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THE DYNAMICAL EVOLUTION OF GEOMETRIC UNCERTAINTY PRINCIPLE FOR SPIN 1/2 SYSTEM

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ABSTRACT. Geometric Quantum Mechanics is a formulation that demonstrates how quantum theory may be casted in the language of Hamiltonian phase-space dynamics. In this framework, the states are referring to points in complex projective Hilbert space, the observables are real valued functions on the space and the Hamiltonian flow is defined by Schrödinger equation. Recently, the effort to cast uncertainty principle in terms of geometrical language appeared to become the subject of intense study in geometric quantum mechanics. One has shown that the stronger version of uncertainty relation i.e. the Robertson-Schrödinger uncertainty relation can be expressed in terms of the symplectic form and Riemannian metric. In this paper, we investigate the dynamical behavior of the uncertainty relation for spin $\frac{1}{2}$ system based on this formulation. We show that the Robertson-Schrödinger uncertainty principle is not invariant under Hamiltonian flow. This is due to the fact that during evolution process, unlike symplectic area, the Riemannian metric is not invariant under the flow.

1. INTRODUCTION

Topology and geometry which have complex and advanced mathematical constructions have been commonly utilized in physics to explain the laws of nature

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[1]. These mathematical theories have found applications in many areas of physics [2,3] and one of the recent development is to apply the geometrical ideas of complex projective space $\mathbb{C}P^n$ to explain the peculiar properties of entanglement phenomena in quantum information theory [4,5].

There are numerous motivations for attempting to express quantum physics in geometrical terms. The fact that classical mechanics, general relativity and others are highly geometrical, for instance, inspired some physicists to cast quantum mechanics in geometrical language in order to better understanding the quantum-classical transition. The research line referred as Geometric Quantum Mechanics was partly motivated by work of Kibble in 1979 [6] which demonstrates how quantum theory may be formulated in the language of Hamiltonian phase-space dynamics. Many researchers have recently contributed to formulate the geometric version of quantum mechanics and demonstrate the significant of this formulation in order to provides us with crucial information about quantum realm and various application in foundations of quantum mechanics such as uncertainty principle, entanglement and many others [7,8].

It is generally accepted that uncertainty principle is a purely quantum concept and cannot be described using classical mechanics. However, this statement is not entirely true when one had shown that the uncertainty principle can naturally arise from the structure of classical mechanics [9]. This is achieved through a topological tool known as symplectic capacity together with the notion of quantum blob. Thus, it is clear that the uncertainty principle in this context is invariant under symplectic transformation since it can be expressed in term of symplectic capacity. In this paper, motivated by this work [9], the possibility of the uncertainty principle in geometric quantum mechanics is invariant under the Hamiltonian flows has been demonstrated since in this formulation the uncertainty principle is partly expressed in terms of symplectic form [10]. This research may becomes a significant step in order to constructs a connection between geometric quantum mechanics and symplectic topology.

In particular, section 2 briefly discuss the derivation of the geometric Robertson-Schrödinger uncertainty relation and the description on how to compute its evolution along Hamiltonian flow. Section 3 and 4 are the authors contribution where the calculation of the results of the geometric formulation of uncertainty principle for the case of spin $\frac{1}{2}$ system. All the author's research findings have been discussed and summarized in section 5.

2. ROBERTSON-SCHRÖDINGER UNCERTAINTY RELATION

Uncertainty principle, firstly discovered by the German theoretical physicist Werner Heisenberg is one of the fundamental concepts that shows the weirdness of quantum mechanics. It sets the limitation of complementary variables such as position and momentum to be measured simultaneously with high precision. After that, this principle was generalized to an arbitrary observables $\hat{A}$ and $\hat{B}$ known as Robertson uncertainty principle given by

$$(\Delta\hat{A})(\Delta\hat{B}) \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|, \tag{2.1}$$

and within a year, the stronger extension named Roberson-Schrödinger uncertainty principle was proposed by adding covariance term to the formulation

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|^2 + \left|\frac{1}{2}\langle[\hat{A},\hat{B}]_+\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2. \tag{2.2}$$

In geometric quantum mechanics, Ashtekar [10] shows that the symplectic form Ω and Riemannian metric G allow one to formulate a geometric version of Roberton-Schrödinger uncertainty principle. Let Ψ be a normalized state vector, the above formula (2.2) can be rephrased in Hilbert space $\mathcal{H}$ as

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \left(\frac{\hbar}{2}\Omega(X_{\hat{A}},X_{\hat{B}})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{A}},X_{\hat{B}}) - AB\right)^2, \tag{2.3}$$

where $(\Delta\hat{A})^2$ denotes a function on $\mathcal{H}$ given by $(\Delta\hat{A})^2(\Psi) := (\Delta\hat{A})^2_\Psi$ and

$$X_{\hat{A}} = -\frac{i}{\hbar}\hat{A}\Psi, \quad X_{\hat{B}} = -\frac{i}{\hbar}\hat{B}\Psi \tag{2.4}$$

are Schrödinger vector fields. We may see how the Robertson-Schrödinger uncertainty principle can also be expressed on the complex projective Hilbert space $P(\mathcal{H})$ i.e. the proper quantum phase space. Let consider the expectation values $A(\Psi)$ and $B(\Psi)$ of two quantum observables $\hat{A}$ and $\hat{B}$ respectively, and let a and b be the corresponding functions on $P(\mathcal{H})$, i.e.,

$$a\circ\Pi = \langle\hat{A}\rangle_\Psi = A(\Psi), \quad b\circ\Pi = \langle\hat{B}\rangle_\Psi = B(\Psi), \tag{2.5}$$

where Π is the canonical projection $\mathcal{H}\to P(\mathcal{H})$. Thus, for any $X_a = \Pi_*(X_{\hat{A}})$ and $X_b = \Pi_*(X_{\hat{B}})$ are elements of tangent vector at point ψ on $P(\mathcal{H})$ i.e. $T_\psi P(\mathcal{H})$, the uncertainty principle may be rephrased as the following equation in terms of symplectic form ω and Riemannian metric g define on $P(\mathcal{H})$:

$$(\Delta a)^2(\Delta b)^2 \geq \frac{\hbar^2}{4}(\omega(X_a, X_b)^2 + g(X_a, X_b)^2), \tag{2.6}$$

where $(\Delta a)^2(\psi) := (\Delta A)^2(\Psi)$ and $(\Delta b)^2(\psi) := (\Delta B)^2(\Psi)$.

In order to carry out this study, the Schrödinger vector field with respect to spin $\frac{1}{2}$ operators has been calculated followed by finding the solution of the state vector of the system. After that, these vector fields have been push-forwarded to one dimensional complex projective Hilbert space $\mathbb{C}P^1$ since it is the proper quantum phase space of spin $\frac{1}{2}$ system. Here, the contraction between push-forward vector fields with symplectic form and the components of Riemannian metrics have been computed. Finally, the evolution of Robertson-Schrödinger uncertainty principle can be obtained by finding its expression along the projection of solution corresponds to a Schrödinger vector field.

3. The evolution of Schrödinger vector fields for spin $\frac{1}{2}$ system

Let us compute the Robertson-Schrödinger uncertainty principles for the case of spin $\frac{1}{2}$ particles where self-adjoint operator are

$$\hat{S}_x = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}, \quad \hat{S}_y = \begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}, \quad \hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}.$$

Consider the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ and (e_1, e_2) represent the orthonormal basis in $\mathbb{C}^2$ satisfy

$$\langle e_\alpha | e_\beta \rangle = \delta_{\alpha\beta}.$$

Then the state of spin $\frac{1}{2}$ particle in $\mathcal{H}$ is expressed as

$$|\Psi\rangle = Z_1 |e_1\rangle + Z_2 |e_2\rangle .$$

First, we start with computing the Schrödinger vector field of the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$. The corresponding expectation value of these operators are

$$S_x(\Psi) = \langle\Psi| \hat{S}_x |\Psi\rangle = \frac{\hbar}{2}(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2); \tag{3.1}$$

$$S_y(\Psi) = \langle\Psi| \hat{S}_y |\Psi\rangle = \frac{i\hbar}{2}(Z_1\bar{Z}_2 - \bar{Z}_1 Z_2); \tag{3.2}$$

$$S_z(\Psi) = \langle\Psi| \hat{S}_z |\Psi\rangle = \frac{\hbar}{2}(|Z_1|^2 - |Z_2|^2). \tag{3.3}$$

Note that, here we we assume that $\mathbb{C}^2$ is complexification of real vector space $\mathbb{R}^4$. Thus the complexified tangent space is spanned by 4 vectors; $\frac{\partial}{\partial Z_1}, \frac{\partial}{\partial Z_2}, \frac{\partial}{\partial \bar{Z}_1}, \frac{\partial}{\partial \bar{Z}_2}$. Therefore, according to equation (2.4) the Schrödinger vector fields correspond to these operators are

$$X_{\hat{S}_x} = -\frac{Z_2}{2i}\frac{\partial}{\partial Z_1} - \frac{Z_1}{2i}\frac{\partial}{\partial Z_2} + \frac{\bar{Z}_2}{2i}\frac{\partial}{\partial \bar{Z}_1} + \frac{\bar{Z}_1}{2i}\frac{\partial}{\partial \bar{Z}_2}; \tag{3.4}$$

$$X_{\hat{S}_y} = \frac{Z_2}{2}\frac{\partial}{\partial Z_1} - \frac{Z_1}{2}\frac{\partial}{\partial Z_2} + \frac{\bar{Z}_2}{2}\frac{\partial}{\partial \bar{Z}_1} - \frac{\bar{Z}_1}{2}\frac{\partial}{\partial \bar{Z}_2}; \tag{3.5}$$

$$X_{\hat{S}_z} = -\frac{Z_1}{2i}\frac{\partial}{\partial Z_1} + \frac{Z_2}{2i}\frac{\partial}{\partial Z_2} + \frac{\bar{Z}_1}{2i}\frac{\partial}{\partial \bar{Z}_1} - \frac{\bar{Z}_2}{2i}\frac{\partial}{\partial \bar{Z}_2}. \tag{3.6}$$

The solutions of Z_1 and Z_2 according to $X_{\hat{S}_x}$ are computed as follows. From equation (3.4), we can show that

$$\frac{dZ_1}{dt} = -\frac{Z_2}{2i}, \quad \frac{dZ_2}{dt} = -\frac{Z_1}{2i}. \tag{3.7}$$

It is obvious that the general solution for equation Z_1 is

$$Z_1(t) = Ae^{\frac{it}{2}} + Be^{-\frac{it}{2}}, \tag{3.8}$$

where A and B are complex numbers. Note that, in this study, we assume that the frequency $\omega = 1$ is to simplify the calculation. Besides, from equation (3.8) we get

$$Z_2 = -2i\frac{d}{dt}(Ae^{\frac{it}{2}} + Be^{-\frac{it}{2}}) = Ae^{\frac{it}{2}} - Be^{-\frac{it}{2}}. \tag{3.9}$$

Furthermore, we calculate the solution for Z_1 and Z_2 with respect to $X_{\hat{S}_y}$. Referring to equation (3.5), it is obvious that

$$\frac{dZ_1}{dt} = \frac{Z_2}{2}, \qquad \frac{dZ_2}{dt} = -\frac{Z_1}{2}. \tag{3.10}$$

Clearly, the general solution for Z_1 and Z_2 are

$$Z_1(t) = Ce^{\frac{it}{2}} + De^{-\frac{it}{2}}; \tag{3.11}$$

$$Z_2(t) = iCe^{\frac{it}{2}} - iDe^{-\frac{it}{2}} \tag{3.12}$$

where $C, D \in \mathbb{C}$. This is not surprising that the solution (3.11) and (3.12) are quite similar to (3.8) and (3.9) respectively since the expression of $\frac{dZ_1}{dt}$ and $\frac{dZ_2}{dt}$ for $X_{\hat{S}_y}$ are difference from $X_{\hat{S}_x}$ by a factor of i.

Lastly, we find a solution of Z_1 and Z_2 for the case of $X_{\hat{S}_z}$. According to equation (3.6), one can show that

$$\frac{dZ_1}{dt} = -\frac{Z_1}{2i}, \qquad \frac{dZ_2}{dt} = \frac{Z_2}{2i}. \tag{3.13}$$

Solve the equations (3.13) we get

$$Z_1(t) = Ee^{\frac{it}{2}}; \qquad Z_2(t) = Fe^{-\frac{it}{2}},$$

where E and F are complex numbers. These solutions are simple compared to the case of $X_{\hat{S}_x}$ and $X_{\hat{S}_y}$ due to the fact that the eigenstates of operator $\hat{S}_z$ are simply basis vectors of $\mathbb{C}^2$ i.e. $|e_1\rangle, |e_2\rangle$ while eigenstates of the operators $\hat{S}_x$ and $\hat{S}_y$ are more complicated which is a combination of the basis vectors. Note that the solutions $Z_1(t)$ and $Z_2(t)$ which correspond to the evolution of vector fields $X_{\hat{S}_x}$, $X_{\hat{S}_y}$ and $X_{\hat{S}_z}$ preserve the initial value of expectation values S_x, S_y and S_z respectively. Note that, any state vector $\Psi, \Phi \in \mathcal{H}$ such that $\Phi = c\Psi, c \in \mathbb{C}$ defines the same physical state. Thus, it is necessary to find the expression of Robertson-Schrödinger uncertainty principle in $\mathbb{C}P^1$ which is the quantum phase space of spin $\frac{1}{2}$ particle.

4. The evolution of geometric uncertainty principle in $\mathbb{C}P^1$

In order to compute the Robertson-Schrödinger uncertainty principle on $\mathbb{C}P^1$, we need to find the pushforward vector fields of $X_{\hat{S}_x}$, $X_{\hat{S}_y}$ and $X_{\hat{S}_z}$ under the map $\Pi_* : T_\Psi\mathcal{H} \to T_\psi P(\mathcal{H})$. Let $\Pi(Z_1, Z_2) = z = \frac{Z_2}{Z_1}$ be a local coordinate of U_1 where $Z_1 \neq 0$. Then, the pushforward vector fields correspond to $X_{\hat{S}_x}$, $X_{\hat{S}_y}$ and $X_{\hat{S}_z}$ are

$$\begin{aligned}\Pi_* X_{\hat{S}_x} &= -\frac{Z_1 z}{2i}\left(-\frac{Z_2}{Z_1^2}\frac{\partial}{\partial z}\right) - \frac{Z_1}{2i}\left(\frac{1}{Z_1}\frac{\partial}{\partial z}\right) + \frac{\bar{Z}_1\bar{z}}{2i}\left(-\frac{\bar{Z}_2}{\bar{Z}_1^2}\frac{\partial}{\partial \bar{z}}\right) + \frac{\bar{Z}_1}{2i}\left(\frac{1}{\bar{Z}_1}\frac{\partial}{\partial \bar{z}}\right)\\ &= \frac{zZ_2}{2iZ_1}\frac{\partial}{\partial z} - \frac{1}{2i}\frac{\partial}{\partial z} - \frac{\bar{z}\bar{Z}_2}{2i\bar{Z}_1}\frac{\partial}{\partial \bar{z}} + \frac{1}{2i}\frac{\partial}{\partial \bar{z}} = \frac{(z^2-1)}{2i}\frac{\partial}{\partial z} - \frac{(\bar{z}^2-1)}{2i}\frac{\partial}{\partial \bar{z}};\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Pi_* X_{\hat{S}_y} &= \frac{Z_1 z}{2}\left(-\frac{Z_2}{Z_1^2}\frac{\partial}{\partial z}\right) - \frac{Z_1}{2}\left(\frac{1}{Z_1}\frac{\partial}{\partial z}\right) + \frac{\bar{Z}_1\bar{z}}{2}\left(-\frac{\bar{Z}_2}{\bar{Z}_1^2}\frac{\partial}{\partial \bar{z}}\right) - \frac{\bar{Z}_1}{2}\left(\frac{1}{\bar{Z}_1}\frac{\partial}{\partial \bar{z}}\right)\\ &= -\frac{zZ_2}{2Z_1}\frac{\partial}{\partial z} - \frac{1}{2}\frac{\partial}{\partial z} - \frac{\bar{z}\bar{Z}_2}{2\bar{Z}_1}\frac{\partial}{\partial \bar{z}} - \frac{1}{2}\frac{\partial}{\partial \bar{z}} = -\frac{(z^2+1)}{2}\frac{\partial}{\partial z} - \frac{(\bar{z}^2+1)}{2}\frac{\partial}{\partial \bar{z}};\end{aligned} \tag{4.2}$$

$$\Pi_* X_{\hat{S}_z} = -\frac{Z_1}{2i}\left(-\frac{Z_2}{Z_1^2}\frac{\partial}{\partial z}\right) + \frac{Z_1 z}{2i}\left(\frac{1}{Z_1}\frac{\partial}{\partial z}\right) + \frac{\bar{Z}_1}{2i}\left(-\frac{\bar{Z}_2}{\bar{Z}_1^2}\frac{\partial}{\partial \bar{z}}\right) - \frac{\bar{Z}_1 \bar{z}}{2i}\left(\frac{1}{\bar{Z}_1}\frac{\partial}{\partial \bar{z}}\right)$$

$$= \frac{Z_2}{2iZ_1}\frac{\partial}{\partial z} + \frac{z}{2i}\frac{\partial}{\partial z} - \frac{\bar{Z}_2}{2i\bar{Z}_1}\frac{\partial}{\partial \bar{z}} - \frac{\bar{z}}{2i}\frac{\partial}{\partial \bar{z}} = -iz\frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial \bar{z}}. \tag{4.3}$$

Note that, in $\mathbb{C}P^1$, the symplectic form ω is expressed as

$$\omega = i\hbar\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = i\hbar\frac{dz \otimes d\bar{z}}{(1+|z|^2)^2} - i\hbar\frac{d\bar{z} \otimes dz}{(1+|z|^2)^2}$$

and the Riemannian metric is denoted by

$$g = \hbar\frac{dz \otimes d\bar{z}}{(1+|z|^2)^2} + \hbar\frac{d\bar{z} \otimes dz}{(1+|z|^2)^2}.$$

Now we are ready to compute the Robertson-Schrödinger uncertainty principle for the case of spin $\frac{1}{2}$ on $\mathbb{C}P^1$. Let us define the Robertson-Schrödinger uncertainty relations for this case as

$$(\Delta\hat{S}_y)^2(\Delta\hat{S}_z)^2 \geq \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2 + g(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2]; \tag{4.4}$$

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_z)^2 \geq \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2 + g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2]; \tag{4.5}$$

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 \geq \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2 + g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2], \tag{4.6}$$

where the contraction of pushforward vector fields with symplectic form ω are

$$\omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z}) = \iota_{\Pi_* X_{\hat{S}_y}}\iota_{\Pi_* X_{\hat{S}_z}}\omega = -\frac{\hbar(\bar{z}+z)}{2(1+|z|^2)}; \tag{4.7}$$

$$\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z}) = \iota_{\Pi_* X_{\hat{S}_x}}\iota_{\Pi_* X_{\hat{S}_z}}\omega = \frac{i\hbar(\bar{z}-z)}{2(1+|z|^2)}; \tag{4.8}$$

$$\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y}) = \iota_{\Pi_* X_{\hat{S}_x}}\iota_{\Pi_* X_{\hat{S}_y}}\omega = \frac{\hbar(|z|^4-1)}{2(1+|z^2|)^2} \tag{4.9}$$

and components of the Riemannian metrics correspond to these vectors are given as

$$g(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z}) = -\frac{i\hbar[(|z|^2-1)(z-\bar{z})]}{2(1+|z|^2)^2}; \tag{4.10}$$

$$g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z}) = \frac{\hbar[(|z|^2-1)(z+\bar{z})]}{2(1+|z|^2)^2}; \tag{4.11}$$

$$g(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y}) = \frac{i\hbar(z^2 - \bar{z}^2)}{2(1+|z|^2)^2}. \tag{4.12}$$

Equations (4.4), (4.5) and (4.6) can now be expressed by

$$(\Delta \hat{S}_y)^2 (\Delta \hat{S}_z)^2 \geq \frac{1}{16}\left[\left(-\frac{\hbar^2[\bar{z}+z]}{1+|z|^2}\right)^2 + \left(-\frac{i\hbar^2[(|z|^2-1)(z-\bar{z})]}{[1+|z|^2]^2}\right)^2\right]; \tag{4.13}$$

$$(\Delta \hat{S}_x)^2 (\Delta \hat{S}_z)^2 \geq \frac{1}{16}\left[\left(\frac{i\hbar^2[\bar{z}-z]}{1+|z|^2}\right)^2 + \left(\frac{\hbar^2[(|z|^2-1)(z+\bar{z})]}{[1+|z|^2]^2}\right)^2\right]; \tag{4.14}$$

$$(\Delta \hat{S}_x)^2 (\Delta \hat{S}_y)^2 \geq \frac{1}{16}\left[\left(\frac{\hbar^2[|z|^4-1]^2}{[1+|z|^2]^2}\right)^2 + \left(\frac{i\hbar^2[z^2-\bar{z}^2]}{[1+|z|^2]^2}\right)^2\right]. \tag{4.15}$$

Thus, the evolution of Robertson-Schrödinger uncertainty principle for the case of

• $\hat{S}_y$ and $\hat{S}_z$ is

$$\begin{aligned}(\Delta \hat{S}_y)^2 (\Delta \hat{S}_z)^2 \geq \frac{1}{16} \big[&(2\hbar^2[|B|^2 - |A|^2])^2 \\ &+ (4i\hbar^2[A^2\bar{B}^2 e^{2it} - \bar{A}^2 B^2 e^{-2it}])^2\big]\end{aligned} \tag{4.16}$$

along the projection of solution associated with Schrödinger vector field $X_{\hat{S}_x}$

$$\Pi(\Psi(t)) = \frac{Z_2(t)}{Z_1(t)} = \frac{Ae^{\frac{it}{2}} - Be^{-\frac{it}{2}}}{Ae^{\frac{it}{2}} + Be^{-\frac{it}{2}}}; \tag{4.17}$$

• $\hat{S}_x$ and $\hat{S}_z$ is

$$\begin{aligned}(\Delta \hat{S}_x)^2 (\Delta \hat{S}_z)^2 \geq \frac{1}{16} \big[&(2\hbar^2[|C|^2 - |D|^2])^2 \\ &+ (4i\hbar^2[C^2\bar{D}^2 e^{2it} - \bar{C}^2 D^2 e^{-2it}])^2\big]\end{aligned} \tag{4.18}$$

along the projection of solution corresponds to Schrödinger vector field $X_{\hat{S}_y}$

$$\Pi(\Psi(t)) = \frac{Z_2(t)}{Z_1(t)} = \frac{i(Ce^{\frac{it}{2}} - De^{-\frac{it}{2}})}{Ce^{\frac{it}{2}} + De^{-\frac{it}{2}}}; \tag{4.19}$$

• $\hat{S}_x$ and $\hat{S}_y$ is

$$(\Delta \hat{S}_x)^2 (\Delta \hat{S}_y)^2 \geq \frac{1}{16}\left[(\hbar^2[|F|^2 - |E|^2])^2 + (i\hbar^2[F^2\bar{E}^2 e^{2it} - \bar{F}^2 E^2 e^{-2it}])^2\right] \tag{4.20}$$

along the projection of solution corresponds to Schrödinger vector field $X_{\hat{S}_z}$

$$\Pi(\Psi(t)) = \frac{Z_2(t)}{Z_1(t)} = \frac{Fe^{-\frac{it}{2}}}{Ee^{\frac{it}{2}}}. \tag{4.21}$$

5. Discussion and Conclusion

Let $i, j, k = x, y, z$ and $i \neq j \neq k$. According to the computations above we show that:

(1) The contraction of $\Pi_* X_{\hat{S}_i}$ and $\Pi_* X_{\hat{S}_j}$ with symplectic form $\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$ is invariant under projection of Hamiltonian flow induced by $X_{\hat{S}_k}$ implies that the area between vectors $\Pi_* X_{\hat{S}_i}$ and $\Pi_* X_{\hat{S}_j}$ is preserved under the transformation. This is because $\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j}) = \Omega(X_{\hat{S}_i}, X_{\hat{S}_j}) = S_k$ and the expectation value S_k is uniquely conserved along $X_{\hat{S}_k}$ since it satisfies the condition $\iota_{X_{\hat{S}_k}} \Omega = dS_k$.

(2) The Riemannian metric $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$ is non-zero and varies under any Hamiltonian flow showing that the magnitude and angle between $\Pi_* X_{\hat{S}_i}$ and $\Pi_* X_{\hat{S}_j}$ are changing under the transformation. However these vectors preserve the symplectic area $\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$. Besides, the Riemannian metric $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})$ represent the covariance since $g(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j}) = G(X_{\hat{S}_i}, X_{\hat{S}_j}) - \frac{2}{\hbar} S_i S_j$. Here, it is clear that the covariance is purely depend on product of the expectation values $S_i S_j$ since $G(X_{\hat{S}_i}, X_{\hat{S}_j}) = 0$.

(3) The right hand side of uncertainty principle can be fully expressed by contraction of symplectic form with Hamiltonian vector fields that is

$$\begin{aligned}
(\Delta \hat{S}_y)^2 (\Delta \hat{S}_z)^2 \geq \frac{\hbar^2}{4} \Big[& \omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2 \\
& + \frac{4}{\hbar^2} \omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2 \omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2 \Big] ; \\
(\Delta \hat{S}_x)^2 (\Delta \hat{S}_z)^2 \geq \frac{\hbar^2}{4} \Big[& \omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2 \\
& + \frac{4}{\hbar^2} \omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2 \omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2 \Big] ;
\end{aligned}$$

$$(\Delta\hat{S}_x)^2(\Delta\hat{S}_y)^2 \geq \frac{\hbar^2}{4}\Big[\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})^2 + \frac{4}{\hbar^2}\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})^2\omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})^2\Big].$$

Thus, it is obvious that the uncertainty principle of spin $\frac{1}{2}$ particle in $\mathbb{C}P^1$ varies with time along any Hamiltonian flows since there is no such Hamiltonian flows that can preserve $\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_z})$, $\omega(\Pi_* X_{\hat{S}_y}, \Pi_* X_{\hat{S}_z})$ and $\omega(\Pi_* X_{\hat{S}_x}, \Pi_* X_{\hat{S}_y})$ simultaneously at any given time. However, if we reduce the result to the case of the Robertson uncertainty principle i.e.

$$(\Delta\hat{S}_i)^2(\Delta\hat{S}_j)^2 \geq \left(\frac{\hbar}{2}\omega(\Pi_* X_{\hat{S}_i}, \Pi_* X_{\hat{S}_j})\right)^2,$$

it will become invariant under projection of Hamiltonian flow generated by $X_{\hat{S}_k}$.This invariant property of uncertainty principle may becomes a significant step in order to constructs a connection between geometric quantum mechanics and symplectic topology.

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