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MATRIX STUDY OF THE EQUATION OF SOLID RIGID MOTIONS

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ABSTRACT. In this article, we described the equations of motion of a rigid solid by a matrix formulation. The matrices contained in our movement description are homogeneous to the same unit. Inertial characteristics are met in a 4x4 positive definite symmetric matrix called "tensor generalized Poinsot." This matrix consists of 3x3 positive definite symmetric matrix called "inertia tensor Poinsot", the coordinates of the center of mass multiplied by the total body mass and the total mass of the rigid body. The equations of motion are formulated as a gender skew 4x4 matrices. They summarize the "principle of fundamental dynamics". The Poinsot generalized tensor appears linearly in this equality as required by the linear dependence of the equations of motion with the ten characteristics inertia of the rigid solid.

1. INTRODUCTION

Several formulations of a rigid body motion equation have been developed. The well known of them is Newton-Euler formulation; it is usually called "classic Euler equations"(Hubert Hahn [1], Ahmed A. Shabana [2], Haruhiko Asada et al. [3]). This formulation yields six scalar equations for a rigid body (Dasgupta et al. [4]). Different approaches have been developed to describe the rigid body

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motion. As an example, many authors for defining velocity, acceleration and dynamics analysis use vectors (Kozlowski [5], tensors (Fayer et al. [6]), screw-theory or 6×6 matrices (Hubert Hahn [1], Hunt [7]). In « Théorie Nouvelle de la rotation des corps » in 1834, Poinsot has shown that the movement of a body is composed of a rotation and a translation. The rigid body motion is described in this chapter taking into account Poinsot approach.

For the motion equations using matrices, particular attention must be paid to the works of Sheth et al. [8] and Legnani et al. [9] present a matrix approach witch can be used to write the kinematics and dynamics equations of a rigid body motion. In their approach, the 3x3 rotation matrix and the 3x1 column translation vector are gathered in a 4x4 matrix called " rotatranslation matrix", forces and torques are gathered in a 4x4 skew symmetric matrix called "action matrix" and the ten inertia characteristics of the body in 4x4 symmetric matrix called " inertia matrix". The matrices appearing in our formulation have the same properties as those described by Legnani et al. [9]

The matrices appearing in our motion description are homogeneous to an identical unity. Our matrix formulation of the rigid body motion equations is based on "virtual work principle". This formulation synthesizes many objects:

- External forces and torques are gathered in a 4x4 skew symmetric matrix
- Dynamics momentum and resulting momentum are also gathered in 4x4 skew symmetric matrix
- Inertial characteristics are gathered in a 4x4 positive definite symmetric matrix called "generalized Poinsot tensor". This matrix is compound of the 3x3 symmetric positive definite matrix called "Poinsot Inertia tensor", the coordinates of the mass center multiplied by the total mass of the body and the total mass of the rigid body
- The motion equations are formulated as equality between 4x4 skew symmetric matrices. They summarize the "Fundamental dynamics principle" (equality between dynamics tensor and external forces tensor). The generalized Poinsot tensor appears linearly in this equality as required by the linear dependence of motion equations with respect to the ten inertia characteristics of the rigid body.

We show at the end of this chapter the equivalence between our matrix formulation and the classic formulation of rigid body motion equations.

2. Position Matrices

We suppose that the rigid body occupied a regular region in the 3-dimensional Euclidean space $\mathbb{E}^3$. An orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is viewed as basis fixed to the body and an orthonormal basis $\{x_1, x_2, x_3\}$ is fixed in the space. For simplicity of the exposition, we assume that the origin of the fixed basis is the point O. The coordinates X^i of any point M of S and the coordinates X_0^i of its initial point are such that

$$\mathbf{OM} = X^i x_i = X^1 x_1 + X^2 x_2 + X^3 x_3 \qquad \mathbf{OM_0} = X_0^i x_i = X_0^1 x_1 + X_0^2 x_2 + X_0^3 x_3.$$

The translation movement of the rigid body is identified by expressing the position of the point A as a time function $T(t) = \mathbf{OA}$.

Denoting by Y^i the components of the vector $\mathbf{AM}$ and by Y_0^i the components of the vector $\mathbf{A_0M_0}$, the rotation movement of S is identified by the rotation matrix $R(t)$ transforming the vector $\mathbf{A_0M_0}$ to the vector $\mathbf{AM}$

$$Y^i = \mathbf{R}(t) Y_0^i.$$

Thereafter, we will write $\mathbf{R}$ for the rotation matrix and $\mathbf{T}$ for the translation vector instead of the matrix $R(t)$ and the vector $\mathbf{T}(t)$.

For more information on the determining of the rotation matrix R, the reader is invited to read the articles of Vallée et al. [10] and Betsch et al. [11].

The coordinates X^i of the point M are related to the components Y_0^i of the vector $\mathbf{A_0M_0}$ and the components T^i of the vector $\mathbf{OA}$ by the affine formula

$$X^i = T^i + R^i_j Y_0^j. \tag{2.1}$$

3. Rigid Body S Kinematics

3.1. **Generalized position vector.** Choosing l as a characteristic length of the rigid body S, we located S by introducing a 4×1 column vector $[X^i \quad l]^t$ which we'll call "*generalized position vector*". This 4×1 column vector is related to the vector $[Y_0^i \quad l]^t$ by the linear formula

$$\begin{pmatrix} \mathbf{X} \\ l \end{pmatrix} = \begin{pmatrix} R & \frac{1}{l}\mathbf{T} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbf{Y}_0 \\ l \end{pmatrix}. \tag{3.1}$$

The 4×4 matrix gathering the 3×3 rotation matrix and the 3×1 column translation vector is called "*rototranslation matrix*" by Legnani et al. [9].

3.2. **Generalized velocity and Acceleration Vectors.** Since the vector $[Y_0^i \quad l]^t$ is independent to the time t, the derivative of respect to the time t is calculated on the rototranslation matrix. Thereby, the generalized vector and the generalized acceleration vector of the point M are defined :

$$\frac{d}{dt}\begin{pmatrix}\mathbf{X}\\ l\end{pmatrix} = \begin{pmatrix}\mathbf{V}(M)\\ 0\end{pmatrix}; \qquad \frac{d^2}{dt^2}\begin{pmatrix}\mathbf{X}\\ l\end{pmatrix} = \begin{pmatrix}\mathbf{a}(M)\\ 0\end{pmatrix}.$$

$\mathbf{V}(M)$ (resp. $\mathbf{a}(M)$) is the velocity vector (resp. acceleration vector) of M.

The generalized velocity and acceleration vectors are defined as the first and the second derivatives of the generalized position vector:

$$\begin{pmatrix}\mathbf{V}(M)\\ 0\end{pmatrix} = \begin{pmatrix}\dot{R} & \frac{1}{l}\dot{\mathbf{T}}\\ \mathbf{0}^t & 0\end{pmatrix}\begin{pmatrix}\mathbf{Y}_0\\ l\end{pmatrix}; \qquad \begin{pmatrix}\mathbf{a}(M)\\ 0\end{pmatrix} = \begin{pmatrix}\ddot{R} & \ddot{\mathbf{T}}/l\\ \mathbf{0}^t & 0\end{pmatrix}\begin{pmatrix}\mathbf{Y}_0\\ l\end{pmatrix}$$

3.3. **Mass distribution.** The mass distribution will be denoted dm_0 in the body S_0. Because of the mass conservation, the mass of any part ω_0 in S_0 is conserved for the same part ω_t in S after translation and rotation transformations. Denoting dm the mass distribution in S, the total mass m of the rigid body satisfies the conservation low

$$m = \iiint_{S_0} dm_0 - \iiint_S dm.$$

The mass conservation is related to the derivative $\dot{R}$ of the transformation of the point M_0 of ω_0 to the point M of ω_0 . The Jacobean of this transformation is equal to 1, and $dm_0(M_0) = dm(M)$.

4. Virtual Works Principle

Statement : The virtual works of external forces field is equal to the inertia forces virtual works of any virtual displacement at each time t.

Rigidifying virtual displacement : For an infinitesimal variation $\delta\mathbf{R}$ of the rotation matrix $\mathbf{R}$ and an infinitesimal variation $\delta\mathbf{T}$ of the translation vector $\mathbf{T}$, the components of the virtual displacement δM are such that

$$\begin{pmatrix} \mathbf{X} \\ l \end{pmatrix} = \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Y}_0 \\ l \end{pmatrix}.$$

We must ensure that these virtual displacements are rigidifying, i.e. the virtual variation of matrix rotation $\delta\mathbf{R}$ must verify the linear equation $(\delta\mathbf{R})^t\mathbf{R}+\mathbf{R}^t\delta\mathbf{R} = \mathbf{0}$ or an equivalent formula for the 4×4 rotation matrix

$$\begin{pmatrix} \delta\mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} + \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} \delta\mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{3\times 3} & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix}.$$

External forces In addition to gravity, external forces act on the rigid body. We will denote $\mathbf{f}$ their mass density in M and set $\mathbf{f} = [f_1\ f_2\ f_3]^t$. The generalized density vector will be denoted $[\mathbf{f}\ 0]^t$

Virtual works principle is used to formulate the motion equations of the rigid body S. The application of this principle needs the calculation of external forces virtual works and inertia forces virtual works. The application of the principle must take into account the rigidity of the body.

4.1. **External Forces Virtual Works.** The virtual works of external forces of the rigid body is given by

$$\tau_f = \iiint_S \mathbf{f}^t \delta\mathbf{X} dm.$$

In the new formulation presented here, the virtual works of external forces is expressed in a 4-dimensional form

$$\tau_f = \iiint_S [\mathbf{f}^t\ 0]\, \delta \begin{pmatrix} \mathbf{X} \\ l \end{pmatrix} dm.$$

As $\mathbf{X} = \mathbf{T} + \mathbf{R}\mathbf{Y}_0$,

$$\tau_f = \mathrm{tr}\left(\begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \iiint_{S_0} \begin{pmatrix} \mathbf{Y} \\ l \end{pmatrix} [\mathbf{f}^t\ 0]\, dm_0 \right). \tag{4.1}$$

In this 4-dimensional formulation of virtual external forces works, appears the 4×4 matrix

$$\mathbf{F}(M) = \begin{pmatrix} \iiint_{S_0} \mathbf{Y}\mathbf{f}^t dm_0 & \mathbf{0} \\ l \iiint_{S_0} \mathbf{f}^t dm_0 & 0 \end{pmatrix}.$$

4.2. **Inertia Forces Virtual Works.** Inertia forces virtual works $\tau_a = \iiint_S \ddot{\mathbf{X}}^t \delta\mathbf{X} dm$ can be expressed in a 4-dimensional case:

$$\tau_a = \iiint_S \begin{pmatrix} \mathbf{a}^t & 0 \end{pmatrix} \delta \begin{pmatrix} \mathbf{X} \\ l \end{pmatrix} = \iiint_{S_0} \begin{pmatrix} \ddot{R} & \mathbf{0} \\ \ddot{\mathbf{T}}^t/l & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Y}_0^t & l \end{pmatrix} \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Y}_0 \\ l \end{pmatrix} dm_0$$
$$= \operatorname{tr}\left\{ \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \ddot{R} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \iiint_{S_0} \begin{pmatrix} \mathbf{Y}_0 \\ l \end{pmatrix} \begin{pmatrix} \mathbf{Y}_0^t & l \end{pmatrix} dm_0 \right\}.$$

In this 4-dimensional formulation of inertia forces virtual works appears a 4×4 symmetric matrix

$$\mathcal{J}_0 = \iiint_{S_0} \begin{pmatrix} \mathbf{Y}_0 \\ l \end{pmatrix} \begin{pmatrix} \mathbf{Y}_0^t & l \end{pmatrix} dm_0 = \begin{pmatrix} \iiint_{S_0} \mathbf{Y}\mathbf{Y}_0^t dm_0 & l \iiint_{S_0} \mathbf{Y}_0 dm_0 \\ l \iiint_{S_0} \mathbf{Y}_0^t dm_0 & l^2 \iiint_{S_0} dm_0 \end{pmatrix}. \tag{4.2}$$

This matrix $\mathcal{J}_0$ is positive definite.

Note that this matrix is called “inertia matrix” by Legnani et al. [9] or “pseudo inertial matrix” by Li and Sankar [12].

4.3. **matrix formulation of rigid body motion equations.** The principle of virtual works requires equality between external forces virtual works τ_f and and inertia forces virtual works τ_a

$$\tau_f = \operatorname{tr}\left\{ \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \right\},$$
$$\tau_a = \operatorname{tr}\left\{ \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix}^t \right\}.$$

Then, it results from this equality that

$$\operatorname{tr}\left\{ \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \left[\mathbf{F} - \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix}^t \right] \right\} = \mathbf{0} \tag{4.3}$$

We can remark that

$$\left[\mathbf{F} - \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix}^t \right] = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{Z}^t & 0 \end{pmatrix}. \tag{4.4}$$

The equation (4.3) equivalent to the nullity of the quantity

$$\operatorname{tr}\left\{ \begin{pmatrix} \delta\mathbf{R} & \delta\mathbf{T}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{Z}^t & 0 \end{pmatrix} \right\} = \operatorname{tr}\left(\left(\delta\mathbf{R}\mathbf{R}^t \right) \mathbf{C} \right) + \frac{1}{l} \mathbf{Z}^t \delta\mathbf{T}$$

for all virtual translation $\delta\mathbf{T}$ and virtual skew symmetric matrix $\delta\mathbf{R}\mathbf{R}^t$ implies that

- the 3x3 matrix $\mathbf{C}$ is symmetric;
- the vector $\mathbf{Z}$ is equal to the vector zero.

It results that the 4×4 matrix $\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{Z}^t & 0 \end{pmatrix}$ is a symmetric matrix:

$$\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{Z}^t & 0 \end{pmatrix}^t - \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{Z}^t & 0 \end{pmatrix} = \mathbf{0}. \tag{4.5}$$

Applying the constraint (4.5) to equation (4.4), we obtain the motion differential equations of the rigid body

$$\begin{aligned} &\begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix}^t - \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}^t \\ &= \begin{pmatrix} \iiint_S \left(\mathbf{Y}\mathbf{f}^t - \mathbf{f}\mathbf{Y}^t\right) dm & l\mathcal{R} \\ -l\mathcal{R}^t & 0 \end{pmatrix} \end{aligned} \tag{4.6}$$

In equation (4.6), the "Poinsot generalized tensor" $\mathcal{J}_0$ appears linearly us the linear dependence of classic motion equations respect to the ten inertia characteristics.

We can reduce the second member of equation (4.6). The double vector product formula

$$(\mathbf{Y} \times \mathbf{f}) \times \mathbf{u} = \mathbf{u} \times (\mathbf{f} \times \mathbf{Y}) = (\mathbf{Y^t u})\mathbf{f} - (\mathbf{f^t u})\mathbf{Y}$$

available for all vector $\mathbf{u}$, implies:

$$\mathbf{Y}\mathbf{f}(\mathbf{M})^{\mathbf{t}} - \mathbf{f}(\mathbf{M})\mathbf{Y}^{\mathbf{t}} = -\mathbf{j}(\mathbf{Y} \times \mathbf{f}(\mathbf{M})).$$

It results that

$$\iiint_S (\mathbf{Y}\mathbf{f}(\mathbf{M})^{\mathbf{t}} - \mathbf{f}(\mathbf{M})\mathbf{Y}^{\mathbf{t}})\mathbf{dm} = -\mathbf{j}(\aleph_{\mathbf{A}}),$$

where the linear skew application j is associated to the *product vector*: given two vectors $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \times \mathbf{v} = \mathbf{u}\mathbf{v}^t = [j(\mathbf{u})]\mathbf{v}$.

The external forces momentum at the privilege point A is

$$\aleph_A = \iiint_S \mathbf{Y} \times \mathbf{f}(\mathbf{M})\mathbf{dm}.$$

The motion equations of the rigid body S can be written in a 4-dimensional form

(4.7) $$\begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix}^t - \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}^t = \begin{pmatrix} j(\aleph_A) & l\mathcal{R} \\ -l\mathcal{R}^t & 0 \end{pmatrix}.$$

One can formulate these matrix equations by Lagrange multipliers techniques. The rigidity of the body is taken into account by introducing 3×3 symmetic matrix containing Lagrange multipliers ([10, 13]). In this case, in addition to external and inertia forces works, one calculates the virtual works taking into account the rigidity constraint. Once Lagrange multipliers matrix eliminated, rigid body motion equations (4.7) are obtained (Atchonouglo [15], Atchonouglo et al. [14]). We note that the reduction element of $\mathcal{R}$ and $\aleph_A$ of external forces tensors are gathered in the 4×4 skew symmetric matrix

$$\begin{pmatrix} j(\aleph_A) & l\mathcal{R} \\ -l\mathcal{R}^t & 0 \end{pmatrix}$$

and the motion equations are formulated as equality between 4×4 skew symmetric matrices.

Rather than gathering reduction elements of $\mathcal{R}$ and $\aleph_A$ in vector with 6 components denoted tensor, the external forces tensor will be the 4×4 skew symmetric matrix

$$\begin{pmatrix} j(\aleph_A) & l\mathcal{R} \\ -l\mathcal{R}^t & 0 \end{pmatrix}.$$

This is the "action matrix" defined by Legnani et al. Similarly, the 4×4 skew symmetric matrix

$$\begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix}^t - \begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \mathcal{J}_0 \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}^t$$

will be called dynamics tensor.

Form the virtual works principle, we have deduced dynamics fundamental principle of the rigid body S: the dynamics tensor and external forces tensor are equals. This matrix formulation is identical to the mechanic classic formulation of the six equations of a rigid solid motion.

Note that in the matrix formulation of motion equations appear six scalar equations independents. indeed, a 4×4 skew symmetric matrix has six independent components such as the six independent equations of rigid body motion.

5. Resulting theorem and resulting momentum theorem

We'll establish in the two following subsections the link between the classic formulation of the rigid body motion equations and the matrix formulation that we have just developed.

The three matrices product below is calculated by blocs

$$\begin{pmatrix} \ddot{\mathbf{R}} & \ddot{\mathbf{T}}/l \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{K}_0 & l\iiint_S \mathbf{Y}_0 dm \\ \iiint_S \mathbf{Y}_0{}^t dm & ml^2 \end{pmatrix} \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \ddot{\mathbf{R}}\mathbf{K}_0 + \ddot{\mathbf{T}}\iiint_S \mathbf{Y}_0{}^t dm & l\left(\ddot{\mathbf{R}}\iiint_S \mathbf{Y}_0 dm + m\ddot{\mathbf{T}}\right) \\ \mathbf{0}^t & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{R}^t & \mathbf{0} \\ \mathbf{0}^t & \mathbf{1} \end{pmatrix}$$
$$= \begin{pmatrix} \ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t + \ddot{T}\left(\mathbf{R}\iiint_{\mathbf{S}} \mathbf{Y_0}^{\mathbf{t}}\mathbf{dm}\right)^t & l\left(\ddot{\mathbf{R}}\iiint_S \mathbf{Y}_0 dm + m\ddot{\mathbf{T}}\right) \\ \mathbf{0}^t & \mathbf{0} \end{pmatrix}.$$

Deducting the transpose of this, we have the 4x4 skew symmetric matrix:

$$\left(\begin{array}{c|c} \ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t - \mathbf{R}^t\mathbf{K}_0\ddot{\mathbf{R}} + \ddot{T}\left(\iiint_S \mathbf{Y} dm\right)^t - \left(\iiint_S \mathbf{Y} dm\right)\ddot{T}^t & l\left(\ddot{\mathbf{R}}\iiint_S \mathbf{Y}_0 dm + m\ddot{\mathbf{T}}\right) \\ \hline -l\left(\ddot{\mathbf{R}}\iiint_S \mathbf{Y}_0 dm + m\ddot{\mathbf{T}}\right) & \mathbf{0} \end{array}\right).$$

The form of this dynamics tensor is the same as the form of external forces tensor. The resulting theorem and momentum resulting theorem are deduced by equalization of these two tensors.

Let's calculate the resulting theorem and the resulting momentum theorem in classic mechanics.

5.1. Resulting Theorem.

The dynamics resulting $\left(\ddot{\mathbf{R}}\iiint_S \mathbf{Y}_0 dm + m\ddot{\mathbf{T}}\right)$ is equal to the external forces resulting

$$m\ddot{\mathbf{T}} + \ddot{\mathbf{R}}\iiint_S \mathbf{Y}_0 dm = \mathcal{R}.$$

Remark The mass center G acceleration $\mathbf{a}(G)$ of the part S is the second derivative of the components of vector $\mathbf{OG} = \mathbf{OA} + \mathbf{AG}$:

$$m\mathbf{a}(G) = m\ddot{\mathbf{T}} + \ddot{\mathbf{R}}\iiint_{S_0} \mathbf{Y}_0 dm_0.$$

It results the classic resulting theorem

$$m\mathbf{a}(G) = \mathcal{R}.$$

5.2. **Resulting Momentum Theorem.** The dynamics momentum at the point A

$$j^{-1}\left(\ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t - \mathbf{R}\mathbf{K}_0\ddot{\mathbf{R}}^t + \ddot{T}\iiint_S \mathbf{Y}^t dm - \left(\iiint_S \mathbf{Y} dm\right)\ddot{\mathbf{T}}^t\right)$$

is equal to the external forces momentum at the point A

$$\ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t - \mathbf{R}\mathbf{K}_0\ddot{\mathbf{R}}^t + \ddot{T}\iiint_S \mathbf{Y}^t dm - \left(\iiint_S \mathbf{Y} dm\right)\ddot{T}^t = j(\aleph_{\mathbf{A}}).$$

Remark: Since $\dot{\mathbf{R}} = j(\mathbf{\Omega})\mathbf{R}$, the second derivative of rotation matrix is

$$\ddot{\mathbf{R}} = j(\mathbf{\Omega})\dot{\mathbf{R}} + \mathbf{j}(\dot{\mathbf{\Omega}})\mathbf{R} = \left(\mathbf{j}(\mathbf{\Omega})\mathbf{j}(\mathbf{\Omega}) + \mathbf{j}(\dot{\mathbf{\Omega}})\right)\mathbf{R}.$$

Then it comes

$$\ddot{\mathbf{R}}^t = \mathbf{R^t}\left(\mathbf{j}(\mathbf{\Omega})\mathbf{j}(\mathbf{\Omega}) - \mathbf{j}(\dot{\mathbf{\Omega}})\right), \mathbf{R}\mathbf{K_0}\ddot{\mathbf{R}}^\mathbf{t} = \mathbf{K}\left(\mathbf{j}(\mathbf{\Omega})\mathbf{j}(\mathbf{\Omega}) - \mathbf{j}(\dot{\mathbf{\Omega}})\right).$$

$$\ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t = \left(j(\mathbf{\Omega})\mathbf{j}(\mathbf{\Omega}) - \mathbf{j}(\dot{\mathbf{\Omega}})\right)\mathbf{K}.$$

From the formula $j(\mathbf{\Omega})\mathbf{j}(\mathbf{\Omega}) = \mathbf{\Omega}\mathbf{\Omega}^\mathbf{t} - (\mathbf{\Omega}^\mathbf{t}\mathbf{\Omega})\mathbf{I}$ we deduce

$$\ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t - \mathbf{R}\mathbf{K}_0\ddot{\mathbf{R}}^t = \Omega(\mathbf{K}\mathbf{\Omega})^\mathbf{t} - (\mathbf{K}\mathbf{\Omega})\mathbf{\Omega}^\mathbf{t} + \mathbf{j}(\dot{\mathbf{\Omega}})\mathbf{K} + \mathbf{K}\mathbf{j}(\dot{\mathbf{\Omega}}).$$

From the trace definition

$$\left(\mathbf{K}\dot{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{W}\right) + \left(\dot{\mathbf{\Omega}}, \mathbf{K}\mathbf{\Omega}, \mathbf{W}\right) + \left(\dot{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{K}\mathbf{W}\right) = (\mathbf{trK})\left(\dot{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{W}\right)$$

available for all vector $\mathbf{W}$, it results that (see scalar product definition)

$$\left([\mathbf{K} - (\mathbf{trK})\mathbf{I}]\dot{\mathbf{\Omega}}, \mathbf{\Omega}, \mathbf{W}\right) = -g\left(\dot{\mathbf{\Omega}} \times (\mathbf{K}\mathbf{\Omega}), \mathbf{W}\right) - g\left(\mathbf{K}(\dot{\mathbf{\Omega}} \times \mathbf{\Omega}), \mathbf{W}\right).$$

where g is the linear application associated to the *scalar product*: given two vectors $\mathbf{u}$ and $\mathbf{v}$: $< \mathbf{u}, \mathbf{v} >= \mathbf{u}^t\mathbf{v} = g(\mathbf{u}, \mathbf{v})$.

Then $j(\dot{\mathbf{\Omega}})\mathbf{K} + \mathbf{K}j(\dot{\mathbf{\Omega}}) = j(\mathbf{J}\dot{\mathbf{\Omega}})$. Since (see double vector product definition)

$$\mathbf{\Omega}(\mathbf{K}\mathbf{\Omega})t - (\mathbf{K}\mathbf{\Omega})\mathbf{\Omega}^\mathbf{t} = j\left((\mathbf{K}\mathbf{\Omega}) \times \mathbf{\Omega}\right) = -j\left((J\mathbf{\Omega}) \times \mathbf{\Omega}\right)$$

then the equation $\ddot{\mathbf{R}}\mathbf{K}_0\mathbf{R}^t - \mathbf{R}\mathbf{K}_0\ddot{\mathbf{R}}^t$ can be rewritten

$$j\left(\mathbf{J}\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times (\mathbf{J}\mathbf{\Omega})\right).$$

Similarly,

$$\ddot{\mathbf{T}}\iiint_S \mathbf{Y}^t dm - \iiint_S \mathbf{Y} dm \ddot{T}^t = j\left(\iiint_S \mathbf{Y} dm \times \ddot{\mathbf{T}}\right).$$

Finally, the equality between dynamics momentum at the point A and the external forces momentum at the point A is equivalent to

$$J\dot{\Omega} + \Omega \times (J\Omega) + \iiint_S \mathbf{Y} dm \times \ddot{\mathbf{T}} = \aleph_{\mathbf{A}}.$$

The first member of this equality is equal to the integral $\iiint_S \mathbf{Y} \times \mathbf{a}(M) dm$. Indeed

$$\begin{aligned}
\iiint_S \mathbf{Y} \times \mathbf{a}(M) dm &= \iiint_S \mathbf{Y} \times \left(\ddot{\mathbf{T}} + \ddot{\mathbf{R}}\mathbf{Y}_0\right) dm \\
&= \iiint_S \mathbf{Y} \times \ddot{\mathbf{T}} dm + \iiint_S \mathbf{Y} \times \left(\ddot{\mathbf{R}}\mathbf{R}^t\mathbf{Y}\right) dm \\
&= \iiint_S \mathbf{Y} \times \ddot{\mathbf{T}} dm + \iiint_S \mathbf{Y} \times [\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{Y})] \mathbf{dm} \\
&\quad + \iiint_S \mathbf{Y} \times [\dot{\mathbf{\Omega}} \times \mathbf{Y}] \mathbf{dm}.
\end{aligned}$$

Remark that $\mathbf{J}\dot{\mathbf{\Omega}} = \iiint_S \mathbf{Y} \times [\dot{\mathbf{\Omega}} \times \mathbf{Y}] dm$

$$\begin{aligned}
\iiint_S \mathbf{Y} \times \mathbf{a}(M) dm &= \iiint_S \mathbf{Y} \times \ddot{\mathbf{T}} dm + \mathbf{J}\dot{\mathbf{\Omega}} - \iiint_S g(\mathbf{Y}, \mathbf{\Omega}) \mathbf{\Omega} \times \mathbf{Y} dm \\
&= \iiint_S \mathbf{Y} \times \ddot{\mathbf{T}} dm + \mathbf{J}\dot{\mathbf{\Omega}} + \iiint_S \mathbf{\Omega} \times [\mathbf{Y} \times (\mathbf{\Omega} \times \mathbf{Y})] dm \\
&= \iiint_S \mathbf{Y} \times \ddot{\mathbf{T}} dm + \mathbf{J}\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \iiint_{\mathbf{S}} \mathbf{Y} \times (\mathbf{\Omega} \times \mathbf{Y}) dm \\
&= \iiint_S \mathbf{Y} \times \ddot{\mathbf{T}} dm + \mathbf{J}\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times (\mathbf{J}\mathbf{\Omega}).
\end{aligned}$$

So, from the matrix formulation developed in this article, one can obtain the classic resulting momentum theorem and the resulting theorem.

6. Conclusion and perspectives

We have formulated the six motion equations of a rigid body in a 4×4 skew symmetric matrix form, involving the generalized Poinsot inertia matrix which is 4×4 symmetric and positive definite.

This formulation has several applications already being exploited in our researches.

It is used to determine the ten inertia characteristics of a rigid body. Indeed, instead of using the formalism of Newton-Euler for determining a 10×1 column vector containing the ten inertia characteristics by introducing some conditions such as the positivity of the body's mass, we determine the generalized Poinsot tensor by a projected gradient algorithm on the cone of symmetric positive definite matrices.

In our formulation, external forces and torques are gathered in a 4x4 skew symmetric matrix. Our matrix formulation can help, in our opinion, the determination of rigid body movement by control optimal theory. This approach is a part of our futures applications.

As a future work, this formulation will be extended to rigid multibody dynamics.

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