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A HARMONIC MEAN INEQUALITY CONCERNING THE GENERALIZED EXPONENTIAL INTEGRAL FUNCTION

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ABSTRACT. In this paper, we prove that for $s \in (0, \infty)$, the harmonic mean of $E_k(s)$ and $E_k(1/s)$ is always less than or equal to $\Gamma(1-k, 1)$. Where $E_k(s)$ is the generalized exponential integral function, $\Gamma(u, s)$ is the upper incomplete gamma function and $k \in \mathbb{N}$.

1. INTRODUCTION

Special functions play a pivotal role in both pure and applied mathematics. In particular, they are frequently encountered in mathematical analysis, mathematical physics, probability and statistics, and engineeering.

The classical exponential integral which is one of the most celebrated special functions is defined for $s > 0$ as [1, p. 228]

$$E(s) = \int_s^\infty \frac{e^{-t}}{t} dt = \int_1^\infty \frac{e^{-st}}{t} dt = \Gamma(0, s),$$

where $\Gamma(u, s)$ is the upper incomplete gamma function defined as

$$\Gamma(u, s) = \int_s^\infty t^{u-1} e^{-t} dt.$$

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It may also be defined as follows

$$E(s) = -\gamma - \ln s + \sum_{r=1}^{\infty} \frac{(-1)^{r+1} s^r}{r! r}$$
$$= -\ln s + e^{-s} \sum_{r=0}^{\infty} \frac{s^r}{r!} \psi(r+1),$$

where γ is the Euler-Mascheroni constant and $\psi(.)$ is the digamma function.

The generalized exponential integral function is defined as [2]

$$E_k(s) = s^{k-1} \int_s^{\infty} \frac{e^{-t}}{t^k} dt$$
$$= \int_1^{\infty} \frac{e^{-st}}{t^k} dt \tag{1.1}$$
$$= s^{k-1} \Gamma(1-k, s), \tag{1.2}$$

where $k \in \mathbb{N}$ is the order of the integral and $E_1(s) = E(s)$. It is also known in some text as the Theis well function [3]. This special function has useful applications in astrophysics, neutron physics, quantum chemistry, hydrology and other applied sciences. As a result of its practical importance, it has been investigated in different directions. For example, see [3], [4], [5], [6], [7], [8], [9], [11] and the references therein.

In a recent work [12], it was established that for $s \in (0, \infty)$, the harmonic mean of $E(s)$ and $E(1/s)$ is always less than or equal to $\Gamma(0,1) = 0.21938393...$. In this work, the goal is to extend this result to the generalized function $E_k(s)$. We present our findings in the following sections.

2. Results

Lemma 2.1. *The function* $\frac{sE_k'(s)}{E_k^2(s)}$ *is strictly decreasing on* $(0, \infty)$.

Proof. By identity (1.2) we obtain

$$E_k'(s) = \frac{(k-1)E_k(s) - e^{-s}}{s}$$

and since $E_k(s)$ is decreasing, then

$$(k-1)E_k(s) < e^{-s}. \tag{2.1}$$

Now let

$$T(s) = \frac{sE_k'(s)}{E_k^2(s)} = \frac{(k-1)E_k(s) - e^{-s}}{E_k^2(s)}$$

for $s \in (0, \infty)$. Then by differentiating and using (1.1) and (2.1), we obtain

$$\begin{aligned} E_k^3(s)T'(s) &= -(k-1)E_k(s)E_k'(s) + e^{-s}E_k(s) + 2e^{-s}E_k'(s) \\ &< -e^{-s}E_k'(s) + e^{-s}E_k(s) + 2e^{-s}E_k'(s) \\ &= e^{-s}\left[E_k(s) + E_k'(s)\right] \\ &= e^{-s}\int_1^\infty \left[\frac{1-t}{t^k}\right] e^{-st}dt \\ &< 0. \end{aligned}$$

Hence, $T'(s) < 0$ and this completes the proof of the lemma. □

Theorem 2.1. *For $s \in (0, \infty)$ and $k \in \mathbb{N}$, the inequality*

$$\frac{2E_k(s)E_k(1/s)}{E_k(s) + E_k(1/s)} \leq \Gamma(1-k, 1) \tag{2.2}$$

is satisfied. Equality holds when $s = 1$.

Proof. The case for $s = 1$ is apparent. So let $Q(s) = \frac{2E_k(s)E_k(1/s)}{E_k(s)+E_k(1/s)}$ and $\chi(s) = \ln Q(s)$ for $s \in (0,1) \cup (1, \infty)$. Then by direct computation, we obtain

$$\chi'(s) = \frac{E_k'(s)}{E_k(s)} - \frac{1}{s^2}\frac{E_k'(1/s)}{E_k(1/s)} - \frac{E_k'(s) - \frac{1}{s^2}E_k'(1/s)}{E_k(s) + E_k(1/s)}$$

and this implies that

$$s\left[E_k(s) + E_k(1/s)\right]\chi'(s) = s\frac{E_k'(s)}{E_k(s)}E_k(1/s) - \frac{1}{s}\frac{E_k'(1/s)}{E_k(1/s)}E_k(s).$$

Further manipulation reveals that

$$s\left[\frac{1}{E_k(s)} + \frac{1}{E_k(1/s)}\right]\chi'(s) = s\frac{E_k'(s)}{E_k^2(s)} - \frac{1}{s}\frac{E_k'(1/s)}{E_k^2(1/s)} = A(s).$$

As a consequence of Lemma 2.1, we conclude that $A(s) > 0$ if $s \in (0,1)$ and $A(s) < 0$ if $s \in (1, \infty)$. This implies that, $\chi'(z) > 0$ if $s \in (0,1)$ and $\chi'(s) < 0$ if $s \in (1, \infty)$. Therefore, $Q(s)$ is increasing on $(0,1)$ and decreasing on $(1, \infty)$. For both cases, we arrive at

$$Q(s) < \lim_{s \to 1} Q(s) = E_k(1) = \Gamma(1-k, 1)$$

and this concludes the proof of the theorem. □

Figure 1 is a graphical illustration of the results of Theorem 2.1 for the special cases where $k = 1$, $k = 2$, $k = 3$, $k = 4$ and $k = 5$.

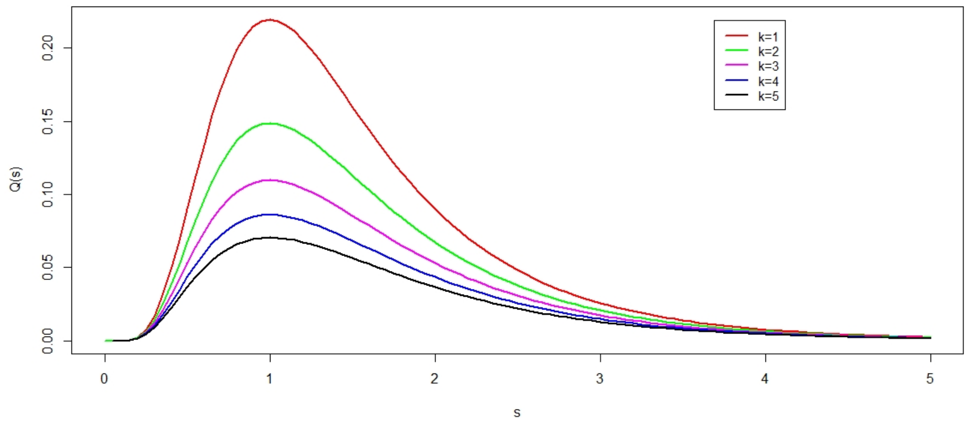

FIGURE 1. Plot of $Q(s)$ for some particular values of k.

3. Concluding Remarks

Let $\mathcal{H}(u,v) = \frac{2uv}{u+v}$, $\mathcal{A}(u,v) = \frac{u+v}{2}$ and $\mathcal{Q}(u,v) = \sqrt{\frac{u^2+v^2}{2}}$ respectively be the harmonic mean, arithmetic mean and root-square mean of u and v. In this paper, we have proved that

$$\mathcal{H}\left(E_k(s), E_k(1/s)\right) \leq \Gamma(1-k,1), \tag{3.1}$$

where $s \in (0,\infty)$. This is equivalent to

$$\mathcal{A}\left(\frac{1}{E_k(s)}, \frac{1}{E_k(1/s)}\right) \geq \frac{1}{\Gamma(1-k,1)}. \tag{3.2}$$

Also, since $Q(u,v) \geq A(u,v)$ [10], then we have

$$\mathcal{Q}\left(\frac{1}{E_k(s)}, \frac{1}{E_k(1/s)}\right) \geq \frac{1}{\Gamma(1-k,1)}, \tag{3.3}$$

which implies that

$$\mathcal{H}\left(E_k^2(s), E_k^2(1/s)\right) \leq \Gamma^2(1-k,1) < \Gamma(1-k,1). \tag{3.4}$$

For the particular case where $k = 1$, inequality (3.1) reduces to the results of [12]. We anticipate that the results of this paper will inspire further studies on the generalized exponential integral function.

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