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COMMUTING SYMMETRIC BI-SEMIDERIVATIONS ON RINGS

Faiza Shujat[1], Abu Zaid Ansari, and Kapil Kumar

ABSTRACT. In the present note, we discuss the notion of symmetric bi-semiderivations on rings and prove some commutativity results for commuting bi-semiderivations. Moreover, we obtain the characterization of symmetric bi-semiderivation on prime ring.

1. INTRODUCTION

The idea of symmetric bi-derivations inaugrated by Maksa [7]. He shown that symmetric bi-derivations are related to general solution of some functional equations. A mapping $D : R\times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $D : R\times R \to R$ is called bi-additive if it is additive in both slot. Now we introduce the concept of symmetric bi-derivations as follows: A bi-additive mapping $D : R \times R \longrightarrow R$ is called a bi-derivation if for every $x \in R$, the map $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a derivation of R. That is, $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$ and $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$. A mapping $d : R \to R$ defined by $d(x) = D(x, x)$, where $D : R \times R \to R$ is a symmetric mapping is called the trace of D.

[1]*corresponding author*

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Bergen [3] define the concept of semiderivations of a ring R. An additive mapping $f : R \longrightarrow R$ is called a semiderivation if there exists a function $g : R \longrightarrow R$ such that $f(ab) = f(a)g(b) + af(b) = f(a)b + g(a)f(b)$ and $f(g(a)) = g(f(a))$ for all $a, b \in R$. In case g is an identity map of R; then all semiderivations associated with g are merely ordinary derivations. On the other hand, if g is a homomorphism of R such that $g \neq 1$ $(identity\ map)$; then $f = g - 1$ is a semiderivation which is not a derivation. In case R is prime and $f \neq 0$; it has been shown by Chang [5] that g must necessarily be a ring endomorphism.

Inspired by the definition in [10], a symmetric bi-additive function $D : R \times R \longrightarrow R$ is called a symmetric bi-semiderivation associated with a function $f : R \longrightarrow R$ (or simply a symmetric bi-semiderivation of a ring R) if

$$D(ab, c) = D(a, c)f(b) + aD(b, c) = D(a, c)b + f(a)D(b, c)$$

and $d(f(a)) = f(d(a))$ for all $a, b, c \in R$, where $d : R \longrightarrow R$ stands for the trace of D.

Example 1. *Consider R is a commutative ring and*

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in R \right\}$$

will be a ring under matrix addition and multiplication. Define $D : S \times S \longrightarrow S$ such that

$$D\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & cf \end{pmatrix}$$

and $f : S \longrightarrow S$ by

$$f\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then D represents a bi-semiderivation on S associated with a function f.

In [11], author proved the following result: Let R be a noncommutative prime ring of characteristic different from two and $D : R \times R \longrightarrow R$ be a symmetric biderivation with trace d. If d is commuting on R, then $D = 0$. Further generalization of this idea presented by Ali et.al. [1]. Very recently authors in [2] generalized the notion in rings with involution. Our theorem 2.2 is the extension of a results proved in [12]. Author in [12] established that: Let R be a prime ring of characteristic not two and three. If D_1, D_2 are the symmetric biderivations of R with trace d_1, d_2, respectively, such that $d_1(x)d_2(x) = 0$ for all x in R, then either $D_1 = 0$ or $D_2 = 0$. For the more related literature reader can look inside [1,2,4,6,7,9] and the references therein. Continuing our investigation, we prove some commutativity results for commuting bi-semiderivations

2. Symmetric bi-semiderivation

Throughout we consider f a surjective function on R. To prove our main theorems, we need the following lemma.

Lemma 2.1. *Let R be a prime ring of characteristic not two, I be a nonzero right (left) ideal of R and D be a symmetric bi-semiderivation having associated function f on R. If $D(x,x) = 0$, for all $x \in I$, then either $D = 0$ on R or R is commutative.*

Proof. Consider $D(x,x) = 0$ for all $x \in I$. Linearization gives that $D(x,y) = 0$ for all $x, y \in I$. Replace yr for y to get $yD(x,r) = 0$ for all $x, y \in I$ and $r \in R$. Again replace x by xs to find $yxD(s,r) = 0$ for all $x, y \in I$ and $r, s \in R$. By simple manipulation we can have $[y,x]RD(s,r) = 0$, for all $x, y \in I$ and $r, s \in R$. Next making two additive sets namely $\mathrm{K} = \{y \in R : [y,x] = 0, \text{ for all } x \in R\}$ and $\mathrm{L} = \{s \in R : D(s,r) = 0, r \in R\}$. Since a group can not be the union of its two subgroups. Therefore, either $\mathrm{K} = 0$ or $\mathrm{L} = 0$ by Brauer's trick. In the first case if $[y,x] = 0$ for all $x, y \in I$. A simple calculation yields that $[y,t]R[s,x] = 0$ for all $x, y \in I$ and $t, s, r \in R$. Hence argument of primeness is sufficient to complete the proof. □

Theorem 2.1. *Let R be a prime ring of characteristic not two, I be a nonzero right ideal of R and D be a symmetric bi-semiderivation having associated function f on R. If $[D(x,x), x] = 0$, for all $x \in I$, then either $D = 0$ or R is commutative.*

Proof. Given that D is commuting on I, that is,

$$[D(x,x), x] = 0, \text{ for all } x \in I. \tag{2.1}$$

Linearization of above equation yields that

$$2[D(x,y), x] + 2[D(x,y), y] + [D(y,y), x] + [D(x,x), y] = 0, \text{ for all } x, y \in I. \tag{2.2}$$

Putting $-y$ in place of y in equation (2.2) and using (2.2) with characteristic condition, we find

$$[D(y,y),x]+2[D(x,y),y]=0, \text{ for all } x,y\in I. \tag{2.3}$$

Substitute xr for x in (2.3) and use (2.3) to get

$$[D(y,y),x]r+2[D(x,y),y]f(r)+2D(x,y)[f(r),y]+2[x,y]D(r,y)=0, \tag{2.4}$$

for all $x,y\in I$, $r\in R$. By surjectiveness of f allow us to write $f(r)=r\in R$ in (2.4) and simplifying in view of (2.3), we obtain

$$D(x,y)[r,y]+[x,y]D(r,y)=0, \tag{2.5}$$

for all $x,y\in I$, $r\in R$.

Again replace x by xs in above equation to find

$$D(x,y)f(s)[r,y]+[x,y]sD(r,y)=0, \text{ for all } x,y\in I,\ r,s\in R. \tag{2.6}$$

In particular, we obtain

$$[x,y]sD(y,y)=0, \text{ for all } x,y\in I,\ s\in R. \tag{2.7}$$

Now proceed in the same way as in Lemma 2.1. This completes the proof. □

Theorem 2.2. *Let R be a prime ring of characteristic not two and D_1,D_2 be two symmetric bi-semiderivations having associated function f on R. If $D_1(x,x)D_2(x,x)=0$, for all $x\in R$, then one of the following conditions hold.*

(1) *R is commutative.*
(2) *$D_1=0$.*
(3) *$D_2=0$.*

Proof. By given hypothesis we can have

$$D_1(x,x)D_2(x,x)=0 \text{ for all } x\in R. \tag{2.8}$$

Linearize the above equation to get

$$\begin{aligned}&D_1(x,x)D_2(x,x)+D_1(x,x)D_2(y,y)+2D_1(x,x)D_2(x,y)\\&+D_1(y,y)D_2(x,x)+D_1(y,y)D_2(y,y)+2D_1(y,y)D_2(x,y)\\&+2D_1(x,y)D_2(x,x)+2D_1(x,y)D_2(y,y)+4D_1(x,y)D_2(x,y)=0,\end{aligned} \tag{2.9}$$

for all $x, y \in R$. Putting $-y$ in place of y in (2.9) and adding the resulting equation with (2.9), we find

$$D_1(x,x)D_2(y,y) + D_1(y,y)D_2(x,x) + 4D_1(x,y)D_2(x,y) = 0 \tag{2.10}$$

for all $x, y \in R$. Again linearize (2.10) in x and using (2.10) along with characteristic condition on R to obtain

$$\begin{aligned} &D_1(x,z)D_2(y,y) + D_1(y,y)D_2(x,z) + 2D_1(z,y)D_2(x,y) \\ &+ 2D_1(x,y)D_2(z,y) = 0 \text{ for all } x, y, z \in R. \end{aligned} \tag{2.11}$$

Replacing z by zu in (2.11), we have

$$\begin{aligned} &D_1(x,z)f(u)D_2(y,y) + zD_1(x,u)D_2(y,y) + D_1(y,y)D_2(x,z)f(u) \\ &+ D_1(y,y)zD_2(x,u) + 2D_1(z,y)f(u)D_2(x,y) \\ &+ 2zD_1(u,y)D_2(x,y) + 2D_1(x,y)D_2(z,y)f(u) \\ &+ 2D_1(x,y)zD_2(u,y) = 0 \text{ for all } u, x, y, z \in R. \end{aligned} \tag{2.12}$$

Comparing (2.11) and (2.12) to find

$$\begin{aligned} &D_1(x,z)[f(u), D_2(y,y)] + 2D_1(z,y)[f(u), D_2(x,y)] + 2[D_1(y,y), z]D_2(x,u) \\ &+ 2[D_1(x,y), z]D_2(u,y) = 0 \text{ for all } u, x, y, z \in R. \end{aligned} \tag{2.13}$$

Particularly, if we take $x = y$ and $f(u) = t \in R$ in (2.13), we obtain

$$\begin{aligned} &D_1(x,z)[t, D_2(x,x)] + 2D_1(z,x)[t, D_2(x,x)] + 2[D_1(x,x), z]D_2(x,u) \\ &+ 2[D_1(x,x), z]D_2(u,x) = 0 \text{ for all } u, x, t, z \in R. \end{aligned} \tag{2.14}$$

Rewrite (2.14) by changing $D_2(x,x)$ for t and evaluate after applying characteristic condition to get

$$[D_1(x,x), z]D_2(u,x) = 0 \text{ for all } u, x, z \in R. \tag{2.15}$$

Putting su in place of u in (2.15), we get

$$[D_1(x,x), z]sD_2(u,x) = 0 \text{ for all } s, u, x, z \in R. \tag{2.16}$$

Now making two subgroups of R namely $\prod = \{y \in R, [D_1(y,y), z] = 0;$ for all $z \in R\}$ and $\coprod = \{y \in R, D_2(y,u) = 0;$ for all $u \in R\}$. Since $\prod$ and $\coprod$ are two additive subgroups of R. By group theory arguments that a group can not be the union of its two subgroups. Hence either $\prod = 0$ or $\coprod = 0$. This implies that either $D_2 = 0$ or $[D_1(y, y), z] = 0$ for all $y, z \in R$. In later case Theorem 2.1 gives the assertion of theorem. $\square$

Example 2. *Let*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$$

where S is any commutative ring. Consider $D : R \times R \longrightarrow R$ is defined as

$$D\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & b_1 b_2 \\ 0 & 0 \end{pmatrix},$$

and

$$f\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

By definition D is a symmetric bi-semiderivation on R. We can see that $DD = 0$ but $D \neq 0$. This implies that primeness will be the essential condition in Theorem 2.2.

3. Characterization of bi-semiderivations

In this section, we discuss the characterization of symmetric bi-semiderivation of a prime ring. A prominent result of Martindale states that if R is a prime ring and $a, b \in R$ satisfy $axb = bxa$ for all $x \in R$, then a and b are linearly dependent over C, the extended centroid of R. This result has proved to be a valuable tool in our learning and investigation. For detailed study about C and concerned topic one advised to look in [8]. To prove our main theorem, we need the following lemma.

Lemma 3.1. *[4] Let R be a prime ring and S be any subset of R. If functions $f, g, h : S \longrightarrow R$ satisfy $f(x)yg(z) = h(x)yf(z)$ for all $x, y, z \in S$ and $f \neq 0$, then $g(x) = h(x) = \lambda f(x)$ for some λ in the extended centroid C of R.*

Theorem 3.1. *Let R be a prime ring of characteristic not two and I be an ideal of R. If D_1, D_2 be two nonzero symmetric bi-semiderivations on R such that $D_1(x, x)D_2(y, y) = D_2(x, x)D_1(y, y)$ for all $x, y \in I$, then there exists $\lambda \in C$ such that $D_2(x, x) = \lambda D_1(x, x)$, for all $x \in I$.*

Proof. By the hypothesis of our theorem, it is given that

(3.1) $$D_1(x,x)D_2(y,y) = D_2(x,x)D_1(y,y) \text{ for all } x, y \in I.$$

Linearization of (3.1) in y yields that for all $x, y, z \in I$

(3.2) $$D_1(x,x)D_2(y,z) = D_2(x,x)D_1(y,z)$$

Substitute rz in place of z in (3.2) and compare the resulting equation with (3.2) to get

(3.3) $$D_1(x,x)rD_2(y,z) = D_2(x,x)rD_1(y,z) \text{ for all } x, y, z \in I, r \in R.$$

In particular, reword the above equation as

(3.4) $$D_1(x,x)rD_2(y,y) = D_2(x,x)rD_1(y,y) \text{ for all } x, y \in I, r \in R.$$

In light of Lemma 3.1, we find $D_2(x,x) = \lambda(x)D_1(x,x)$ for all $x \in I$ and $\lambda(x) \in C$. With this form of D_2, rewrite the equation (3.1) like

(3.5) $$D_1(x,x)\lambda(y)D_1(y,y) - \lambda(x)D_1(x,x)D_1(y,y) = 0 \text{ for all } x, y \in I.$$

A simple calculation gives that

(3.6) $$(\lambda(y) - \lambda(x))rD_1(x,x)D_1(y,y) = 0 \text{ for all } x, y \in I, r \in R.$$

Since $D_1 \neq 0$, hence we get $\lambda(x) = \lambda(y)$. This implies that $D_2(x,x) = \lambda D_1(x,x)$, for all $x \in I$.

□

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TAIBAH UNIVERSITY, MADINAH
SAUDI ARABIA.
Email address: `faiza.shujat@gmail.com`

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ISLAMIC UNIVERSITY OF MADINAH
SAUDI ARABIA.
Email address: `ansari.abuzaid@gmail.com`

DEPARTMENT OF MATHEMATICS
K. V. S. COLLEGE OF SCIENCE
SWAMI VIVEKANAND SUBHARTI UNIVERSITY, MEERUT
INDIA.
Email address: `01.kapilmathsamu@gmail.com`