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ON POINTWISE APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY SINGULAR INTEGRALS

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ABSTRACT. We prove a theorem on weighted pointwise convergence of multi-dimensional integral operators with radial kernels to generating function of several variables, which are in general non-integrable in n-dimensional Euclidean space E_n in the sense of Lebesgue. Main result holds at almost every point of E_n.

1. INTRODUCTION

It is well known that classical singular integrals, such as Abel-Poisson, Gauss-Weierstrass and Picard are widely used approximate identities in the theory of approximation of functions which belong to Lebesgue spaces. In particular, we know that this approximation is valid at each Lebesgue point of indicated Lebesgue integrable function by the previous literature (see, e.g., [20]). Some of the results in the literature concerning approximation by singular integrals may be given as follows: approximation by convolution type [5, 10]; convergence of singular integrals depending on two parameters [12, 21]; approximation by non-convolution type singular integrals [1, 6, 13]; after the initial study given by Taberski [22], approximation of functions of two variables by

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three-parameter singular integrals [16–18, 24, 26]; approximation of functions in weighted sense [1, 23].

Throughout the years, above-mentioned type results were extended to the multivariate case by constructing $n-$dimensional generalizations of integral operators with kernels satisfying suitable conditions. Among these works, we refer the reader to [7, 9, 15] and fundamental monographies [8, 19, 20]. Also, for some of the contemporary research results, we refer the reader to [2–4, 25].

Let E_n denote $n-$dimensional Euclidean space and $L_1(E_n)$ denote the set of all functions which are integrable on whole Euclidean space E_n in the sense of Lebesgue. In the present paper, we consider the following problem: Let f be a non-integrable function together with $\frac{f}{\rho} \in L_1(E_n)$ for a positive (weight) function ρ belonging to $L_1(E_n)$. In this case, we consider the problem of approximation of function f at Lebesgue points of functions ρ and $\frac{f}{\rho}$ by the family of multidimensional integral operators with non-negative radial kernels. We note that these kinds of operators have been studied for the one-dimensional case in [10].

Let us recall some notations which will be used in the sequel. Let E_n denote $n-$dimensional Euclidean space and $L_1(E_n)$ denote the set of all functions which are integrable on whole Euclidean space E_n in the sense of Lebesgue.We use the notations $x = (x_1, x_2, \ldots, x_n)$ and $t = (t_1, t_2, \ldots, t_n)$ for the elements of E_n. The operators are defined by the family of multidimensional integral operators with non-negative radial kernels given by

$$T_\lambda(f, x_1, x_2, \ldots, x_n) = \int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} f(x_1+t_1, \ldots, x_n+t_n) K_\lambda(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n,$$

and their simplified forms are given as

$$T_\lambda(f, x) = \int\limits_{E_n} f(x+t) K_\lambda(t) dt,$$

where $dt = dt_1 dt_2 \ldots dt_n$ denotes the ordinary Lebesgue measure.

We say that $K_\lambda(t)$ is a radial function, if there is $K_\lambda(t) = K_\lambda(\sqrt{t_1 + t_2 + \ldots + t_n}) = K_\lambda(r)$ with $0 \leq r < \infty$.

In the present article, we will prove the pointwise convergence of the operators:

$$T_\lambda(f,x^*)=\int\limits_{E_n} f(x^*+t)K_\lambda(t)dt \tag{1}$$

under suitable conditions to $f\notin L_1(E_n)$ at Lebesgue points x^* of functions ρ and $\frac{f}{\rho}$. We use the approach similar to used in [11, 14].

2. APPROXIMATION

In this section, we shall investigate the approximation of non-integrable functions of several variables by singular integral $T_\lambda(f,x^*)$ in (1). Let $\rho\in L_1(E_n)$ with $\rho>0$ and A_ρ is Lebesgue points set of ρ. Also let $A_{\frac{f}{\rho}}$ is Lebesgue points set of $\frac{f}{\rho}$ with $f\notin L_1(E_n)$ and $\frac{f}{\rho}\in L_1(E_n)$. We let $A=A_{\frac{f}{\rho}}\cap A_\rho$, here A is the set of Lebesgue points of ρ and $\frac{f}{\rho}$.

Moreover, we consider the following function α which is connected with the function ρ :

$$\alpha(\delta):=\sup_{\substack{x\in E_n\\ |y|\le\delta}}\frac{\rho(x+y)}{\rho(x)}<\infty.$$

Obviously that $\alpha(\delta)$ is monotonically increasing function and for any y, $\alpha(\delta)\ge 1$.

The main theorem of this paper is as follows:

Theorem 2.1. *Let $\frac{f}{\rho}\in L_1(E_n)$ for $\rho\in L_1(E_n)$ with $\rho>0$ and $f\notin L_1(E_n)$. Suppose that the following conditions are satisfied:*

a) *$\rho(x^*+t)K_\lambda(t)$ is non-negative and bounded for any fixed $x^*\in E_n$, $\lambda>0$ and for all $t\in E_n$, and $\lim\limits_{\lambda\to\infty}\int\limits_{E_n}K_\lambda(t)dt=1$.*

b) *$K_\lambda(t)$ is non-negative and radial (i.e., $K_\lambda(t)=K_\lambda(\sqrt{t_1^2+\ldots+t_n^2})=K_\lambda(r)\ge 0$) and $\alpha(r)K_\lambda(r)$ is non-increasing on $[0,\infty)$.*

c) *For any positive number δ and $\rho>0$, $\rho\in L_1(E_n)$,*

$$\lim_{\lambda\to\infty}\sup_{|t|\ge\delta}\left(\rho(x^*+t)K_\lambda(t)\right)=0.$$

for any fixed x^*.

d) *$\int\limits_0^\infty r^{n-1}\alpha(r)K_\lambda(r)dr$ is finite.*

e) *For any fixed* $\delta > 0$,

$$\lim_{\lambda\to\infty} \int\limits_{|t|\geq\delta} \rho(x^* + t)K_\lambda(t)dt = 0.$$

f) *For any fixed* $\delta > 0$,

$$\lim_{\lambda\to\infty} \int_\delta^\infty K_\lambda(r)dr = 0$$

and

$$\lim_{\lambda\to\infty} K_\lambda(\delta) = 0.$$

Then, we have

$$\lim_{\lambda\to\infty} T_\lambda(f, x^*) = f(x^*)$$

at every Lebesgue point $x^* \in A$.

First, we show the existence of the operators.

Lemma 2.1. *Let* $\frac{f}{\rho} \in L_1(E_n)$, $f \notin L_1(E_n)$, $\rho > 0$ *and* K_λ *verify condition (a) of Theorem 2.1. Then,* $T_\lambda(f, x^*)$ *is finite in* E_n *for fixed* $\lambda > 0$ *and* $x^* \in E_n$.

Proof. We can write

$$\begin{aligned} T_\lambda(f, x^*) &= \int\limits_{E^n} f(x^* + t)K_\lambda(t)dt \\ &\leq \sup_{t\in E_n} (\rho(x^* + t)K_\lambda(t)) \left\| \frac{f}{\rho} \right\|_{L_1(E_n)}. \end{aligned}$$

Thus the proof is completed. □

Now, let us give the proof of the theorem.

Proof. For integral (1), we can write

$$\begin{aligned} T_\lambda(f, x^*) - f(x^*) &= \int\limits_{E_n} \left[\frac{f(x^* + t)}{\rho(x^* + t)} - \frac{f(x^*)}{\rho(x^*)} \right] \rho(x^* + t)K_\lambda(t)dt \\ &\quad + \frac{f(x^*)}{\rho(x^*)} \left[\int\limits_{E_n} \rho(x^* + t)K_\lambda(t)dt - \rho(x^*) \right] \end{aligned}$$

and in view of (b), we have

$$\begin{aligned}|T_\lambda(f,x^*)-f(x^*)| &\le \int_{E_n}\left|\frac{f(x^*+t)}{\rho(x^*+t)}-\frac{f(x_0)}{\rho(x_0)}\right|\rho(x^*+t)K_\lambda(t)dt \\ &\quad+\left|\frac{f(x^*)}{\rho(x^*)}\right|\left[\int_{E_n}\rho(x^*+t)K_\lambda(t)dt-\rho(x^*)\right] \\ &= I_1(x^*,\lambda)+I_2(x^*,\lambda).\end{aligned}$$

Since $\rho \in L_1(E_n)$ and x^* is a Lebesgue point of ρ, we know that

$$\lim_{\lambda\to\infty}\int_{E_n}\rho(x^*+t)K_\lambda(t)dt=\rho(x^*)$$

holds at every Lebesgue point x^* of ρ. Therefore, $I_2(x^*,\lambda)\to 0$ as $\lambda\to\infty$.

Now, we will calculate the $I_1(x^*,\lambda)$. For any fixed $\delta > 0$, we can write to $I_1(x^*,\lambda)$ as follows.

$$\begin{aligned}I_1(x^*,\lambda) &= \int_{|t|\ge\delta}\left|\frac{f(x^*+t)}{\rho(x^*+t)}-\frac{f(x^*)}{\rho(x^*)}\right|\rho(x^*+t)K_\lambda(t)dt \\ &\quad+\int_{|t|<\delta}\left|\frac{f(x^*+t)}{\rho(x^*+t)}-\frac{f(x^*)}{\rho(x^*)}\right|\rho(x^*+t)K_\lambda(t)dt \\ &= I_{11}(x^*,\lambda)+I_{12}(x^*,\lambda).\end{aligned}$$

It is sufficent to show that terms on right hand side of the last expression tends to zero as $\lambda\to\infty$.

First, we consider $I_{11}(x^*,\lambda)$,

$$I_{11}(x^*,\lambda)=\int_{|t|\ge\delta}\left|\frac{f(x^*+t)}{\rho(x^*+t)}-\frac{f(x^*)}{\rho(x^*)}\right|\rho(x^*+t)K_\lambda(t)dt.$$

By condition (c), we get

$$\begin{aligned}I_{11}(x^*,\lambda) &\le \sup_{|t|\ge\delta}\rho(x^*+t)K_\lambda(t)\int_{|t|\ge\delta}\left|\frac{f(x^*+t)}{\rho(x^*+t)}\right|dt+\left|\frac{f(x^*)}{\rho(x^*)}\right|\int_{|t|\ge\delta}\rho(x^*+t)K_\lambda(t)dt \\ &\le \sup_{|t|\ge\delta}\rho(x^*+t)K_\lambda(t)\left\|\frac{f}{\rho}\right\|_{L_1(E_n)}+\left|\frac{f(x^*)}{\rho(x^*)}\right|\int_{|t|\ge\delta}\rho(x^*+t)K_\lambda(t)dt.\end{aligned}$$

According to (e) and (c), we obtain $I_{11}(x^*,\lambda)$ tends to zero as $\lambda\to\infty$.

Next we consider $I_{12}(x^*, \lambda)$.

By the definition of Lebesgue point

$$\lim_{r\to 0} \frac{1}{r^n} \int\limits_{|t|<r} \left| \frac{f(x^*+t)}{\rho(x^*+t)} - \frac{f(x^*)}{\rho(x^*)} \right| dt = 0.$$

This means

$$\lim_{r\to 0} \frac{1}{r^n} \int\limits_0^r \sigma^{n-1} \int\limits_{S^{n-1}} \left| \frac{f(x^*+\sigma\theta)}{\rho(x^*+\sigma\theta)} - \frac{f(x^*)}{\rho(x^*)} \right| d\theta d\sigma = 0,$$

where $d\theta$ is element of area of sphere S^{n-1}.

Therefore, for any given $\varepsilon > 0$, there exists a positive number δ such that for all $r \leq \delta$, one has

$$\int\limits_0^r \sigma^{n-1} \int\limits_{S^{n-1}} \left| \frac{f(x^*+\sigma\theta)}{\rho(x^*+\sigma\theta)} - \frac{f(x^*)}{\rho(x^*)} \right| d\theta d\sigma < \varepsilon r^n.$$

Now, we let

$$g(\sigma) = \int\limits_{S^{n-1}} \left| \frac{f(x^*+\sigma\theta)}{\rho(x^*+\sigma\theta)} - \frac{f(x^*)}{\rho(x^*)} \right| d\theta,$$

and

$$G(r) = \int\limits_0^r \sigma^{n-1} g(\sigma) d\theta.$$

One obtains that

$$G(r) \leq \varepsilon r^n, \tag{2}$$

for all $r \leq \delta$. It is clear that for any fixed x^*

$$\rho(x^*+t) \leq \alpha(|t|)\rho(x^*).$$

Thus, by above inequality, we get

$$I_{12}(x^*, \lambda) \leq \rho(x^*) \int\limits_{|t|<\delta} \left| \frac{f(x^*+t)}{\rho(x^*+t)} - \frac{f(x^*)}{\rho(x^*)} \right| \alpha(|t|) K_\lambda(t) dt$$

and from the inequality (2) we can write

$$I_{12}(x^*, \lambda) \leq \rho(x^*) \int\limits_0^\delta r^{n-1} g(r) \alpha(r) K_\lambda(r) dr.$$

Using integration by parts, we obtain the following inequality:

$$\begin{aligned} I_{12}(x^*,\lambda) &\leq G(r)\alpha(r)K_\lambda(r)\mid_0^\delta - \int_0^\delta G(r)d(\alpha(r)K_\lambda(r)) \\ &\leq \varepsilon\delta^n\alpha(r)K_\lambda(r)\mid_0^\delta - \varepsilon\int_0^\delta r^n d(\alpha(r)K_\lambda(r)). \end{aligned}$$

Using integrating by parts again, we get

$$I_{12}(x^*,\lambda) \leq \varepsilon n \int_0^\delta r^{n-1}\alpha(r)K_\lambda(r)dr.$$

It is follows from condition (d) that $I_{12}(x^*,\lambda) \to 0$ as $\lambda \to \infty$. Hence, collecting the estimates, we have $I_{11}(x^*,\lambda) + I_{12}(x^*,\lambda)$ tends to zero as $\lambda \to \infty$ for arbitrary $\varepsilon > 0$. That is, we obtain

$$\lim_{\lambda\to\infty} T_\lambda(f,x^*) = f(x^*)$$

and this completes the proof. □

3. Applications

Now, we give examples about the functions ρ, f and $\frac{f}{\rho}$ satisfiying above properties.

Example 1. *Let us consider the function*

$$f(t) = \left\{ \begin{array}{cc} \frac{1}{t_1.t_2...t_n(1+t_1^2.t_2^2...t_n^2)}, & \textit{if } 0 < t_k < 1 \textit{ or } t_k \geq 1, k = 1,\ldots,n \\ 0, & \textit{for other t's} \end{array} \right\}.$$

Then, we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(t)dt > \int_{0}^{1}\cdots\int_{0}^{1}\frac{dt_1dt_2\ldots dt_n}{t_1.t_2\ldots t_n(1+t_1^2.t_2^2\ldots t_n^2)}$$
$$> \frac{1}{2}\int_{0}^{1}\cdots\int_{0}^{1}\frac{dt_1dt_2\ldots dt_n}{t_1.t_2\ldots t_n}=\infty.$$

We say that $f\notin L_1(E_n)$.

Now, we let

$$\rho(t)=\left\{\begin{array}{ll}\frac{1}{\sqrt{t_1.t_2\ldots t_n}(1+t_1.t_2\ldots t_n)}, \text{ if } 0<t_k<1 \text{ or } t_k\geq 1, k=1,\ldots,n \\ e^{-(|t_1|+\ldots+|t_n|)}, & \text{for other } t's\end{array}\right\},$$

with $t=(t_1,t_{2,},..,t_n)$. *We will show that* $\rho\in L_1(-\infty,\infty)$.

Let $P^n=E_n/(0,1)X\ldots X(0,1,)U(1,\infty)X\ldots(1,\infty)$. *It is clear that*

$$\int_{E_n}\rho(t)dt=\int_{0}^{1}\cdots\int_{0}^{1}\frac{dt_1\ldots dt_n}{\sqrt{t_1.t_2\ldots t_n}}+\int_{1}^{\infty}\cdots\int_{1}^{\infty}\frac{dt_1\ldots dt_n}{t_1^{\frac{3}{2}}.t_2^{\frac{3}{2}}\ldots t_n^{\frac{3}{2}}}+\int_{P^n}e^{-(|t_1|+\ldots+|t_n|)}<\infty.$$

Hence, we showed that $\rho\in L_1(E_n)$.

Now, we look $\frac{f}{\rho}$. *That is*

$$\frac{f(t)}{\rho(t)}=\left\{\begin{array}{ll}\frac{(1+t_1.t_2\ldots t_n)}{\sqrt{t_1.t_2\ldots t_n}(1+t_1^2.t_2^2\ldots t_n^2)}, \text{ if } 0<t_k<1 \text{ or } t_k\geq 1, k=1,\ldots,n \\ 0, & \text{for otherwise } t's\end{array}\right\}.$$

i) For $0<t_k<1, k=1,\ldots,n$, *the inequality*

$$\frac{f(t)}{\rho(t)}<\frac{2}{\sqrt{t_1.t_2\ldots t_n}}$$

is satisfied.

ii) For $t_k\geq 1$, *we have*

$$\frac{f(t)}{\rho(t)}<\frac{2}{t_1^{\frac{3}{2}}.t_2^{\frac{3}{2}}\ldots t_n^{\frac{3}{2}}}.$$

Hence,

$$\int\limits_{E_n} \frac{f(t)}{\rho(t)}dt < \int\limits_0^1 \cdots \int\limits_0^1 \frac{2dt_1 \ldots dt_n}{\sqrt{t_1.t_2 \ldots t_n}} + \int\limits_1^\infty \cdots \int\limits_1^\infty \frac{dt_1 \ldots dt_n}{t_1^{\frac{3}{2}}.t_2^{\frac{3}{2}} \ldots t_n^{\frac{3}{2}}} + 0.$$

Then, we obtain $\frac{f}{\rho} \in L_1(E_n)$.

Now, we will examine $\alpha(\delta) = \sup\limits_{\substack{x_0 \in E_n \\ |t| \le \delta}} \frac{\rho(x_0+t)}{\rho(x_0)}$. *We can write*

$$\frac{\rho(x_0+t)}{\rho(x_0)} = \left\{ \begin{array}{cc} \frac{\sqrt{x_{01}\ldots.x_{0n}}(1+x_{01}\ldots x_{0n})}{\sqrt{(t_1+x_{01})\ldots(t_n+x_{0n})}(1+(t_1+x_{01})\ldots(t_n+x_{0n}))}, & \begin{array}{c} \text{if } 0 < t_k < 1 \\ \text{or} \\ t_k \ge 1, k = 1, \ldots, n, \ |t| \le 1 \end{array} \\ \frac{e^{-(|t_1+x_{01}|+\ldots+|t_n+x_{0n}|)}}{e^{-(|x_{01}|+\ldots+|x_{0n}|)}}, & \text{for other } t's \end{array} \right\}$$

at a fixed point $x_0 = (x_{01} \ldots x_{0n})$. *It is clear that* $\alpha(\delta) = \sup\limits_{|t| \le \delta} \frac{\rho(x_0+t)}{\rho(x_0)} = 1$.

Now, we take $K_\lambda(t) = \frac{\lambda^n}{\pi^{\frac{n}{2}}} e^{-\lambda^2(t_1^2+\ldots+t_n^2)}$ *(kernel of Gauss-Weierstrass). It is clear that these functions are non-negative for every* $t \in E_n$.

On the other hand, for integral $\frac{\lambda^n}{\pi^{\frac{n}{2}}} \int_{E_n} e^{-\lambda^2(t_1^2+\ldots+t_n^2)} dt_1 \ldots dt_n$, *if the transformation* $\lambda t = u$ $(u = (u_1, \ldots, u_n))$ *is made, we have*

$$\frac{\lambda^n}{\pi^{\frac{n}{2}}} \int_{E_n} e^{-\lambda^2(t_1^2+\ldots+t_n^2)} dt_1 \ldots dt_n = \frac{1}{\pi^{\frac{n}{2}}} \int_{E_n} e^{-\lambda^2(u_1^2+\ldots+u_n^2)} du_1 \ldots du_n = 1.$$

Also, $K_\lambda(t)$ *is radial. Hence, the functions* $K_\lambda(r)$ *are in the form of* $K_\lambda(r) = \frac{1}{\pi^{\frac{n}{2}}} e^{-\lambda^2 r^2}$.

We will investigate other conditions of the theorem.

a) First, we look $\lim_{\lambda\to\infty} \sup_{|t|\ge\delta}(\rho(x_0+t)K_\lambda(t)) = 0$.

i) For $0 < t_k < 1$ *or* $t_k \geq 1, k = 1, \ldots, n$*, we have*

$$\begin{aligned}
&\sup_{|t|\geq\delta}(\rho(x_0+t)K_\lambda(t)) \\
= &\sup_{|t|\geq\delta}\Big(\frac{\lambda^n}{\pi^{\frac{n}{2}}}e^{-\lambda^2(t_1^2+\ldots+t_n^2)} \\
&.\frac{1}{\sqrt{(t_1+x_{01})\ldots(t_n+x_{0n})}(1+(t_1+x_{01})\ldots(t_n+x_{0n}))}\Big) \\
\leq &\frac{\lambda^n}{\pi^{\frac{n}{2}}}e^{-\lambda^2\delta^2}\frac{1}{\sqrt{(x_{01})\ldots(x_{0n})}(1+x_{01}\ldots x_{0n})}.
\end{aligned}$$

ii) For other $t's$*, we get*

$$\begin{aligned}
\sup_{|t|\geq\delta}(\rho(x_0+t)K_\lambda(t)) &= \sup_{|t|\geq\delta}\Big(\frac{\lambda^n}{\pi^{\frac{n}{2}}}e^{-\lambda^2(t_1^2+\ldots+t_n^2)}e^{-(|t_1+x_{01}|+\ldots+|t_n+x_{0n}|)}\Big) \\
&\leq \frac{\lambda^n}{\pi^{\frac{n}{2}}}e^{-\lambda^2\delta^2}e^{-(|x_{01}|+\ldots+|x_{0n}|)}.
\end{aligned}$$

Therefore, we obtain

$$\lim_{\lambda\to\infty}\sup_{|t|\geq\delta}(\rho(x_0+t)K_\lambda(t)) = 0.$$

b) $\int\limits_0^\infty r^{n-1}\alpha(r)K_\lambda(r)dr$ *is finite since*

$$\begin{aligned}
\int\limits_0^\infty r^{n-1}\alpha(r)K_\lambda(r)dr &\leq \frac{1}{\pi^{\frac{n}{2}}}\int\limits_0^\infty r^{n-1}e^{-\lambda^2r^2}dr \\
&= \frac{1}{\pi^{\frac{n}{2}}}\left\{\begin{array}{ll} \frac{(2k-1)!!}{(2\lambda)^{2k+1}}\sqrt{\pi}, & n-1=2k \\ \frac{k!}{2\lambda^{2k+2}}, & n-1=2k+1 \end{array}\right\}.
\end{aligned}$$

c) We will show that $\lim\limits_{\lambda\to\infty}\int\limits_{|t|\geq\delta}\rho(x_0+t)K_\lambda(t)dt = 0.$

i) For $0 < t_k < 1$ *or* $t_k \geq 1, k = 1, \ldots, n$, *we have*

$$\begin{aligned}
&\int\limits_{|t|\geq\delta} \rho(x_0+t)K_\lambda(t)dt \\
&= \frac{\lambda^n}{\pi^{\frac{n}{2}}} \int\limits_{|t|\geq\delta} \frac{e^{-\lambda^2(t_1^2+\ldots+t_n^2)}}{\sqrt{(t_1+x_{01})\ldots(t_n+x_{0n})}(1+(t_1+x_{01})\ldots(t_n+x_{0n}))} dt_1\ldots dt_n \\
&\leq \frac{\lambda^n}{\pi^{\frac{n}{2}}} \frac{1}{\sqrt{x_{01}\ldots x_{0n}}(1+x_{01}\ldots x_{0n})} \int\limits_{|t|\geq\delta} e^{-\lambda^2(t_1^2+\ldots+t_n^2)} dt_1\ldots dt_n \\
&= \frac{\lambda^n}{\pi^{\frac{n}{2}}} \frac{1}{\sqrt{x_{01}\ldots x_{0n}}(1+x_{01}\ldots x_{0n})} \int\limits_{\delta}^{\infty} \int\limits_{S_{n-1}} r^{n-1}e^{-\lambda^2r^2}drdt' \\
&= \frac{1}{\pi^{\frac{n}{2}}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{\sqrt{x_{01}\ldots x_{0n}}(1+x_{01}\ldots x_{0n})} \int\limits_{\lambda\delta}^{\infty} r^{n-1}e^{-r^2}dr.
\end{aligned}$$

Here, S_{n-1} *denotes the surface of the unit sphere and* $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ *is the surface area of the unit sphere* S_{n-1} *of* E_n.

ii) For other t' *s,*

$$\begin{aligned}
\int\limits_{|t|\geq\delta} \rho(x_0+t)K_\lambda(t)dt &< \frac{\lambda^n}{\pi^{\frac{n}{2}}} e^{-(|x_{01}|+\ldots+|x_{0n}|)} \int\limits_{\delta}^{\infty} \int\limits_{S_{n-1}} r^{n-1}e^{-\lambda^2r^2}drdt' \\
&= \frac{\lambda^n}{\pi^{\frac{n}{2}}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} e^{-(|x_{01}|+\ldots+|x_{0n}|)} \int\limits_{\delta}^{\infty} r^{n-1}e^{-\lambda^2r^2}dr \\
&= \frac{2}{\Gamma(\frac{n}{2})} e^{-(|x_{01}|+\ldots+|x_{0n}|)} \int\limits_{\lambda\delta}^{\infty} r^{n-1}e^{-r^2}dr.
\end{aligned}$$

Hence, since $\int\limits_{\lambda\delta}^{\infty} r^{n-1}e^{-r^2}dr$ *tends to zero as* $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} \int\limits_{|t|\geq\delta} \rho(x_0+t)K_\lambda(t)dt = 0.$$

Therefore, all of the conditions of main result are satisfied.

Example 2. *Let* $\rho(t) = \left\{ \begin{array}{ll} \frac{1}{|t|^{\frac{n}{2}}}, & |t| \leq 1 \\ \frac{1}{|t|^{2n}}, & |t| > 1 \end{array} \right\}$ *with* $t = (t_1, t_{2,}, .., t_n)$. *We will show that* $\rho \in L_1(-\infty, \infty)$.

It is clear that

$$\int\limits_{E_n} \rho(t)dt = \int\limits_{|t|\leq 1} \frac{1}{|t|^{\frac{n}{2}}} dt + \int\limits_{|t|>1} \frac{1}{|t|^{2n}} dt < \infty.$$

Hence, we showed that $\rho \in L_1(E_n)$.

We let $f(t) = \left\{ \begin{array}{ll} \frac{1}{|t|^{n}}, & |t| \leq 1 \\ \frac{1}{|t|^{4n}}, & |t| > 1 \end{array} \right\}$. *Then, we can write*

$$\int\limits_{E_n} f(t)dt = \int\limits_{|t|\leq 1} \frac{1}{|t|^{n}} dt + \int\limits_{|t|>1} \frac{1}{|t|^{4n}} dt = \infty$$

and we see that $f \notin L_1(E_n)$.

Now, we look $\frac{f}{\rho}$, *that is,* $\frac{f(t)}{\rho(t)} = \left\{ \begin{array}{ll} \frac{1}{|t|^{\frac{n}{2}}}, & |t| \leq 1 \\ \frac{1}{|t|^{2n}}, & |t| > 1 \end{array} \right\}$. *Then, we obtain* $\frac{f}{\rho} \in L_1(E_n)$.

Also, we will examine $\alpha(\delta) = \sup\limits_{\substack{x_0 \in E_n \\ |t| \leq \delta}} \frac{\rho(x_0+t)}{\rho(x_0)}$. *We can write*

$$\frac{\rho(x_0+t)}{\rho(x_0)} = \left\{ \begin{array}{ll} \frac{|x_0|^{\frac{n}{2}}}{|x_0+t|^{\frac{n}{2}}}, & |t| \leq 1 \\ \frac{|x_0|^{2n}}{|x_0+t|^{2n}}, & |t| > 1 \end{array} \right\}$$

at a fixed point $x_0 = (x_{01} \ldots x_{0n})$. *It is clear that* $\alpha(\delta) = \sup\limits_{\substack{x_0 \in E_n \\ |t| \leq \delta}} \frac{\rho(x_0+t)}{\rho(x_0)} = 1$.

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