{n_k} \to l$, all but finitely many $x_{n_k} \in I$. From $\bar{u}(\{n_k : k \in N\}) = \epsilon$ we conclude that $\bar{u}(\{n_k : x_{n_k} \in I\}) = \epsilon > m(I)$, and consequently that $\bar{u}(\{n : x_n \in I\}) \geq \bar{u}(\{n_k : x_{n_k} \in I\}) > m(I)$, a contradiction since x is well distributed and therefore by Lemma 4.1, $u(\{n : x_n \in I\}) = m(I)$. This completes the proof that x has no uniform statistical limit points. Since the set of statistical limit points is contained in the set of uniform statistical limit points, the proof is complete. □

We see that the conclusion in Theorem 4.2 is stronger than in Theorem 3.2. In fact, we give an example to show that a sequence that is equidistributed, but not well distributed in $(0, 1]$ may have a uniform statistical point.

Example 2. *Suppose that $x = \{x_n\}$ is any fixed equidistributed sequence (for instance the one in Example 1). We construct a new sequence $y = \{y_n\}$ by inserting some $1\text{J}s$ into the sequence x the following way:*

$$x_1, x_2, 1, x_3, x_4, x_5, x_6, 1, 1, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, 1, 1, 1, \ldots,$$

i.e., we follow 2^n consecutive members of x by n consecutive $1's$ and continue this way for $n = 1, 2, 3 \ldots$. It is easy to see that the new sequence y is still equidistributed since the asymptotic density of the inserted $1's$ is 0. However 1 is a uniform statistical limit point of y since $\bar{u}(\{n : y_n = 1\}) = 1$ as y contains arbitrarily long strings of consecutive $1's$. Hence y is an equidistributed sequence that has a uniform statistical limit point.

5. Some additional connections

In his paper [13], Miller proved the following theorem regarding the size of the set of equidistributed subsequences of a sequence. Additionally in [17] we obtained a corollary of it.

Theorem 5.1. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0, 1]$. The set of $t \in (0, 1]$ for which $x(t)$ is equidistributed is meager (first Baire category).*

Corollary 5.1. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0, 1]$. The set of $t \in (0, 1]$ for which $x(t)$ is well distributed is meager.*

In light of the results about statistical and uniform statistical limit points, we can offer a new proof for Millers result. For this purpose we also recall an earlier result of the author and co-authors, see [11].

Theorem 5.2. *Suppose $x = \{x_n\}$ is a bounded sequence of reals and L is the set of its limit points. Then the set of $t \in (0, 1]$ for which $x(t)$ has all $l \in L$ as statistical (and consequently uniform statistical) limit points is comeager.*

Now we can give a new and short proof of Theorem 5.1 and Corollary 5.1.

Proof. Suppose $x = \{x_n\}$ is a sequence of reals in $(0, 1]$. If L denotes the set of limit poits of x, L is a non-empty closed subset of $[0, 1]$. By Theorem 5.2, the set of $t \in (0, 1]$ for which $x(t)$ has all $l \in L$ as statistical limit points is comeager. Therefore the set of $t \in (0, 1]$ for which $x(t)$ has no statistical limit points is meager. From Theorem 3.2 we conclude that the set of $t \in (0, 1]$ for which $x(t)$ is equidistributed is contained in the set of $t \in (0, 1]$ for which $x(t)$ has no statistical limit points, and therefore is meager. This completes the proof of Theorem 5.1 and consequently Corollary 5.1. □

Finally we can bring up one more new insight related to the connection of well distributed sequences and uniform statistical limit points, this time using Lebesgue measure. In [17] we proved the following:

Theorem 5.3. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0, 1]$. The set of $t \in (0, 1]$ for which $x(t)$ is well distributed has Lebesgue measure 0.*

The same of course is not true if we replace well distributed with equidistributed, see [13], [17]. However looking closely at the argument in the proof of Theorem 5.3 we can obtain the following new result.

Theorem 5.4. *Suppose $x = \{x_n\}$ is a sequence of reals in $(0, 1]$. Then there exists $l \in [0, 1]$ such that the set of $t \in (0, 1]$ for which l is a uniform statistical limit point of $x(t)$ has Lebesgue measure 1.*

Proof. Clearly $\bar{u}(\{i, x_i \in [0, \frac{1}{2}]\} \geq \frac{1}{2}$ or $\bar{u}(\{i, x_i \in [\frac{1}{2}, 1]\} \geq \frac{1}{2}$. Without loss of generality we can assume that $\bar{u}(\{i, x_i \in [0, \frac{1}{2}]\} \geq \frac{1}{2}$.

In the proof of Theorem 5.3, we showed that in this case, for $n \in \mathbb{N}$,

$$X_n = \{t \in (0,1], x(t) \text{contains } n \text{ consecutive terms in } [0, \frac{1}{2}]\}$$

has measure 1 (in that proof a half-open interval was used, but using a closed interval does not change the argument).

We can continue the argument by dividing the interval $[0, \frac{1}{2}]$ into two halves $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$ and in a complete analogous way as in the proof of Theorem 5.3 show that for one of these two intervals:

For any $m \in \mathbb{N}$, the set of $t \in (0,1]$ for which $x(t)$ contains m consecutive terms inside the interval has measure 1.

If we continue dividing the interval into two halves inductively, we obtain a nested sequence of intervals I_n of length $\frac{1}{2^n}$, $I_{n+1} \subseteq I_n$ such that: For any m, n the set of $t \in (0,1]$ for which $x(t)$ contains m consecutive terms inside I_n has measure 1. Now introduce the following notation. Let

$$Y_n = \{t \in (0,1], x(t) \text{contains } n \text{ consecutive terms in } I_n\}.$$

From the above discussion for any n, Y_n has Lebesgue measure 1, and consequently $Y = \bigcap_n Y_n$ has measure 1. Now let $l = \bigcap_n I_n$ (I_n are nested closed intervals with lengths going to 0). Suppose $t \in Y$. It is easy to construct a subsequence of $x(t)$ that converges to l with upper statistical uniform density 1 (simply take the union of segments of n consecutive terms of $x(t)$ contained in I_n, over n). Hence for any $t \in Y$, l is a uniform statistical limit point of $x(t)$. Since Y has measure 1, the theorem is proved. □

Theorem 5.3 and Theorem 5.4 share a similar proof and Theorem 5.3 could now be seen as a corollary of Theorem 5.4. However it is also important to mention that the analogue of Theorem 5.4 for statistical limit points does not hold. Namely as Miller proved in [13], if $x = \{x_n\}$ is a equidistributed sequence of reals in $(0,1]$, then the set of $t \in (0,1]$ for which $x(t)$ is equidistributed has Lebesgue measure 1. As already discussed in Theorem 3.2, equidistributed sequences have no statistical limit points. Hence, if $x = \{x_n\}$ is a equidistributed sequence of reals in $(0,1]$, then the set of $t \in (0,1]$ for which $x(t)$ has no statistical limit points has Lebesgue measure 1. The discussion here is also related to results in [15] and [16].

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Faculty of Engineering and Natural Sciences
International University of Sarajevo
Hrasnička 15, Sarajevo, Bosnia and Herzegovina
Email address: lmiller@ius.edu.ba