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Γ-LANGUAGES AND Γ-AUTOMATA

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ABSTRACT. The aim of this paper is to extend the notion of an automaton as a triple made of a set of states, a free monoid on some set, and an action of this monoid on the set of states, to the case where the free monoid is replaced by a free Γ-monoid, and the action is replaced by the action of this Γ-monoid on the set of states. We call the respective triple a Γ-automaton. This concept leads to another new concept, that of a Γ-language, which is a subset of a free Γ-monoid. Also we define recognizable Γ-languages and prove that they are exactly those Γ-languages that are recognized by a finite Γ-automaton. In the end, in analogy with the standard theory, we relate the recognizability of a Γ-language with the concept of division of semigroups.

1. INTRODUCTION AND PRELIMINARIES

Let S and Γ be two non empty sets. Every map from $S\times\Gamma\times S$ to S will be called a Γ-multiplication in S and is denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [4], a Γ-semigroup S is an ordered pair $(S, (\cdot)_\Gamma)$ where S and Γ are non empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S which satisfies the following property

$$\forall(a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

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One can thus regard Γ-semigroups, as semigroups with several multiplications, one for each "operator" $\gamma \in \Gamma$. It is also clear that for each such $\gamma \in \Gamma$ we have already a plain semigroup $S_\gamma = (S, \circ_\gamma)$ defined where $\circ_\gamma$ is given by $x \circ_\gamma y = x\gamma y$. If $(S, (\cdot)_\Gamma)$ and $(T, (\cdot)_\Gamma)$ are two Γ-semigroups, then a map $\varphi : S \to T$ is called a Γ-homomorphism if for every $x, y \in S$ and $\gamma \in \Gamma$ we have $\varphi(x\gamma y) = \varphi(x)\gamma\varphi(y)$. It is folklore the definition of the free Γ-semigroup relative to Γ and a nonempty set A. For sake of completeness we will describe it below. We denote by A_Γ^+ the set A together with the set of all strings

$$a_1\gamma_1 a_2 \ldots a_{n-1}\gamma_{n-1}a_n,$$

where $n \geq 2$, $a_i \in A$ for $1 \leq i \leq n$, and $\gamma_i \in \Gamma$ for all $1 \leq i \leq n-1$. A_Γ^+ becomes a Γ-semigroup if we define for all $a_1\gamma_1 a_2 \ldots a_{n-1}\gamma_{n-1}a_n \in A_\Gamma^+$, $a'_1\gamma'_1 a'_2 \ldots a'_{m-1}\gamma'_{m-1}a'_m \in A_\Gamma^+$ and $\gamma \in \Gamma$, a Γ-multiplication by

$$\begin{aligned}&(a_1\gamma_1 \ldots a_{n-1}\gamma_{n-1}a_n)\gamma(a'_1\gamma'_1 \ldots a'_{m-1}\gamma'_{m-1}a'_m)\\ &= a_1\gamma_1 \ldots a_{n-1}\gamma_{n-1}a_n\gamma a'_1\gamma'_1 \ldots a'_{m-1}\gamma'_{m-1}a'_m.\end{aligned}$$

It can be proved very easily that A_Γ^+ satisfies the following universal property. For every Γ-semigroup $(S, (\cdot)_\Gamma)$ and every map $f : A \to A$, there is a unique Γ-homomorphism $\varphi : A_\Gamma^+ \to S$ such that $\varphi|_A = f$. We remark here that there is no known notion of free Γ-monoids as there are no Γ-monoids at all as we will explain shortly below. Indeed, if there is a Γ-semigroup $(S, (\cdot)_\Gamma)$ with a unit e, then for every $\alpha, \beta \in \Gamma$ and $x, y \in S$ we would have

$$x\alpha y = x\alpha(e\beta y) = (x\alpha e)\beta y = x\beta y.$$

But this would imply that the multiplications α and β coincide, and so we would end up with a semigroup with a unit element.

The reason we are interested in free Γ-monoids is our intention to see how notions such as languages and finite state machines, which are closely related to free monoids, generalize to the case of Γ-semigroups.

Below we give a few basic definitions from the theory of languages and machines. Let A be a nonempty finite alphabet. We say that L is a language with letters from A if $L \subseteq A^*$ where A^* is the free monoid on A. We say that a monoid M recognizes L if there is a monoid morphism $\eta : A^* \to M$ and a subset $P \subseteq M$ such that $\eta^{-1}(P) = L$. A language is called recognizable if there is a monoid M which recognizes L. Let A and Q be finite nonempty sets and let $\cdot : Q \times A^* \to Q$

be a monoid action. In such a case we call the triple $\mathcal{A} = (Q, A, \cdot)$ a finite state automaton. The set Q is called the set of states and A is called the alphabet. It is clear that for every $w \in A^*$ we have a map $[w] : Q \to Q$ defined by $q \mapsto q \cdot w$. The composition $[v] \circ [u]$ of any two such maps gives in fact $[vu]$, a map of the same sort, therefore we have a submonoid $M(\mathcal{A})$ of the full transformation monoid $\mathcal{T}(Q)$ whose unit is $[1]$. This monoid $M(\mathcal{A})$ is called the transition monoid of $\mathcal{A}$. Also there is an obvious morphism $A^* \to M(\mathcal{A})$. It is interesting to mention that the languages that are recognized from the transition monoid $M(\mathcal{A})$ and those that are recognized from the automaton $\mathcal{A} = (Q, A, \cdot)$ match to each other. We recall here that $L \subseteq A^*$ is recognized by $\mathcal{A} = (Q, A, \cdot)$ if there is a $q_0 \in Q$ (called the initial state), and $F \subseteq Q$ (called the set of final states) such that $u \in L$ if and only if $q_0 \cdot u \in F$.

Of a very importance notion in the theory is the syntactic congruence $\sim_L$ of a language $L \subseteq A^*$. It is defined by setting $u \sim_L v$ if and only if $C(u) = C(v)$ where

$$C(u) = \{(x, y) \in A^* \times A^* : xuy \in L\}.$$

The quotient monoid $A^*/\sim_L$ is called the syntactic monoid of L and it is denoted by $M(L)$. It is proved in [3] that a monoid M recognizes L if and only if $M(L)$ divides M.

For everything related with semigroups, languages and machines which is not explained here we refer the reader to [1] and [3].

2. Free Γ-monoids and Γ-languages

Throughout this paper we assume that Γ and A are two nonempty sets and $\lambda : \Gamma \to A$ is an injection. For every $\gamma \in \Gamma$ we write a_γ instead of $\lambda(\gamma)$. We have already defined the free Γ-semigroup A^+_Γ on A. Now, we let A^*_Γ, or simply A^* if no confusion arises, the quotient of A^+_Γ by the equivalence generated by pairs

$$(a_\gamma\gamma(a_1\gamma_1 \dots \gamma_{n-1}a_n), a_1\gamma_1 \dots \gamma_{n-1}a_n), ((a_1\gamma_1 \dots \gamma_{n-1}a_n)\gamma a_\gamma, a_1\gamma_1 \dots \gamma_{n-1}a_n)$$

for all strings $a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n$ and $\gamma \in \Gamma$. Note that A^* is still a Γ-semigroup where the Γ-multiplication is defined by

$$cls(a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n)\gamma cls(a'_1\gamma'_1 \dots a'_{m-1}\gamma'_{m-1}a'_m) = cls(a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n\gamma a'_1\gamma'_1 \dots a'_{m-1}\gamma'_{m-1}a'_m).$$

Also we note that for each $\gamma \in \Gamma$, $A^*_\gamma = (A^*, \circ_\gamma)$ is a monoid with unit a_γ. We call A^* the free Γ-monoid on the set A.

The Γ-semigroup A^* has the following universal property in the category of Γ-semigroups (S, Γ) where each S_γ is a monoid with unit e_γ and for every pair $(\gamma, \gamma') \in \Gamma \times \Gamma$ we have $e_\gamma \gamma' e_\gamma = e_\gamma$. For every such Γ-semigroup (S, Γ) and every map $f : A \to S$ such that $f(a_\gamma) = e_\gamma$, there is a unique Γ-homomorphism $\varphi : (A^*, \Gamma) \to (S, \Gamma)$ such that $\varphi|_A = f$. Indeed, f extends to $\phi : A^+_\Gamma \to S$ where for every $a \in A$,

$$\phi(a) = f(a),$$

and for every string $a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n$ we have

$$\phi(a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n) = f(a_1)\gamma_1 \dots f(a_{n-1})\gamma_{n-1}f(a_n).$$

It is easy to see that ϕ is the only Γ-homomorphism which matches with f in A. Further we see that ϕ agrees with the defining relations of A^*, for if

$$(a_\gamma\gamma(a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n), a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n)$$

is such a relation, then

$$\begin{aligned}
\phi(a_\gamma\gamma(a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n)) &= f(a_\gamma)\gamma(f(a_1)\gamma_1 \dots f(a_{n-1})\gamma_{n-1}f(a_n)) \\
&= e_\gamma\gamma(f(a_1)\gamma_1 \dots f(a_{n-1})\gamma_{n-1}f(a_n)) \\
&= f(a_1)\gamma_1 \dots f(a_{n-1})\gamma_{n-1}f(a_n) \\
&= \phi(a_1\gamma_1 \dots a_{n-1}\gamma_{n-1}a_n).
\end{aligned}$$

The proof for the other relation $((a_1\gamma_1 \dots \gamma_{n-1}a_n)\gamma a_\gamma, a_1\gamma_1 \dots \gamma_{n-1}a_n)$ is similar and is omitted. It follows that ϕ induces a Γ-homomorphism $\varphi : (A^*, \Gamma) \to (S, \Gamma)$ such that $\varphi|_A = f$, which likewise ϕ, is the unique Γ-homomorphism with the above property.

Definition 2.1. *Let A^* be the free Γ-monoid on A. Any nonempty subset $L \subseteq A^*$ is called a Γ-language.*

Definition 2.2. *We say that a Γ-language $L \subseteq A^*$ is recognized by a monoid M if there is a subset $\hat{L} \subseteq \Gamma \times A^*$ such that $\pi_2(\hat{L}) = L$ where π_2 is the projection in the second coordinate, a semigroup morphism $\eta : \Gamma \times A^* \to M$ such that for every $\gamma \in \Gamma$, $\eta(\gamma, a_\gamma) = 1_M$, and a subset $P \subseteq M$, such that $\hat{L} = \eta^{-1}(P)$.*

Definition 2.3. *We say that a* Γ*-language* $L \subseteq A^*$ *is recognizable if it is recognized by a finite monoid.*

3. The definition of a Γ-automaton and a basic property

Definition 3.1. *Let* A *be a finite alphabet and* Γ *be a nonempty set such that there is an injection* $\lambda : \Gamma \to A$*, and let* A^* *be the free* Γ *monoid on* A *defined with these data. We call a triple* $\mathcal{A} = (Q, A, \Gamma)$ *a* Γ*-automaton if there is a map*

$$\cdot : Q \times \Gamma \times A^* \to Q$$

$$(q, \gamma, w) \mapsto q\gamma w,$$

which satisfies the following properties:

(i) *For every* $q \in Q$ *and* $\gamma \in \Gamma$*,* $q\gamma a_\gamma = q$*;*
(ii) *For every* $q \in Q$*,* $\alpha, \beta \in \Gamma$ *and* $u, v \in A^*$*,* $(q\alpha u)\beta v = q\alpha(u\beta v)$*.*

The set Q *is called the set of states of* $\mathcal{A}$*. When* Q *is finite, we say that* $\mathcal{A}$ *is a finite state* Γ*-automaton. We remark here that the action of* A^* *on the set of states* Q *is nothing but the one defined in [5] apart from the fact that we do not assume the existence of a unit element for* A^**.*

Definition 3.2. *Let* $\mathcal{A} = (Q, A, \Gamma)$ *be a* Γ*-automaton. We say that a* Γ*-language* $L \subseteq A^*$ *is recognized by* $\mathcal{A}$ *if there is a state* $q_0 \in Q$ *(called the initial state) and a set of states* F *(called the set of final states) such that* $u \in L$ *if and only if there is* $\gamma \in \Gamma$ *such that* $q_0\gamma u \in F$*.*

Every Γ-automaton $\mathcal{A} = (Q, A, \Gamma)$ gives rise to a submonoid $\mathcal{M}_\Gamma(\mathcal{A})$ of the full transformation monoid $\mathcal{T}(Q)$ of the set Q. The elements of $\mathcal{M}_\Gamma(\mathcal{A})$ are maps

$$[\gamma, w] : Q \to Q \text{ such that } q \mapsto q\gamma w,$$

for every $\gamma \in \Gamma$ and $w \in A^*$. The composition $[\gamma', w'] \circ [\gamma, w]$ of two such maps is the map $[\gamma, w\gamma' w']$ since for every $q \in Q$,

$$\begin{aligned}([\gamma', w'] \circ [\gamma, w])(q) &= [\gamma', w'](q\gamma w)\\ &= (q\gamma w)\gamma' w'\\ &= q\gamma(w\gamma' w')\\ &= [\gamma, w\gamma' w'](q).\end{aligned}$$

This shows that $\mathcal{M}_\Gamma(\mathcal{A})$ is closed under $\circ$, and therefore a subsemigroup of $\mathcal{T}(Q)$. Note that for every $\gamma \in \Gamma$, the maps $[\gamma, a_\gamma]$ coincide and equal to id_Q, so $\mathcal{M}_\Gamma(\mathcal{A})$ is a submonoid of $\mathcal{T}(Q)$ which will be called the transition monoid of the Γ-automaton $\mathcal{A}$.

On the other hand we can consider the set $\Gamma \times A^*$ equipped with the multiplication

$$(\gamma', w') \cdot (\gamma, w) = (\gamma, w\gamma' w').$$

It forms a semigroups since

$$\begin{aligned}((\gamma'', w'') \cdot (\gamma', w')) \cdot (\gamma, w) &= (\gamma', w'\gamma'' w'') \cdot (\gamma, w)\\ &= (\gamma, w\gamma' w'\gamma'' w'')\\ &= (\gamma'', w'') \cdot (\gamma, w\gamma' w')\\ &= (\gamma'', w'') \cdot ((\gamma', w') \cdot (\gamma, w)).\end{aligned}$$

Finally, the map

$$\eta : \Gamma \times A^* \to \mathcal{M}_\Gamma(\mathcal{A}) \text{ such that } (\gamma, w) \mapsto [\gamma, w]$$

is a homomorphism of semigroups since

$$\begin{aligned}\eta((\gamma', w') \cdot (\gamma, w)) &= \eta((\gamma, w\gamma' w'))\\ &= [\gamma, w\gamma' w']\\ &= [\gamma', w'] \circ [\gamma, w]\\ &= \eta((\gamma', w')) \circ \eta((\gamma, w)).\end{aligned}$$

Remark here that η has an important property, it sends every (γ, a_γ) to the unit of $\mathcal{M}_\Gamma(\mathcal{A})$. This follows from our previous remark that $[\gamma, a_\gamma] = id_Q$. We call η the canonical morphism of $\mathcal{A}$.

Theorem 3.1. *A Γ-language $L \subseteq A^*$ is recognized by a Γ-automaton $\mathcal{A} = (Q, A, \Gamma)$ if it is recognized by the transition monoid $\mathcal{M}_\Gamma(\mathcal{A})$ of this Γ-automaton. Moreover, L is recognized by a finite Γ-automaton if and only if it is recognizable.*

Proof. Suppose that L is recognized by $\mathcal{A}$ and let q_0 be the initial state and F the set of final states. Consider now the set

$$\hat{L} = \{(\gamma, u) \in \Gamma \times A^* : q_0\gamma u \in F\}.$$

From Definition 3.2 we see that $\pi_2(\hat{L}) = L$. Let now $P = \eta(\hat{L})$ where $\eta : \Gamma \times A^* \to \mathcal{M}_\Gamma(\mathcal{A})$ is the canonical morphism of the automaton. If we show that $\eta^{-1}(P) = \hat{L}$, then from Definition 2.2 it follows that L is recognized by the monoid $\mathcal{M}_\Gamma(\mathcal{A})$. Let $(\gamma, u) \in \eta^{-1}(P)$, hence $[\gamma, u] \in P = \eta(\hat{L})$ and then there is $(\gamma', u') \in \hat{L}$ such that $[\gamma, u] = [\gamma', u']$. It follows that

$$q_0\gamma u = [\gamma, u](q_0) = [\gamma', u'](q_0) = q_0\gamma' u'.$$

But $u' \in L$, hence $q_0\gamma' u' \in F$ and consequently $q_0\gamma u \in F$. This proves that $(\gamma, u) \in \hat{L}$. We have thus proved that $\eta^{-1}(P) \subseteq \hat{L}$, and since the reverse inclusion is obvious, we have the equality. If it happens that $\mathcal{A}$ is finite, then $\mathcal{M}_\Gamma(\mathcal{A})$ is finite and L is thus recognizable.

Conversely, if L is recognizable, then (i) there exists a finite monoid $(M, \cdot)$, (ii) there is some $\hat{L} \subseteq \Gamma \times A^*$ with $\pi_2(\hat{L}) = L$, (iii) there is $P \subseteq M$, and (iv) a morphism $\eta : \Gamma \times A^* \to M$, such that $\eta(\gamma, a_\gamma) = 1_M$ and $\eta^{-1}(P) = \hat{L}$. With these data we define a finite state Γ-automaton $\mathcal{A} = (M, A, \Gamma)$ where for each $m \in M$, $\gamma \in \Gamma$ and $u \in A^*$, we have $m\gamma u = \eta(\gamma, u) \cdot m$. It is easy to check that $\mathcal{A} = (M, A, \Gamma)$ is indeed a Γ-automaton. Take now 1_M as the initial state and P as the set of final states of $\mathcal{A}$. For this choice we see that $\mathcal{A}$ recognizes L since

$$\begin{aligned} u \in L &\Leftrightarrow \exists\gamma \in \Gamma, (\gamma, u) \in \hat{L} = \eta^{-1}(P) \\ &\Leftrightarrow \exists\gamma \in \Gamma, \eta(\gamma, u) \in P \\ &\Leftrightarrow \exists\gamma \in \Gamma, 1_M\gamma u \in P, \end{aligned}$$

and then Definition 3.2 applies. □

4. Recognizable Γ-languages

We recall from [2] that for some fixed $\gamma_0 \in \Gamma$ it is defined a semigroup Σ_{γ_0} which is a factor of the free semigroup $F(\Gamma \cup A^*)$ on $\Gamma \cup A^*$ by the congruence generated from relations of the form

$$((\alpha, \beta), \alpha), ((u, \alpha, v), u\alpha v), ((u, v), u\gamma_0 v),$$

for every $\alpha, \beta \in \Gamma$ and $u, v \in A^*$. Letting $\nu : F(\Gamma \cup A^*) \to \Sigma_{\gamma_0}$ be the quotient morphism, we can regard $\nu(A^*)$ as a subsemigroup of Σ_{γ_0} where the multiplication of two classes $\nu(a_1\alpha_1 \ldots \alpha_i a_i)$, $\nu(b_1\beta_1 \ldots \beta_j b_j)$ of $\nu(A^*)$ is given by

$$\nu(a_1\alpha_1 \ldots \alpha_i a_i) \cdot \nu(b_1\beta_1 \ldots \beta_j b_j) = \nu(a_1\alpha_1 \ldots \alpha_i a_i\gamma_0 b_1\beta_1 \ldots \beta_j b_j).$$

Let $L \subseteq A^*$ be a Γ-language. For every $s \in \nu(A^*)$ we let

$$C(s) = \{(x, y) \in \nu(A^*) \times \nu(A^*) : xsy \in \nu(L)\},$$

and in similarity with [3] we define a relation $\sim_L$ in $\nu(A^*)$ by setting

$$(s,t) \in \sim_L \text{ if and only if } C(s) = C(t).$$

It is straightforward that $\sim_L$ is an equivalence relation and that it is compatible with the multiplication in $\nu(A^*)$. We call $\sim_L$ the syntactic congruence in $\nu(A^*)$ defined by L. Consider now that quotient semigroup $A^*(L) = \nu(A^*)/\sim_L$ and let $\eta : \nu(A^*) \to A^*(L)$ be the canonical epimorphism.

Definition 4.1. *We say that a Γ-language $L \subseteq A^*$ is recognized by a semigroup M if there is a semigroup morphism $\eta : \nu(A^*) \to M$, and a subset $P \subseteq M$, such that $\eta^{-1}(P) = \nu(L)$.*

Theorem 4.1. *Let $L \subseteq A^*$ be a Γ-language. Then the following hold true.*

(1) *A semigroup M recognizes L if and only if $A^*(L)$ divides M.*
(2) *If M recognizes L and if M divides N, then N recognizes L.*

Proof. (a) We prove first that $A^*(L)$ recognizes L. For this we consider the canonical morphism $\eta : \nu(A^*) \to A^*(L)$: $u \mapsto \eta(u)$ and put $P = \eta(\nu(L))$. If we show that $\eta^{-1}(P) = \nu(L)$ then it follows from Definition 4.1 that $A^*(L)$ recognizes L. The inclusion $\nu(L) \subseteq \eta^{-1}(P)$ is obvious. To prove the reverse inclusion we let $\nu(u) \in \eta^{-1}(P)$, then there is $\nu(v) \in \nu(L)$ such that $\eta(\nu(u)) = \eta(\nu(v))$. From the definition of $\sim_L$ we see that $C(\nu(u)) = C(\nu(v))$. But $(\nu(a_{\gamma_0}), \nu(a_{\gamma_0})) \in C(\nu(v))$ since

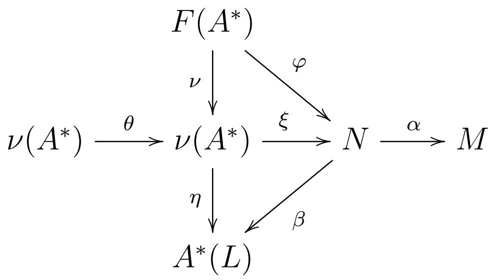
$$\begin{aligned}\nu(a_{\gamma_0})\nu(v)\nu(a_{\gamma_0}) &= \nu(a_{\gamma_0})\gamma_0\nu(v)\gamma_0\nu(a_{\gamma_0})\\ &= \nu(a_{\gamma_0}\gamma_0 v\gamma_0 a_{\gamma_0})\\ &= \nu(v) \in \nu(L),\end{aligned}$$

consequently we have that $(\nu(a_{\gamma_0}), \nu(a_{\gamma_0})) \in C(\nu(u))$ and then

$$\begin{aligned}L \ni \nu(a_{\gamma_0})\nu(u)\nu(a_{\gamma_0}) &= \nu(a_{\gamma_0})\gamma_0\nu(u)\gamma_0\nu(a_{\gamma_0})\\ &= \nu(a_{\gamma_0}\gamma_0 u\gamma_0 a_{\gamma_0})\\ &= \nu(u).\end{aligned}$$

This shows that $\eta^{-1}(P) \subseteq \nu(L)$ and we are done.

(b) Assume now that $A^*(L)$ divides M, then there is a semigroup N, an injective morphism $\alpha : N \to M$ and a surjective morphism $\beta : N \to A^*(L)$. It follows that there is a morphism $\varphi : F(A^*) \to N$ such that $\beta\varphi = \eta\nu$ where $\nu : F(A^*) \to \nu(A^*)$ is the canonical morphism. Consider the following commutative diagram



where ξ is defined on generators by $\xi(\nu(u)) = \varphi(u)$ with $u \in A^*$ and $\xi(\nu(\gamma)) = \varphi(\gamma)$ with $\gamma \in \Gamma$, and $\theta = 1_{\nu(A^*)}$. It is easy to check that ξ is a well defined morphism of semigroups. Since $L \subseteq A^* \subseteq F(A^*)$, we can define $P := \alpha\beta^{-1}\eta\nu(L)$, and then chasing the diagram we see that

$$
\begin{aligned}
\theta^{-1}\xi^{-1}\alpha^{-1}(P) &= \theta^{-1}\xi^{-1}\alpha^{-1}(\alpha\beta^{-1}\eta\nu(L)) \\
&= \theta^{-1}\xi^{-1}\beta^{-1}\eta\nu(L) && (\alpha \text{ is mono}) \\
&= \theta^{-1}\eta^{-1}\eta\nu(L) && (\text{since } \eta = \beta\xi) \\
&= \theta^{-1}(\nu(A^*) \cap \nu(L)) && (\theta \text{ is mono and } \mathrm{Im}(\theta) = \nu(A^*)) \\
&= \theta^{-1}(\nu(L)) \\
&= \nu(L).
\end{aligned}
$$

This shows that $\nu(L)$ satisfies the conditions of Definition 4.1 relative to the morphism $\alpha\xi\theta$, thus M recognizes L.

(c) Suppose that M recognizes L, hence there is a morphism $\psi : \nu(A^*) \to M$ and a subset $P \subseteq M$ such that $\psi^{-1}(P) = \nu(L)$. If we let $N = \mathrm{Im}(\Psi)$, N is a subsemigroup of M and to prove our claim it remains to find a surjective morphism $\pi : N \to A^*(L)$. To achieve this we prove first the implication

$$((\forall u, v \in \nu(A^*)), (\psi(u) = \psi(v))) \Rightarrow (u \sim_L v).$$

Let $(x, y) \in C(u)$, then $xuy \in \nu(L)$. Since both $x, y \in \nu(A^*)$, then

$$xuy = x\gamma_0 u\gamma_0 y.$$

From the assumption that $\psi(u) = \psi(v)$ it follows that

$$\psi(x\gamma_0 v\gamma_0 y) = \psi(x\gamma_0 u\gamma_0 y) \in \psi(\nu(L)) = P,$$

which implies that $xvy = x\gamma_0 v\gamma_0 y \in \psi^{-1}(P) = \nu(L)$, and as a result $xvy \in C(v)$. Thus we have the inclusion $C(u) \subseteq C(v)$. The reverse inclusion is proved similarly and we have $u \sim_L v$. Now proposition 1.4 of [3] implies the existence of a surjective morphism $\pi : N \to A^*(L)$. So in conclusion we have that $A^*(L)$ divides M.

Combining (b) and (c) one gets (1), while (2) is a direct consequence of (1). □

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